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0. Introduction

The notion of collectionwise Hausdorff (cwH) spaces is an interesting by-product of recent work on the normal Moore space problem. A collection Y of points of a space X is *closed, discrete* if every $x \in X$ has a neighborhood U such that $\text{card}(U \cap Y) \leq 1$. We say that a closed discrete collection Y can be *screened* if it can be simultaneously separated by disjoint open sets. A space X is cwH ($<\lambda$cwH) [$\leq\lambda$cwH] if every closed discrete collection Y of points of X, ($|Y| < \lambda$), [$|Y| \leq \lambda$], can be screened. In this paper we investigate when "X is $<\lambda$cwH" implies "X is $\leq\lambda$cwH."

Metric spaces are cwH. The Moore road space and the Jones road space are not $\leq\omega_1$ cwH. The author has argued [F_1] that this is the "real" reason that these Moore spaces are not metrizable. Bing's example G [Bi] was the first example of a normal, not collectionwise normal space. It is not collectionwise normal for the simplest of reasons; it is not $\leq\omega_1$cwH.

The question arose of whether $\leq\omega_1$cwH spaces are cwH, or in more detail, of whether $\leq\lambda$cwH spaces were $\leq\lambda^+$cwH, (for $\lambda \geq \omega_1$). Blair [Bl], assuming GCH, and Pryzmusiński [P], modified Bing's example G to give counterexamples. Their techniques in fact give $<\lambda$cwH, not $\leq\lambda$cwH spaces for all regular $\lambda \geq \omega_1$. Using Pryzmusiński's technique, the author [F_2] gave an example of a $<\lambda$cwH, not $\leq\lambda$cwH space, for

$\lambda > cf\lambda > \omega$. All these examples have the strongest collectionwise normal properties possible, e.g. a $\leq \omega_1$ collectionwise normal, not $\leq\omega_2$cwH space, and a $\leq\omega_1$ collectionwise normal, $<\aleph_{\omega 2}$cwH, not $\leq\aleph_{\omega 2}$cwH space.

In section 1, we give an example of a regular $<\lambda$cwH, not $\leq\lambda$cwH space, for $\lambda > cf\lambda = \omega$. Again this is the strongest separation property possible.

Since cwH is a by-product of the normal Moore space question, it is natural to ask whether first countable normal spaces need be cwH. The affirmative answer is consistent with [F_3] and independent of [MS] the usual axioms of set theory. An interesting open question is whether one can prove without extra axioms of set theory (or perhaps only using GCH) that first countable, normal $\leq\omega_1$cwH (perhaps $\leq c$ cwH) spaces are cwH.

When we drop normality and retain first countability, we get that for large (weakly compact) cardinals λ, $<\lambda$cwH spaces are $\leq\lambda$cwH, (Theorem 1). For regular, not weakly compact cardinals, assuming extra axioms of set theory there are examples of $<\lambda$cwH not $\leq\lambda$cwH spaces [F_1]. These are very nice spaces--tree spaces of height ω+1. In the other direction, in the model obtained by collapsing a supercompact cardinal to ω_2, first countable tree spaces (i.e. of height $\leq\omega_1$) are cwH iff $\leq\omega_1$cwH.

For singular cardinals, we have no results for first countable spaces, but we have a result for a new cardinal function g, related to the loacl cellularity. If $\lambda > cf\lambda$, X is $<\lambda$cwH, and $g(x) < \lambda$, then X is $\leq\lambda$cwH (Theorem 3). This function g is exactly what is needed to give as corollary 5 a theorem of Shelah on the coloring number of a graph.

We conclude with a discussion of a "wide Cantor tree."

1. A $<\lambda$cwH not $\leq\lambda$cwH Space for $\mathrm{cf}\lambda = \omega$

Let $\lambda > \mathrm{cf}\lambda = \omega$. In this section we construct a $<\lambda$cwH not $\leq\lambda$cwH space.

The spaces of Pryzmusiński [P] and [F_2] are products of discrete spaces, indexed by the desired separations, with the topology strengthened by isolating all but a few special points. The not $\leq\kappa$cwH is demonstrated using the Δ-system lemma.

Δ-*System lemma* (Erdös-Rado). *If* $\kappa > \omega$ *is a regular cardinal, and* $\{F_\alpha: \alpha < \kappa\}$ *is a family of finite sets, then there are* $S \subset \kappa$, *card* $S = \kappa$, *and a finite set* R *such that* $\alpha, \beta \in S$, $\alpha \neq \beta$, *implies* $F_\alpha \cap F_\beta = R$. (We say that R is the root of the Δ-system.)

(Blair's examples were analogues of Bing's example G constructed from products of two point spaces with the $<\kappa$ box topology. This method uses a version of the Δ-system lemma with infinite F_α's.)

These spaces are $<\kappa$cwH in the following strong sense: a closed discrete collection of $<\kappa$ points can be separated by a discrete (rather than merely disjoint) family of open sets. When $\mathrm{cf}\lambda = \omega$, this strong $<\lambda$cwH implies $\leq\lambda$cwH. The prevention of strong $<\lambda$cwH gives us the clue needed to construct the desired space. We will use one-point compactifications in place of discrete spaces.

Recall that $\lambda > \mathrm{cf}\lambda = \omega$. For $a \subset \lambda$, let X_a be the space with point set $a \cup \{\infty\}$ (where $\infty \notin \lambda$). Elements of a are isolated; a neighborhood of ∞ contains ∞ and all but finitely many elements of a. The point set of the desired space Z will be $\Pi\{X_a: a \subset \lambda, \text{card } a < \lambda\}$. The topology is obtained by strengthening the usual product topology by isolating all points but the special points $\hat{\alpha}$, $\alpha \in \lambda$, where $\hat{\alpha}$ is defined by $\hat{\alpha}(a) = \alpha$ if $\alpha \in a$, $\hat{\alpha}(a) = \infty$ if $\alpha \notin a$.

Lemma. Let $a \subset \lambda$, *card* $a = \omega_1$, *and let* $\{U_\alpha: \alpha \in a\}$ *be a family of open sets of* Z, *with* $\hat{\alpha} \in U_\alpha$. *Let* H = *closure* $(\cup\{U_\alpha: \alpha \in a\})$. *Then card* $(\{\beta \in \lambda: \hat{\beta} \notin H\}) < \lambda$.

Proof. We may assume that the U_α's are basic open sets of the usual product topology. We apply the Δ-system lemma to the supports of the U_α's. We get a Δ-system $\{\text{supp } U_\alpha: \alpha \in a'\}$, card $a' = \omega_1$, with root R. Card $\cup R < \lambda$. If $\beta \notin \cup R$, then every neighborhood of $\hat{\beta}$ meets all but finitely many U_α's, $\alpha \in a'$.

That Z is not $\leq\lambda$cwH follows immediately from the lemma. We have in fact shown that Z is not weakly λcwH in the Tall sense [T].

2. Large Cardinals

We define, for κ a cardinal, Y a set, card $Y \geq \kappa$, and $y \in Y$:

$$[Y]^{<\kappa} = \{a \subset Y: \text{card } a < \kappa\}$$

$$P_y = \{a \in [Y]^{<\kappa}: y \in a\}.$$

A κ-*complete field* on I is a subfamily of $\mathcal{P}(I)$ closed under complementation and unions of cardinality $<\kappa$. A filter is κ-*complete* if it is closed under intersections of cardinality $<\kappa$. If κ is inaccessible, the κ-complete field generated by κ subsets of I has cardinality κ.

A cardinal κ is *strongly (weakly) compact* if a κ-complete filter on a κ-complete field (of cardinality κ) can be extended to an ultrafilter.

Theorem 1. *Let* X *be* $<\kappa$cwH, $\chi(X) < \kappa$.

a) If κ *is weakly compact, then* X *is* $\leq\kappa$cwH.

b) If κ *is strongly compact, then* X *is* cwH.

Proof. Let $Y \subset X$ be closed discrete, card $Y = \kappa$ in case a). We must screen Y. For each $y \in Y$, let $\{B(y,\nu): \nu < \chi(X)\}$ be a neighborhood base for y. For each $a \in [Y]^{<\kappa}$, let S_a be a set of basic open sets screening a. Such an S_a exists because X is $<\kappa$cwH.

Let $\mathcal{A}$ be the κ-complete field on $[Y]^{<\kappa}$ generated by $\{P_y: y \in Y\} \cup \{\{a: B(y,\nu) \in S_a\}: y \in Y, \nu < \chi(X)\}$. In case a), card $\mathcal{A} = \kappa$. Let F be the κ-complete filter generated by $\{P_y: y \in Y\}$, let U be a κ-complete ultrafilter extending F. For each y there is $\nu(y)$ such that $\{a: B(y,\nu(y)) \in S_a\} \in U$ because $P_y \in U$ and U is κ-complete. Then $\{B(y,\nu(y)): y \in Y\}$ screens Y.

The author conjectures that these "compactness w.r.t. cwH" properties for first countable spaces are preserved by the Levy collapse. However, he can only prove, using the technique of [Ba 7.4, 7.10] (see [F_4]):

Theorem 2. *Let* M *be a model obtained by Levy collapsing a supercompact cardinal to* ω_2. *In* M, *a first countable tree space* (i.e., *of height* $\leq\omega_1$) *is* cwH *iff* $<\omega_2$cwH.
(This theorem has also been obtained by Shelah and Litman. For a stronger conclusion, see Theorem 2 of [Sh_3].)

3. Singular Cardinals: cwH and Coloring Numbers of Graphs

In this section we introduce a cardinal function g, and use it to give a sufficient condition for $<\lambda$cwH to imply $\leq\lambda$cwH for singular cardinals. We apologize for introducing such a strange function (van Douwen suggests that g stands for gek [= crazy, Dutch]). In [F_2] we used local cellularity, but proving Shelah's theorem as a corollary seems to require using g rather than local cellularity.

(Actually g is more closely related to the local Souslin number than the local cellularity. The local Souslin number of a space X is the least cardinal κ such that every point of X has a neighborhood in which every family of disjoint open sets has cardinality strictly less than κ.)

Definition. Let X be a space, $Y \subset X$ a closed discrete collection of points, and $\mathcal{U} = \{U_y: y \in Y\}$ a family of open sets with $y \in U_y$. We say $\mathcal{V}$ partly screens $\mathcal{U}$ at z, and write $\mathcal{V} < \mathcal{U}$, if $\mathcal{V} = \{V_y: y \in Z\}$ is a family of disjoint open sets with $y \in V_y \subset U_y$, $z \in Z \subset Y$, and every V_y meets U_z. We define g(X) as the least cardinal such that for all closed discrete $Y \subset X$ there is an open family $\mathcal{U}$ such that if $\mathcal{V} < \mathcal{U}$, then card $\mathcal{V} < g(X)$.

This definition involves three alternating quantifiers, so we give some examples.

A. X is cwH iff $g(X) = 2$.

B. If X is a tree space of height $\omega+1$, $g(X) \leq \omega$.

C. If X has local Souslin number κ, $g(X) \leq \kappa$.

D. If X is regular and $g(X) < \omega$, then $g(x) = 2$.

E. Let κ be an uncountable cardinal. We topologize

$X = \{(n,\alpha): n < \omega \leq \alpha < \kappa\} \cup \kappa$ as follows. Points (n,α) are isolated. A neighborhood of $m \in \omega$ contains all but finitely many (m,α)'s; a neighborhood of $\beta \in \kappa - \omega$ contains all but finitely many (n,β)'s. X is regular, $g(X) = \omega$, the local Souslin number of X is κ^+.

Theorem 3. *If* λ *is singular,* X *is* $<\lambda$cwH, *and* $g(X) < \lambda$, *then* X *is* $\leq\lambda$cwH.

Proof. Let $Y \subset X$ be closed discrete, card $Y = \lambda > \mathrm{cf}\lambda = \delta$. The obvious thing to do is to split Y into δ pieces, each of cardinality less than λ, screen each piece, and try to arrange that the interaction of the various screenings is reparable. This in fact is our plan. It will be more convenient to consider the union of the first α pieces than the αth piece. We define a standard decomposition to be a set $\{Y_\alpha: \alpha < \delta\}$ satisfying

i) $Y_\alpha \subset Y_\beta$ if $\beta > \alpha$.

ii) $Y_\nu = \cup\{Y_\alpha: \alpha < \nu\}$ if ν is a limit ordinal.

iii) $Y = \cup\{Y_\alpha: \alpha < \delta\}$.

iv) card $Y_\alpha = c_\alpha < \lambda$.

It is not hard to verify (see Lemma 1 of [F_3]) that to screen Y it is sufficient to find a standard decomposition $\{Y_\alpha: \alpha < \delta\}$ and an increasing family $\mathcal{W} = \{W_\alpha: \alpha < \delta\}$ of open sets satisfying

(1) $Y_\alpha \subset W_\alpha$, $Y_\alpha = (\text{closure } W_\alpha) \cap Y$.

Because of the examples discussed in Section 1, we must use our hypothesis $g(X) < \lambda$ somehow. Let $\mathcal{U} = \{U_y: y \in Y\}$ be such that

(2) if $\mathcal{V} < \mathcal{U}$, then card $\mathcal{V} < g(X)$.

Our plan is to set $W_\alpha = \cup\{U_y: y \in Y_\alpha\}$. Let $A \subset Y$. If A is a subset of Y_α, then (closure $\cup\{U_y: Y \in A\}) \cap Y$, which we define to be A^*, must also be a subset of Y_α. With these conventions, (1) can be rewritten

(3) $Y_\alpha^* = Y_\alpha$.

Our plan is to take any standard decomposition, say $\mathcal{A}^0 = \{A_\alpha^0: \alpha < \delta\}$, and improve it $\mu+1$ times, where, without loss of generality $\mu = c_0$ is regular and greater than both $g(X)$ and δ. We will define standard decompositions $\mathcal{A}^\beta$, $\beta \leq \mu$, $\mathcal{A}^\beta = \{A_\alpha^\beta: \alpha < \delta\}$ such that

(4) Card $A_\alpha = c_\alpha$

(5) $A_\alpha^{\beta}* \subset A_\alpha^{\beta+1}$, $A_\alpha^\nu = \cup\{A_\alpha^\beta: \beta < \nu\}$, if ν is a limit ordinal

(6) $A_\alpha^{\mu}* = A_\alpha^\mu$ for all $\alpha < \delta$.

In order to establish (5) it is tempting to simply set $A_\alpha^{\beta+1} = A_\alpha^{\beta}*$, but this does not guarantee that $A_\nu^{\beta+1} = \cup\{A_\alpha^{\beta+1}: \alpha < \nu\}$ for limit ordinals ν. However, to establish (4) and (5) it is sufficient to show

(7) if $g(X) < \text{card } A < \lambda$, then card A = card A^*.

For then by (7), there is a map $F(A_\alpha^\beta)$ from c_α onto $A_\alpha^{\beta}*$. Set $A_\alpha^{\beta+1} = \cup\{F(A_\gamma^\beta)"c_\alpha: \gamma < \delta\}$, where $F"S = \{F(s): s \in S\}$ is the image of S under F.

We turn to the proof of (7). Aiming for a contradiction, suppose $A \subset B \subset A^*$, $g(x) < \text{card } A < \text{card } B < \lambda$. Recall that X is $<\lambda$cwH, so there is $\mathcal{V} = \{V_b: b \in B\}$, a family of disjoint open sets screening B. Because $B \subset$ closure $\cup\{U_a: a \in A\}$, for each b there is a(b) such that $V_b \cap U_{a(b)} \neq \emptyset$. By the pigeon-hole principle, some a# is a(b) for more than card A b's.

But this contradicts (2).

We now prove (6). Again aiming for a contradiction, let $z \in A_\alpha^{\mu}* - A_\alpha^{\mu}$. Note that by (5), $z \notin A_\alpha^{\beta}*$ for any $\beta < \mu$, so that for all $\beta < \mu$, U_z - (closure $\cup\{U_y: y \in A_\alpha^{\beta}\}$) is a nonempty open set. For the same reason, for every $\beta < \mu$ there is ν, $\beta < \nu < \mu$ and $a \in A_\alpha^{\nu}$ with U_a - (closure $\cup\{U_y: y \in A_\alpha^{\nu}\}$ an open set meeting U_z. Thus, since $\mu = cf\mu > g(X)$, we can inductively define $\mathcal{V} < \mathcal{U}$, card $\mathcal{V} = g(X)$. This again contradicts (2) establishing (6) and completing the proof of Theorem 3.

Definition. A graph G consists of a set V of vertices and a set E of unordered pairs of elements of V, called edges. The chromatic number of G is the least cardinal χ, such that the vertices can be painted colors with no two vertices of the same color connected by an edge. The coloring number of G, introduced by T. Rado, is the least cardinal κ such that there is a well ordering < of V such that for all $a \in V$

$$\text{card } \{(a,b) \in E: b < a\} < \kappa.$$

The chromatic number is not greater than the coloring number: use the hypothesized well ordering to inductively paint the vertices. The two notions are not the same: consider a checkerboard.

In [Sh_1] Shelah proved, among other things, Corollary 5 below. In this paper we present a simpler but less general proof.* An alternative proof is to repeat the proof of Theorem 3, setting $A^* = A \cup \{b \in V: \text{card } \{(b,a) \in E: a \in A\} \geq \kappa\}$. We feel that Theorem 4 is of interest itself.

*[Sh_2] is simpler and more general than [Sh_1].

Corollary 5 follows immediately from Theorems 3 and 4.

Definition. For G a graph, with vertices V and edges E, and κ an infinite cardinal, let $X(G,\kappa)$ be the space described below. The point set of $X(G,\kappa)$ is $V \cup E$. Points of E are isolated. N is a neighborhood of $a \in V$ iff N contains all but less than κ edges connected to a; i.e.

$$\text{card}(\{b: (a,b) \in E - N\}) < \kappa.$$

This definition immediately gives $g(X(G,\kappa)) \leq \kappa$.

Theorem 4. $X(G,\kappa)$ *is* λcwH *iff every subgraph of* G *spanned by less than* λ *vertices has coloring number* $\leq\kappa$.

Proof. If. Clearly we may assume that Y, the closed discrete set that we are to separate, is a subset of V. Let $<$ be the hypothesized well ordering of Y. Set $U_y = \{y\} \cup \{(y,z) \in E: y < z\}$. $\{U_y: y \in Y\}$ screens Y.

Only if. Let $Y \subset V$, card $Y < \lambda$. Let $\{U_y: y \in Y\}$ screen Y. The obvious thing to do is to set $y < z$ iff $(z,y) \in U_z$, but this fails in several ways to be a well-ordering. But we can use $<$ to define the desired $<''$.

First, we set $y <' z$ iff there is a chain $y = y_0 < y_1 < \cdots < y_n = z$. Every element has less than κ $<$ predecessors, so every element has at most κ $<'$ predecessors. We choose an element $y_0 \in Y$, well order by $<''$ it and its predecessors type less than or equal κ. If we are not done, we choose $y_1 \in Y$ - field $<''$, well order y_1 and its predecessors not in field $<''$ type less than or equal κ, and "add" this well ordering to $<''$. Continuing in this way, we obtain the desired well-ordering.

Corollary 5 (Shelah). *A graph* C *of singular cardinaltiy* λ *has coloring number* $\leq\kappa$ *iff every subgraph spanned by* $<\lambda$ *vertices has coloring number* $\leq\kappa$.

4. A Wide Cantor Tree

Recall that the Cantor tree T (described, for example, in [R]) is $\leq\omega$cwH, not $\leq\omega_1$cwH and has a countable dense set I of isolated points. The points of I are (or are indexed by) finite sequences of 0's and 1's, equivalently, by finite sequences of ordinals less than ω. We ask whether there is an analogous space with $\aleph_\omega$ in place of ω.

To be more specific, let I' be the set of finite sequences σ such that $\sigma(n) < \aleph_n$. Points at the top of the wide tree will be functions f from ω to $\aleph_\omega$ satisfying $f(n) < \aleph_n$. Points of I' will be isolated. The nth basic open neighborhood of a point f at the top of the tree will be $\{f\} \cup \{f|m: m \geq n\}$.

We cannot add all the functions, for then our space would contain the Cantor tree, and thus not be $\leq\omega_1$ cwH. Our question is, then, is there a set S of $\aleph_{\omega+1}$ functions such that if we add the functions of S to the top of the tree, the space is $\leq\aleph_\omega$ cwH. In a letter to the author Shelah wrote that he and Litman had showed that the yes answer is independent of and consistent with ZFC + GCH.

(The author confesses that he did not realize that Theorem 2 clearly applies in this situation to give the nonexistence of such a tree. The other direction is not so obvious, but still not difficult. V = L, specifically, Jensen's Gap-1 Two Cardinal Theorem, implies that such a tree exists).

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