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Introduction

A *continuum* is a compact connected metric space; a *tree* is a continuum which is the union of a finite collection of arcs, and contains no simple closed curve. A continuum M is *tree-like* provided that for each $\varepsilon > 0$ there is a tree T and a map $f: M \to T$ such that the inverse image of each point of T has diameter less than ε (such maps are called ε-maps). It follows from [8] that a continuum is tree-like if, and only if, it is the inverse limit of a sequence of trees with bonding maps which are surjections.

In [1] Bing has asked if tree-like continua have the fixed-point property i.e. does each map of a tree-like continuum into itself have a fixed point? Affirmative answers for special cases of this question may be found in [2], [4], [5], and [6]. This work culminates in [7], where Manka shows that a hereditarily decomposable and hereditarily unicoherent continuum has the fixed point property. (A continuum is *hereditarily unicoherent* provided that the intersection of each pair of subcontinua is a continuum. Tree-like continua are hereditarily unicoherent; if a continuum is hereditarily decomposable, the converse holds [3].)

Principal Theorem

We proceed to prove our main result. We are indebted to L. Wayne Goodwyn, who suggested this approach.

Theorem. If there is a tree-like continuum M *and a fixed-point-free map* $f\colon M \to M$, *then there is an indecomposable tree-like continuum* X *and a homeomorphism* $h\colon X \to X$ *such that* h *does not send any proper subcontinuum of* X *into itself.*

Proof. Using the Brouwer Reduction Theorem, one can see that there is a subcontinuum Y of M which is minimal with respect to being mapped into itself. Clearly, $f[Y] = Y$ and, as a subcontinuum of M, Y is tree-like. Let Z be the inverse limit of the sequence (Y_i, f_i), where, for each i, $Y_i = Y$ and f_i is f restricted to Y. Then Z is a continuum. We will show that Z is tree-like by showing that for each $\varepsilon > 0$, there is an ε-map of Z onto a tree. Suppose then that $\varepsilon > 0$ is fixed, and for each i, let Q_i be the projection map of Z onto Y_i. There is a positive integer j so that Q_j is an ε-map. Using the compactness of Y_j, it is easy to see that there is a $\delta > 0$ so that if A is a subset of Y_j of diameter less than δ, then $\operatorname{diam}(Q_j^{-1}(A)) < \varepsilon$. Since Y_j is tree-like, there is a tree T and a δ-map $p\colon Y_j \to T$. Then $p \circ Q_j$ is an ε-map of Z into T, so Z is tree-like.

We now define a homeomorphism $h\colon Z \to Z$ by $h(z_1, z_2, z_3, \cdots) = (z_2, z_3, \cdots)$. As a function into a product space, h is clearly continuous. Moreover, if $h(z) = h(x)$, then $(z_2, z_3, z_4, \cdots) = (x_2, x_3, x_4, \cdots)$ so $z_i = x_i$ if $i \geq 2$. Then $z_1 = f(z_2) = f(x_2) = x_1$, so $z = x$, and h is one-to-one. Also, h is fixed-point free, since if $h(x) = x$ then $(x_2, x_3, x_4, \cdots) = (x_1, x_2, x_3, \cdots)$ and so $x_2 = x_1 = f(x_2)$ and f fixes x_2, which contradicts our assumption about f.

We can again apply the Brouwer Reduction Theorem, and obtain a (necessarily tree-like) subcontinuum X of Z so that

$h[X] = X$ and no proper subcontinuum of X is carried into itself by h. To conclude our argument, we will use a technique of Gray to show that X must be indecomposable. Suppose, to the contrary, that there are proper subcontinua A_0 and B_0 of X such that $X = A_0 \cup B_0$.

For each positive integer i, let $A_i = h^{-i}[A_0]$ and $B_i = h^{-i}[B_0]$. The two sequences $A_0, A_1, \cdots$ and $B_0, B_1, \cdots$ have the following properties:

1) For each i, A_i and B_i are continua and $X = A_i \cup B_i$

2) $A_m \cap A_n \neq \phi$ if, and only if, $A_{m+1} \cap A_{n+1} \neq \phi$

3) $B_m \cap B_n \neq \phi$ if, and only if, $B_{m+1} \cap B_{n+1} \neq \phi$.

Applying [5, Lemma 2], we conclude that either $\cap\{A_n: n \geq 0\} \neq \phi$ or $\cap\{B_n: n \geq 0\} \neq \phi$. Let $L = \cap\{A_n: n \geq 0\}$ and suppose that $L \neq \phi$. Since X is hereditarily unicoherent, L is a continuum. Also, $L \subset A_0$, so L is a proper subcontinuum of X. Clearly, $h[A_i] \subset A_{i-1}$ if $i \geq 1$, so $h[L] \subset L$, contradicting the fact that no proper subcontinuum of X is mapped into itself. This concludes the proof.

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