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by

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STEENROD HOMOTOPY THEORY, HOMOTOPY INDEMPOTENTS, AND HOMOTOPY LIMITS

Harold M. Hastings[1]

1. Introduction

In 1940, N. E. Steenrod [26] introduced a homology theory ${}^{S}H_*$ on compact metric pairs, which is exact on *all* pairs (X,A). The continuity axiom of Čech homology is replaced by a short-exact sequence ([26], and J. Milnor [23]):

$$(1.1)\quad 0 \to \lim_n^1\{H_{i+1}(X_n)\} \to {}^{S}H_i(X) \to \check{H}_i(X) \to 0.$$

In (1.1), $\{X_n\}$ is any tower (inverse sequence) of polyhedra whose inverse limit is X. D. A. Edwards and the author [11, Ch. VIII] observed that any generalized homology theory yields a "Steenrod" homology theory on the category of towers of spaces; in fact, on Grothendieck's category pro-Top of inverse systems of spaces. See M. Artin and B. Mazur [1, Appendix] for pro-Top. Our joint work required a strong (Steenrod) homotopy theory of pro-spaces [11, Ch. III]. Although the precise definition of Steenrod homotopy theory is fairly complex, we can relate Čech (Artin-Mazur, [4]) and Steenrod [11, 12, 13] homotopy theory in §2 below. Motivated by the Brown-Douglas-Fillmore [2,3] theory of normal operators, D. S. Kahn, J. Kaminker and C. Schochet gave a different, independent development of generalized Steenrod homology theories [16, 17].

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The rest of this paper is organized as follows. §§3-5 survey Steenrod homotopy theory. Homotopy limits, largely following [11, Ch. IV] are described in §3. §4 recalls the Edwards-Geoghegan [10] result that "idempotents split in pro-categories." J. Dydak and P. Minc [8], and P. Freyd and A. Heller [14] independently obtained a *non*-split idempotent in unpointed homotopy theory. Dydak observed an important consequence in pro-homotopy: a map which is an equivalence in Čech homotopy theory but *not* in Steenrod homotopy theory. We summarize these results in §5 to complete the relation between Čech and Steenrod homotopy theory.

Finally, we give "geometric" (Artin-Mazur [1] type) formulations of a coherent completion functor (§6) and a strong shape functor (§7). We conclude with a "dual" construction of a coherent Quillen [26] + - construction in §8.

We thank D. A. Edwards, A. Heller, G. Kozlowski, Vo Thanh Liem, S. Mardešić and D. Puppe for helpful conversations.

2. Cech and Steenrod Homotopy Theory

T. A. Chapman's [5] beautiful *complement theorem* relating the shape theory of compacta K in the pseudo-interior $s = \Pi_{i=1}^{\infty} (-1,1)$ of the Hilbert cube $Q = \Pi_{i=1}^{\infty} [-1,1]$ and the homeomorphism type of $Q \backslash K$ has the following corollary [5]: a category isomorphism between the shape category of such compacta and the *weak* proper homotopy category of their complements. Later, Edwards and the author [11, pp. 228-232] obtained a similar relationship between strong shape theory [11, especially Ch. VI and VIII] and the more geometric proper homotopy theory, which together with Chapman's correspondence homotopy theory, which together with Chapman's correspondence yields a commutative square

$$
(2.1)\qquad
\begin{array}{ccc}
\left\{\begin{array}{l}\text{strong shape}\\ \text{(Steenrod homotopy)}\\ \text{category of compacta}\\ K \subset s \subset Q\end{array}\right\} & \xrightarrow{\ \pi\ } & \left\{\begin{array}{l}\text{shape}\\ \text{(Čech homotopy)}\\ \text{category of compacta}\\ K \subset s \subset Q\end{array}\right\} \\
\downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\
\left\{\begin{array}{l}\text{proper homotopy cate-}\\ \text{gory of complements}\\ Q\backslash K,\ K \subset s \subset Q\end{array}\right\} & \xrightarrow{\ \pi'\ } & \left\{\begin{array}{l}\text{weak proper homotopy}\\ \text{category of complements}\\ Q\backslash K,\ K \subset s \subset Q\end{array}\right\}.
\end{array}
$$

In diagram (2.1), Steenrod homotopy theory refers to the strong homotopy theory of inverse systems Ho (pro-Top) of Edwards and the author [11, especially Ch. VIII]; Čech homotopy theory to the Artin-Mazur theory [1] pro-Ho (Top). The vertical map ϕ is Chapman's isomorphism cited above; similarly, ϕ' is the isomorphism of [11, *loc. cit.*]. The maps π and π' are natural quotient maps. The Čech nerve Top $\to$ pro-Ho (Top) yields shape theory (see, e.g. Edwards [9]); a Vietoris functor Top $\to$ Ho (pro-Top) (T. Porter [24]) yields strong shape theory [11]. The distinction between Čech and Steenrod homotopy theory was first recognized by D. Christie [7], although he lacked D. Quillen's abstract homotopy theory [25] needed to define Ho (pro-Top) [11]. We shall give a more "geometric" version of strong shape theory (still using [11]) in §7--some of whose properties were obtained in a conversation with Kozlowski and Liem. Details and applications will be described elsewhere. *Added in proof.* See joint work with A. Calder [30]. J. Dydak and J. Segal [31] and Y, Kodama and J. Ono [32] recently gave independent equivalent descriptions of strong shape theory.

Although the relationship between Ho (pro-Top) and pro-

Ho (Top) appears quite complicated [11], useful results are available for towers (countable inverse systems). Let Top_* be the category of pointed spaces and maps. In 1974, J. Grossman [15], and Edwards and the author [11, Theorem (5.2.1)] independently proved the following.

(2.2) *Theorem. Let* $\{X_m\}$ *and* $\{Y_n\}$ *be towers of pointed spaces. Then there is a short-exact sequence of pointed sets.*

$$* \to \lim_n^1 \operatorname{colim}_m \{[\Sigma X_m, Y_n]\} \to Ho(towers\text{-}Top_*)(\{X_m\},\{Y_n\}) \overset{\pi}{\to} towers\text{-}Ho(Top_*)(\{X_m\},\{Y_n\}) \to *.$$

The functor π is also onto in unpointed pro-homotopy. The appropriate derived functor $\lim^1$ for towers of (non-abelian) groups was defined by Bousfield and Kan [4, p. 251].

Chapman and L. Siebenmann [6] asked whether every weak-proper-homotopy-equivalence is a proper-homotopy-equivalence. A useful partial answer appears in [11, Theorem (5.2.9)]; similar results hold for pointed spaces [11, *loc. cit.*], and for proper homotopy [12].

(2.3) *Theorem* [11]. *Let* f: $\{X_m\} \to \{Y_n\}$ *be a map in Ho (towers-Top) which is invertible in towers-Ho (Top). Then there is an isomorphism* g: $\{X_m\} \to \{Y_n\}$ *in Ho (towers-Top) with* g *equivalent to* f *in towers-Ho (Top).*

(2.4) *Corollary* [11, Corollary 5.2.17]. *The isomorphism classification problems in Ho (towers-Top) and towers-Ho (Top) are equivalent.*

(2.5) *Caution: non-equivalent* maps in Ho (towers-Top) may become equivalent in towers-Ho (Top).

Dydak [8] recently observed that the map f of (2.3) need *not* itself be invertible in Ho (towers-Top). This result involves homotopy limits (§3) and splitting idempotents (§4), and will be discussed in §5.

3. Homotopy Limits

It is easy to see that even towers do *not* have limits in homotopy theory. D. Puppe gave the following example in a 1976 lecture in Dubrovnik. Let

$$K = \{K(Z,2) \overset{3}{\leftarrow} K(Z,2) \overset{3}{\leftarrow} \cdots\} ,$$

where "3" denotes a degree 3 map. Suppose that K had a limit $\overline{K}$ in pro-Ho (Top). Then the Barratt-Puppe sequence

$$S^2 \overset{2}{\rightarrow} S^2 \rightarrow C \rightarrow S^3 \rightarrow \cdots$$

would yield an exact sequence

$$\begin{array}{ccccc} [S^3,\overline{K}] & \longrightarrow & [C,\overline{K}] & \longrightarrow & [S^2,\overline{K}] \\ \| & & \| & & \| \\ \lim\{[S^3,K(Z,2)],3\} & \rightarrow & \lim\{[C,K(Z,2)],3\} & \rightarrow & \lim\{S^2,K(Z,2)],3\} \\ \| & & \| & & \| \\ 0 & \longrightarrow & Z_2 & \longrightarrow & 0, \end{array}$$

an obvious contradiction. However, *homotopy limits* exist in Ho (pro-Top); for a pro-space $Y = \{Y_\alpha\}$, the functor

$$\text{Ho (pro-Top)} \ (-,\{Y_\alpha\})$$

on Top $\subset$ pro-Top is represented by holim$\{Y_\alpha\}$ (Edwards and the author [11, Ch. IV]):

$$\text{Ho (pro-Top)}(X,\{Y_\alpha\}) = \text{Ho (Top)}(X,\text{holim}\{Y_\alpha\}).$$

The construction of [11], reminescent of J. Milnor's [20] mapping telescope, consists of replacing $\{Y_\alpha\}$ by a *fibrant* object (using S. Mardešić [17], and [11]) Y'_β and applying the ordinary inverse limit to Y'_β. Other constructions were given by A. K. Bousfield and D. M. Kan [4, Ch. X], and R. Vogt [29].

4. Splitting Homotopy Idempotents

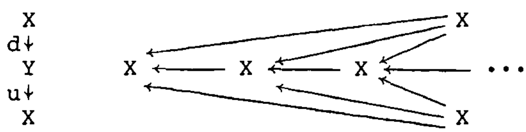
D. A. Edwards and R. Geohegan, in their work [10] on a Wall obstruction on shape theory, showed that "idempotents split in pro-categories." Let $r: X \to X$ be a homotopy idempotent, i.e., $r^2 \simeq r$. If there is a diagram

$$X \underset{u}{\overset{d}{\rightleftarrows}} Y$$

with $du \simeq id_Y$ and $ud \simeq r$, then r is said to *split*. Let Y be the tower

$$Y = \{X \overset{r}{\leftarrow} X \overset{r}{\leftarrow} X \overset{r}{\leftarrow} \cdots\}.$$

Then r induces maps $X \underset{u}{\overset{d}{\rightleftarrows}} Y$ in Čech homotopy theory (towers-Ho (Top))



(all arrows r)

which split r. We may replace Y by a tower of fibrations, and then replace u and d by strict maps (maps in Steenrod homotopy Ho (towers-Top)) [10], see also [11]. Suppose $du \simeq id_Y$ in Ho (towers-Top). Then r splits in Ho (towers-Top) [10] because holim is a functor:

$$X \simeq \text{holim } X \underset{\text{holim } u}{\overset{\text{holim } d}{\rightleftarrows}} \text{holim } Y.$$

The Dydak-Minc [8], Freyd-Heller [14] example of a non-split idempotent in unpointed homotopy (described in §5) thus shows that a weak equivalence need not be a strong equivalence [8], compare Theorem (2.3) (Edwards and the author), above.

5. The Dydak-Minc-Freyd-Heller Example [8, 14]

Let G be the group

$$\langle g_1, g_2, \cdots | g_i^{-1} g_j g_i = g_{j+1}, i < j \rangle .$$

Let f: G → G be the monorphism defined by

$$f(g_i) = g_{i+1}.$$

Then $f^2(g) = g_1^{-1} f(g) g_1$, so that f is conjugate to f, and the induced map

$$r = K(f,1): K(G,1) \to K(G,1)$$

is an *unpointed* homotopy idempotent [8, 14]. Dydak gives a straight-forward argument that r does not split--we sketch his argument here. If r splits, r splits through a K(H,1). In the resulting diagram

$$K(G,1) \underset{u}{\overset{d}{\rightleftarrows}} K(H,1)$$

d is both mono and epi on π_1 by construction, hence d is a homotopy equivalence by the Whitehead theorem. This implies Imf = G, an evident contradiction.

Freyd and Heller [14] have obtained a wealth of interesting results about G.

6. Pro-Finite Completions

Artin and Mazur [1] introduced the following pro-finite completion in order to prove comparison theorems in étale homotopy theory. Let Y be a finite, pointed CW complex. The pro-finite completion of Y, $\hat{Y}$, is the category whose objects are (homotopy classes of pointed) maps

$$Y \to Y_\alpha, \text{ with } \pi_i(Y_\alpha) = \begin{cases} 0, & \text{almost all } i \\ \text{finite}, & \text{otherwise}, \end{cases}$$

and whose morphisms are *homotopy-commutative* diagrams

(6.1)

$$\begin{array}{ccc} & Y & \\ \swarrow & & \searrow \\ Y_\alpha & \longrightarrow & Y_\beta \end{array}$$

This yields a completion functor ^:Ho (finite pointed complexes) $\to$ pro-Ho (Top) as follows. Given a map $X \to Y$, associate to each object $Y \to Y_\alpha$ in the completion $\hat{Y}$ of Y the composite map $X \to Y \to Y_\alpha$, an object in $\hat{X}$. This yields a map $\hat{f}$: $\hat{X} \to \hat{Y}$, see [1, Appendix].

D. Sullivan [28] showed that pro-Ho (Top)(-,Y) is representable, that is,

$$\text{pro-Ho (Top)}(-,\hat{Y}) = [-,\overline{Y}].$$

Later, A. K. Bousfield and D. M. Kan [4, Ch. I] introduced a different, *rigid*, completion, the R-completion $\{R_sY\}$, and observed that $\{R_sY\}$ is cofinal in an Artin-Mazur type R-completion. Here R is a commutative ring with identity; we call $\{R_sY\}$ rigid because the construction of $\{R_sY\}$ yields a functor into Ho (pro-Top).

In developing the "genetics of homotopy theory" [28], D. Sullivan remarked that a simple rigid completion functor could prove useful. We shall rigidify (i.e., lift to Ho (gpro-Top) the Artin-Mazur completion functor by a simple trick. Objects of gpro-Top are inverse systems of spaces which are filtering *up to homotopy*. See [30].

(6.2) *Definition*. The rigid pro-finite completion of a (finite, pointed) complex Y is the category $\hat{Y}_{rig}$ whose objects are pointed maps

$$Y \to Y_\alpha, \text{ with } \pi_i(Y_\alpha) = \begin{cases} 0, & \text{almost all } i \\ \text{finite}, & \text{otherwise}, \end{cases}$$

and whose morphisms are *strictly commutative* diagrams

$$\begin{array}{ccc} & Y & \\ \swarrow & & \searrow \\ Y_\alpha & \xrightarrow{\text{fibration}} & Y_\beta \end{array}$$

in which the bottom map is a *fibration.*

Because the pullback of a diagram

$$\begin{array}{ccc} & & Y_\alpha \\ & & \downarrow{\scriptstyle\text{fibration}} \\ Y_\beta & \xrightarrow{\text{fibration}} & Y_\gamma \end{array}$$

is also a "homotopy pullback," $\hat{Y}_{rig}$ has weak equalizers. It follows easily that $\hat{Y}_{rig}$ is filtering up to homotopy. Further, the functor

$$\text{Ho (pro-Top)}[-,\hat{Y}_{rig}] \cong [-,\text{holim } \hat{Y}_{rig}]$$

is clearly representable, and the Bousfield-Kan spectral sequence [3, Ch. XI] shows that holim $\hat{Y}_{rig} \simeq \overline{Y}$, see (6.1).

(6.3) *Remarks.* The rigid pro-finite completion $\hat{\ }_{rig}$ induces a *reflection* Ho(gpro-Top) $\to$ Ho(gpro-Top), i.e., $(\hat{X}_{rig})\hat{\ }_{rig} = \hat{X}_{rig}$, *always.* In contrast, for the Bousfield-Kan completion, $Z_\infty RP^2$ and $(Z_\infty)^2\, RP^2$ are not equivalent, thus, RP^2 is called Z-bad [4, Ch. I]. Further, there should be an induced homotopy theory (closed model structure [25]) on the image of $\hat{\ }_{rig}$ under which $\hat{\ }_{rig}$ preserves fibration and cofibration sequences. Sullivan's completion functor cannot preserve both types of sequences [28]. Note however, that the inverse limit lim:gpro-Top $\to$ Top preserves fibration sequences but *not* cofibration sequences.

7. Strong Shape Theory

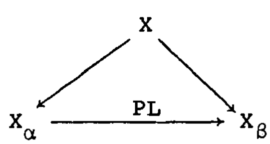
S. Mardešić [21] introduced the following Artin-Mazur approach to the shape theory. The shape of a topological space X, sh(x), is the category whose objects are homotopy classes of maps $X \to X_\alpha$, where X_α is an ANR, and whose morphisms are *homotopy-commutative* triangles of the form (6.1). Let f: $X \to Y$ be a continuous map. Then each map $Y \to Y_\alpha$ in sh(Y) induces a map $X \to Y_\alpha$ by composition with f; this yields a *shape functor*

$$\text{sh: Top} \to \text{pro-Ho(ANR)} \subset \text{pro-Ho(Top)}.$$

One can replace "ANR" by "polyhedron" (possibly infinite) in the Mardešić definition.

We rigidify the Mardešić shape functor (i.e., lift sh to Ho (pro-Top)) by a trick analogous to (6.1), and briefly describe the resulting geometric strong shape theory. A Vietoris functor approach to strong shape theory is developed in Porter [24] and [11, Ch. VIII].

(7.1) *Definition*. The *strong shape* of a topological space X, s - sh(X), is the category whose objects are *maps* $X \to X_\alpha$, with X_α a polyhedron, and whose morphisms are strictly *commutative triangles*



in which the bottom map is PL.

(7.2) *Proposition*. *This construction yields a functor* s - sh: *Top* $\to$ *pro-(polyhedra)* $\subset$ *pro-Top*. *Further, the composite functor* $\pi \circ$ s - sh: *Top* $\to$ *pro-Ho(Top) is equivalent* *to Mardešić's shape functor* sh.

Proof. Observe that the equalizer of two PL maps of polyhedra is a polyhedron. Thus s - sh(X) has equalizers. The rest is easy and omitted.

(7.3) *Proposition. The functor* s - sh *induces a functor on homotopy categories*

$$s - sh: \; Ho(Top) \to Ho(pro\text{-}Top).$$

Proof. Let $H: X \times I \to Y$ be a homotopy, with $H_0 = f$ and $H_1 = g$. Form the commutative diagram in pro-Top

$$\begin{array}{ccc} s - sh(X\times 0) & \xrightarrow{\;s - sh(f)\;} & \\ \downarrow & & \\ s - sh(X\times I) & \xrightarrow{\;s - sh(H)\;} & s - sh(Y) \\ \uparrow & & \\ s - sh(X\times 1) & \xrightarrow{\;s - sh(g)\;} & \end{array}$$

Each map $\phi_\alpha: X\times I \to Z_\alpha$ in s - sh(X×I) factors as

$$X\times I \xrightarrow{(\phi_\alpha,\ proj_I)} Z_\alpha \times I \xrightarrow{\quad proj \quad} Z_\alpha,$$

hence the map s - sh(X×O) → s - sh(X×I) is represented by the inverse system of maps

$$\begin{array}{ccc} X\times O & \longrightarrow & X\times I \\ \vdots & & \vdots \\ \{Z_\alpha \times O & \longrightarrow & Z_\alpha \times I\} \end{array}$$

Thus the map s - sh(X×O) → s - sh(X×I) is a trivial cofibration (i.e. cofibration and equivalence in Ho (pro-Top)), similarly for X×1. (Note: we are *not* asserting that the maps $X\times I \to Z_\alpha \times I$ factor as g×id, only that the *bonding maps* $Z_\alpha \times I \to Z_\beta \times I$ factor in this way). The conclusion follows.

We now restrict the domain of s - sh to the category CM of *compact metric spaces*. We may then assume s - sh takes values in pro-(finite polyhedra).

(7.4) *Proposition. For* X *in* CM, $X \cong lim \circ s - sh(X)$.

Proof. It suffices to prove that natural map $p: X \to \lim \circ s - sh(X)$ is bijective. Because any two distinct points of X are separated by a map of X into [0,1], p is injective. Further any map $X \to X_\alpha$ in s - sh(X) which misses a point * in X_α factors through a subpolyhedron $X'_\alpha \subset X_\alpha$ with $* \notin X'_\alpha$. The conclusion follows.

Propositions (7.2)-(7.4) are summarized in the following diagram--which justifies calling s - sh a *strong-shape functor*--

$$\begin{array}{ccc}
\text{CM} & \underset{\lim}{\overset{s - sh}{\rightleftarrows}} & \text{pro-(finite polyhedra)} \\
\downarrow & & \downarrow \\
\text{Ho(CM)} & \xrightarrow{s - sh} & \text{Ho(pro-(finite polyhedra))} \\
& \searrow^{sh} & \downarrow \\
& & \text{pro-Ho(finite polyhedra)}.
\end{array}$$

(7.5) *Proposition (with Kozlowski-Liem). For any compact metric pair* (X,A), *the sequence*

$$s - sh(A) \to s - sh(X) \to s - sh(X/A)$$

is a cofibration sequence in pro-top.

Proof. Consider the inverse system whose objects are commutative diagrams

$$\begin{array}{ccccc}
A & \text{———} & X & \text{———} & X/A \\
\downarrow & & \downarrow & & \downarrow \\
A_\alpha & \text{———} & X_\alpha & \text{———} & X_\alpha/A_\alpha
\end{array}$$

with (X_α, A_α) a finite polyhedral pair, and whose bonding maps are defined analogously with (7.1). The induced systems $\{X_\alpha\}$ and $\{X_\alpha/A_\alpha\}$ are clearly cofinal in s - sh(X) and s - sh(X/A), respectively: given $X \to X_\alpha$ in s - sh(X), let $A_\alpha = X_\alpha$, and given $X/A \to P_\alpha$ in s - sh(X/A), let A_α be a point, and let $X_\alpha = P_\alpha$.

Finally Kozlowski remarked that any solid-arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & P \\ \downarrow & & \downarrow \\ X & \longrightarrow & CP \end{array}$$

(where CP is the cone on P) with (X,A) a compact metric pair admits a filler (compare Kuratowski's extension lemma for Čech nerves [18, p. 122]). This implies that the induced inverse system A is cofinal in s - sh(A). The conclusion follows.

Propositions (7.2) and (7.5) imply the following (compare D. A. Edwards and the author [11, Ch. VIII]).

(7.6) *Proposition. For any homology theory* h_* *on pro-Top, the composite* $h_* \circ$ s - sh *is a homology theory on* CM.

By comparing s - sh with the Vietoris functor [24], and using the machinery of [11, Ch. VIII], we can prove the appropriate continuity formula

(7.7) s - sh($\lim\{X_n\}$) $\simeq$ *lim* s - sh($\{X_n\}$) *in Ho-pro (Top) for* s - sh. Formula (7.7) implies the following.

(7.8) *Proposition (compare* [11, *Theorem* (8.2.21)]) *The composite functor* h s - sh *is a generalized Steenrod homology theory.*

(7.9) *Remarks.* (a) Formula (7.7) is analogous to the Steenrod-Milnor short exact sequence (1.1).

(b) The relationship between the strong shape category and the shape category is analogous to the relation between Steenrod and Čech homotopy theory, see (2.2)-(2.5), above.

D. S. Kahn, J. Kaminker, and C. Schochet [16] developed yet another independent approach to Steenrod homology theory--see L. Brown, R. Douglas, and P. Fillmore [2,3] and compare the Mardešić-J. Segal [21] natural transformation approach to shape theory.

Unfortunately we do not have a purely geometric proof of (7.7).

8. A Rigid + - Construction

We outline a rigidification of Quillen's + - construction [26] using techniques of Edwards and the author [11], dual to §§3-7, above. For simplicity, let Top_P be the category of pointed spaces with perfect fundamental groups. We define a functor

$$+: Top_P \to Top_P$$

such that our X^+ is equivalent to Quillen's X^+, and such that the diagram

$$(8.1)\qquad \begin{array}{ccc} Top_P & \xrightarrow{\quad + \text{ (ours)} \quad} & Top_P \\ \downarrow & & \downarrow \\ Ho(Top_P) & \xrightarrow{\quad + \text{ (Quillen)} \quad} & Ho(Top_P) \end{array}$$

commutes. In fact our techniques work for pairs (X,H) where X is a pointed space and H a normal subgroup of $\pi_1(X)$ containing $[\pi_1(X), \pi_1(X)]$.

First associate to X the category + (X) whose objects are maps $X \to X_\alpha$ with and whose morphisms are commutative triangles

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ X_\alpha & \xrightarrow{\text{cofib.}} & X_\beta \end{array}$$

in which the bottom map is a cofibration. It is easy to check that + (X) is a *direct* system, filtering *up to homotopy* (see [30], reverse the arrows in the Artin-Mazur definition [1, Appendix] of an (*inverse*) filtering category).

Next define

(8.2) $$X^+ = \text{hocolim } (+(X))$$

where hocolim is the homotopy colimit ([11, pp. 169-171] the dual of the homotopy limit sketched in §3 or Bousfield-Kan [4, Ch. XII]). It is easy to check that Definitions (8.2)-(8.3) yield the required properties. Details are omitted.

References

1. M. Artin and B. Mazur, *Étale homotopy*, Lecture Notes in Math. 100, Springer, Berlin-Heidelberg-New York, 1969.

2. L. Brown, R. Douglas, and P. Fillmore, *Unitary equivalence modulo the compact operators and extensions of* C-*algebras*, in Proc. Conf. on Operator Theory, Lecture Notes in Math. 345, Springer, Berlin-Heidelberg-New York, 1973, 58-128.

3. ________, *Extensions of* C-*algebras, operators with compact self-commutators, and* K-*homology*, Bull. Amer. Math. Soc. 79 (1973), 973-978.

4. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions, and localizations*, Lecture Notes in Math. 304, Springer, Berlin-Heidelberg-New York, 1973.

5. T. A. Chapman, *On some applications of infinite dimensional manifolds to the theory of shape*, Fund. Math. 76

(1972), 191-193.

6. ________, and L. Siebenmann, *Finding a boundary for a Hilbert cube manifold*, to appear.
7. D. Christie, *Net homotopy theory for compacta*, Trans. Amer. Math. Soc. 56 (1944), 275-308.
8. J. Dydak, *A simple proof that pointed, connected* FANR *spaces are regular fundamental retracts of* ANR*'s*, Bull. Acad. Polon. Sci., ser. math. astr. phys. 25 (1977), 55-62.
9. D. A. Edwards, *Étale homotopy and shape*, in *Algebraic and geometric methods in topology*, ed. L. McAuley, Lecture Notes in Math. 428, Springer, Berlin-Heidelberg-New York, 1974, pp. 58-107.
10. ________, and R. Geoghegan, *Shapes of complexes, ends of manifolds, homotopy limits, and the Wall obstruction*, Ann. of Math. (2) 101 (1975), 521-535.
11. D. A. Edwards and H. M. Hastings, *Cech and Steenrod homotopy theories with applications to geometric topology*, Lecture Notes in Math. 542, Springer, Berlin-Heidelberg-New York, 1976.
12. ________, *Every weak proper homotopy equivalence is weakly properly homotopic to a proper homotopy equivalence*, Trans. Amer. Math. Soc. 221 (1976), 239-248.
13. ________, *Why the* R*-completion works*, Gen. Topology and Applics. 7 (1977), 179-184.
14. P. Freyd and A. Heller, *On splitting idempotents*, to appear.
15. J. Grossman, *Homotopy classes of maps between pro-spaces*, Mich. Math. J. 21 (1974), 355-362.
16. D. S. Kahn, J. Kaminker, and C. Schochet, *Generalized homology theories on compact metric spaces*, Mich. Math. J. 24 (1977), 203-224.
17. J. Kaminker and C. Schochet, K*-theory and Steenrod homology: Applications to the Brown-Douglas-Fillmore theory of operator algebras*, Trans. Amer. Math. Soc. 227 (1977), 63-107.
18. C. Kuratowski, *Topologie* II, Monografje Matematyczne, Warsaw, 1948.
19. S. Mardešić, *On the Whitehead theorem in shape theory*, Fund. Math. 91 (1976), 51-64.

20. ________, *Shapes for general topological spaces*, Gen. Top. and its Appl. 2 (1972), 17-23.

21. ________, and J. Segal, *Shapes of compacta and* ANR *systems*, Fund. Math. 72 (1971), 61-68.

22. J. Milnor, *On the Steenrod homology theory*, Mimeo, Berkeley, 1961.

23. ________. *On axiomatic homology theory*, Pacific J. Math. 12 (1962), 337-341.

24. T. Porter, *Cech homotopy* I, J. London Math. Soc. (2) 6 (1973), 429-436.

25. D. Quillen, *Homotopical algebra*, Lecture Notes in Math. 43, Springer, Berlin-Heidelberg-New York, 1967.

26. ________, *Cohomology of groupes*, Actes Congrès Intern. 2, (1970), 47-51.

27. N. E. Steenrod, *Regular cycles on compact metric spaces*, Ann. of Math. (2) 41 (1940), 833-851.

28. D. Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math (2) 100 (1974), 1-79.

29. R. Vogt, *Homotopy limits and colimits*, Math. Z. 134 (1973), 11-52.

30. A. Calder and H. M. Hastings, *Realizing strong shape equivalences*, to appear.

31. J. Dydak and J. Segal, *Strong shape theory*, to appear.

32. Y. Kodama and J. Ono, *On fine shape theory, I, II*, to appear.

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