THE STONE-ČECH COMPACTIFICATION AND SHAPE DIMENSION

by

James Keesling

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Introduction

This paper is a continuation of the work of Keesling [8] and Keesling and Sher [9] in studying the shape properties of the Stone-Čech compactification of a space. For a compact space X, shape theory is the study of those properties of X that can be deduced by considering the homotopy classes of mappings of X into various polyhedra P, [X,P]. Here we are concerned with P = M^n, a closed manifold which is a $K(\pi,1)$ and also P = $S^n \vee S^1$. By considering maps of the Stone-Čech compactification onto these polyhedra we are able to draw certain conclusions about the shape dimension of the Stone-Čech compactification. Our specific results are as follows. Let f: X → Y be a map. Then f is *homotopically onto* provided that if g: X → Y is homotopic to f, then g is onto Y. Let n ≥ 1 be a fixed integer and let M^n be a closed manifold having the property that the universal covering space for M^n is R^n. Suppose that X is locally compact and σ-compact such that for every compact set K ⊂ X, there is a compact set L ⊂ X-K such that dim L ≥ n. Then there is a map f: βX → M^n which is homotopically onto. Also f|βX-X is homotopically onto M^n as well. This implies that the shape dimension of βX is at least n and that the shape dimension of βX-X is at least n also. If X is a Lindelöf space and K is a compactum contained in βX-X, then if dim K ≥ n and M^n is as above, then

there is a map f: K → M^n which is homotopically onto. This

implies that for compacta K ⊂ βX-X, dimension and shape di-

mension coincide. This is generally not true since the di-

mension of the closed unit ball B^n in R^n is n, but the shape

dimension of B^n is 0.

The above results are used to show the following. For

every n ≥ 2, there is a polyhedron P_n of dimension n and a

map f: P_n → T^n to the n-dimensional torus T^n such that f is

homotopically onto and f*: $H^k(T^n)$ → $H^k(P_n)$ is the zero

homomorphism for all k ≥ 2. Also for every n ≥ 2, there is

a metric continuum X_n such that the shape dimension of X_n is

n and dim X_n = n, but the shape dimension of the suspension

ΣX_n is equal to 2. This last example answers a question of

S. Nowak [11].

We then consider maps onto $S^n \vee S^1$ and obtain similar re-

sults. However, in this case we encounter a dimension theo-

retic difficulty and the results are not quite so general as

are the results about mappings onto manifold K(π,1)'s.

It appears from the results in this paper and in [8] and

[9] that shape theory can be a useful tool in studying the

Stone-Čech compactification. The techniques and results

will likely have wider application in the future.

Preliminaries

The reader is assumed to be familiar with shape theory.

The paper by S. Mardešić is a good reference [12]. We will

only be concerned about compact spaces in this paper, how-

ever, and thus will not need the full generality of [12]

which develops shape theory for all topological spaces. The

reader is assumed to be familiar with the basic properties
of the Stone-Čech compactification. Either Gillman and
Jerison [2] or Walker [14] would be a good reference. In
dimension theory we need only the basic results which we
quote from [4] and the Appendix of [10]. We now state the
most important of these for easy reference. By dim X we
mean the Lebesgue covering dimension.

0.1. *Theorem* ([4, 19, p. 85]). *The dimension of a*
normal space X *is the supremum of the integers* n *such that*
there is an essential mapping onto I^n.

Here an *essential mapping* f: X → I^n is a map such that
$f|f^{-1}(BdI^n)$ cannot be extended to all of X with values in
BdI^n. We will also need the Sum Theorem for dimension.

0.2. *Theorem* ([4, Theorem 7, p. 148]). *If* X *is a*
normal space with X = $\cup_{i=1}^{\infty}X_i$ *with* X_i *closed for all* i, *then*
dim X = *sup*{*dim* X_i}.

We will also need the Hopf Extension Theorem.

0.3. *Theorem* ([10, 36-17, p. 207]). *Let* X *be a para-*
compact space of covering dimension ≤ n+1 *and* A *a closed*
subset of X. *Let* f: A → S^n. *Then* f *is extendable over* X
if and only if $f*(H^n(S^n)) \subset i*(H^n(X))$ *where* i: A → X *is the*
inclusion map.

If X is a compact space, then the *shape dimension* of X
is the least integer n such that there is a space Y with
dim Y = n such that Y shape dominates X. We will denote this
by Sd X = n. If X is a compact metric space, then this is

the same as K. Borsuk's fundamental dimension, Fd X. Re-
sults from [11] will be used concerning fundamental dimension.
We will now state and prove a useful elementary result about
shape dimension. (For a stronger result see the discussion
preceding Lemma 2.3 of [15] by J. Dydak.)

0.4. *Theorem. Let* X *be a compact space with* SdX $\leq$ n.
Let f: X $\rightarrow$ P *be a map with* P *a finite polyhedron. Then
there is a finite polyhedron* Q *with dim* Q $\leq$ n *and maps*
g: X $\rightarrow$ Q *and* r: Q $\rightarrow$ P *such that* r$\circ$g *is homotopic to* f.

Proof. Let Y shape dominate X with dim Y = n. Let
f: X $\rightarrow$ P be a map. Let F: X $\rightarrow$ Y and G: Y $\rightarrow$ X be shape
morphisms with G$\circ$F = S(id$_X$). Let f': Y $\rightarrow$ P be a map such
that S(f') = S(f)$\circ$G. Because Y is n-dimensional, there is
an open cover $\mathcal{U}$ of Y of order at most n+1 such that the
barycentric map g$_{\mathcal{U}}$: Y $\rightarrow$ N($\mathcal{U}$) to the nerve of $\mathcal{U}$ factors the
map f' up to homotopy. That is, there is a map r: N($\mathcal{U}$) $\rightarrow$ P
such that r$\circ$g$_{\mathcal{U}}$ is homotopic to f'. Then Q = N($\mathcal{U}$) will have
dimension at most n. Let g: X $\rightarrow$ Q be a map such that
S(g) = S(g$_{\mathcal{U}}$)$\circ$F. Then S(r$\circ$g) = S(r)$\circ$S(g) = S(r)$\circ$S(g$_{\mathcal{U}}$)$\circ$F =
S(f')$\circ$F = S(f)$\circ$G$\circ$F = S(f). Thus S(r$\circ$g) = S(f). This implies
that r$\circ$g and f are homotopic since P is an ANR.

We let H^n(X,A) denote n-dimensional Čech cohomology with
integer coefficients where X is a paracompact space and A is
a closed subset of X. We assume a basic knowledge of Čech
cohomology. The following theorem follows from the Hopf
Extension Theorem.

0.5. *Theorem. If* X *is a paracompact space of covering*

dimension n *and* A *is a closed subset of* X, *then* $H^n(X,A)$ *is*

isomorphic to $[(X,A), (S^n,p)]$.

1. Maps onto Manifold K (Π,1)'s

Let M^n be a closed n-manifold whose universal covering

space is R^n. For example, M^n could be the n-dimensional

torus T^n. However, many more possibilities exist, see for

instance [5]. It is an unsolved problem whether the cover-

ing space of a closed n-manifold $K(\pi,1)$ must be R^n [3, §3, p.

423]. For n = 3, this is related to the Poincaré Conjecture.

Our proofs seem to require that the universal covering space

for M^n be R^n and so we make this assumption rather than that

M^n be a closed manifold $K(\pi,1)$. In this first section of the

paper we study maps from βX onto M^n. The results give us a

better understanding of the shape of βX and of compacta con-

tained in $\beta X-X$. The main results in this section are Theorems

1.2, 1.3, 1.6, and 1.8. In the next section we give some

further applications of these theorems. First we prove an

important lemma.

1.1. *Lemma. Suppose that* P *is a finite polyhedron and*

that H: $\beta X \times I \to P$ *is a homotopy. Suppose that* $\tilde{P}$ *is the uni-*

versal covering space for P. *Let* K *be a compact set in* $\tilde{P}$.

Then there is an open set U *in* $\tilde{P}$ *containing* K *such that the*

closure of U *in* $\tilde{P}$ *is compact and such that for each* $x \in X$ *if*

h: $\{x\} \times I \to \tilde{P}$ *is any lift of the path* $H|\{x\} \times I$, *then if*

$h(\{x\} \times I) \cap K \neq \emptyset$, *then* $h(\{x\} \times I) \subset U$.

Proof. Suppose not. Suppose it is false for the compact

set $K \subset \tilde{P}$. Let $\{U_i\}_{i=1}^{\infty}$ be a sequence of open sets in $\tilde{P}$ such

that (1) $K \subset U_1$, (2) $\bar{U}_i \subset U_{i+1}$ for all i, (3) $\bar{U}_i$ compact for

all i, and (4) $\tilde{P} = \cup_{i=1}^{\infty} U_i$. Since the lemma is assumed false

for K, for each i there is a point $x_i \in X$ such that some

lift h_i of $H|\{x_i\} \times I$ has the property that $h_i(\{x_i\} \times I) \cap$

$K \neq \emptyset$ and $h_i(\{x_i\} \times I) \cap (\tilde{P} - U_i) \neq \emptyset$. Let $t_i(1) \in I$ be such

that $h_i(x_i, t_i(1)) \in K$ and $t_i(2) \in I$ be such that $h_i(x_i,$

$t_i(2)) \notin U_i$. Then let $x \in \beta X$ be a limit point of the set

$\{x_i\}_{i=1}^{\infty}$. Let $\{i_\alpha\}$ be a subnet of the positive integers such

that (1) $x_{i_\alpha} \to x$, (2) $t_{i_\alpha}(1) \to t(1) \in I$, and (3) $t_{i_\alpha}(2) \to$

$t(2) \in I$. Such a net and points $t(1)$ and $t(2) \in I$ can be

found routinely and we omit the details. Now let 0 be a

contractible open set in P containing $H(x, t(1))$ such that the

covering map c: $\tilde{P} \to P$ has the property that c restricted to

each component of $c^{-1}(0)$ is a homeomorphism onto 0. By the

continuity of H: $\beta X \times I \to P$, there must be a β such that for

all $\alpha \geq \beta$, $H(\{x_{i_\alpha}\} \times I_{i_\alpha}) \subset 0$ where I_{i_α} is the subinterval of

I joining $t_{i_\alpha}(1)$ to $t(1)$. So we may suppose that $H(\{x_{i_\alpha}\} \times$

$I_{i_\alpha}) \subset 0$ for all α. Now K is compact in $\tilde{P}$ and since the

components of $c^{-1}(0)$ form a discrete set in $\tilde{P}$, there can be

at most a finite number of these components which intersect

K. Call these components $\{0_1, \cdots, 0_k\}$. For one of these com-

ponents 0_j there must be a subnet of $\{i_\alpha\}$ such that $h_{i_\alpha}(x_{i_\alpha},$

$t_{i_\alpha}(1)) \in 0_j$ for all i_α of the subnet. By renaming the sub-

net let us assume that $h_{i_\alpha}(x_{i_\alpha}, t_{i_\alpha}(1)) \in 0_j$ for all α. Let

p: $0 \to 0_j$ be the inverse of $c|0_j$. Then let $B = \{x_{i_\alpha}\} \cup$

$\{x\} \subset \beta X$ and let f: $B \to \tilde{P}$ be defined by $f \equiv p \circ H|B \times \{t(1)\}$.

Then f is a continuous lift for $H|B \times \{t(1)\}$. Thus there is

a unique lift of $H|B \times I$, $\tilde{H}$: $B \times I \to \tilde{P}$, such that $\tilde{H}|B \times$

$\{t(1)\} \equiv f$. Now by the definition of $f = \tilde{H}|B \times \{t(1)\}$,

$\tilde{H}(x_{i_\alpha}, t(1)) \in 0_j$ for all α. Also $H(\{x_{i_\alpha}\} \times I_{i_\alpha}) \subset 0$ for all

α. Thus $\tilde{H}(x_{i_\alpha}, t_{i_\alpha}(1)) \in O_j$ also. Thus $\tilde{H}|\{x_{i_\alpha}\} \times I \equiv h_{i_\alpha}$

for all α. Now $\tilde{H}(x, t(2)) \in U_i$ for some i. But since $\tilde{H}$ is

continuous, there must be a β such that for all $\alpha \geq \beta$,

$\tilde{H}(x_{i_\alpha}, t_{i_\alpha}(2)) \in U_i$. There is also a γ such that for all

$\alpha \geq \gamma$, $i_\alpha > i$. Then let $\alpha \geq \beta$ and $\alpha \geq \gamma$. Then for this α,

$\tilde{H}(x_{i_\alpha}, t_{i_\alpha}(2)) \in U_i$ and $i_\alpha > i$. This is a contradiction since

$\tilde{H}(x_{i_\alpha}, t_{i_\alpha}(2)) = h_{i_\alpha}(x_{i_\alpha}, t_{i_\alpha}(2)) \notin U_{i_\alpha}$ and $\bar{U}_i \subset U_{i_\alpha}$. This

contradiction proves the lemma.

We now proceed directly to the first main theorem of

this section.

1.2. *Theorem. Let* M^n *be a closed* n-*manifold whose*

covering space is homeomorphic to R^n. *Suppose that* X *is a*

locally compact σ-*compact space such that for every compact*

set $K \subset X$, *there is a compact set* $L \subset X-K$ *with dim* $L \geq n$.

Then there is a map f: $\beta X \to M^n$ *which is homotopically onto.*

Proof. Let $X = \cup_{i=1}^\infty K_i$ where each K_i is compact and

$K_i \subset$ int K_{i+1} for all i. Let $L_1 \subset X-K_1$ be a compact set with

dim $L_1 \geq n$. Let $L_{i+1} \subset X - (\cup_{j=1}^i L_j \cup \cup_{j=1}^{i+1} K_j)$ with dim L_{i+1}

$\geq n$. Then $\{L_i\}_{i=1}^\infty$ will be a sequence of disjoint compact

subsets of X with dim $L_i \geq n$ for all i and with $L_i \subset X-K_i$

for all i. Now if B is any compact set in X, then there is

an i with $B \subset$ int K_i. Thus L_i will have the property that

$L_i \subset X-B$. Now let B^n be the closed unit ball in R^n. Let

g_i: $L_i \to B^n$ be an essential map (which is guaranteed to exist

by Theorem 0.1). Then define g: $\cup_{i=1}^\infty L_i \to R^n$ by $g(x) = kg_k(x)$

for $x \in L_k$. Now $\cup_{i=1}^\infty L_i$ is closed in X and X is Lindelöf,

hence normal. Thus there is an extension of g to all of X

which we also call g. Let c: $R^n \to M^n$ be the covering map.

Then define f to be the Čech extension of $c \circ g: X \to M^n$.
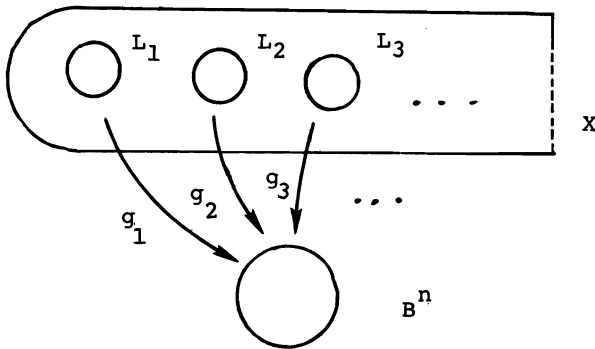
Now $g: X \to R^n$ is onto and thus $f: \beta X \to M^n$ is also onto. We will now show that any map homotopic to f must also be onto.

Claim 1. *The map* $f: \beta X \to M^n$ *is homotopically onto.*

Proof of Claim 1. Suppose that $H: \beta X \times I \to M^n$ be a homotopy from f to h. Note that $g: X \to R^n$ is a lift for the map $f|X: X \to M^n$. Thus there is a unique map $\tilde{H}: X \times I \to R^n$ such that $c \circ \tilde{H} = H|X \times I$ and $\tilde{H}|X \times \{0\} \equiv g$. Let $\tilde{h} = \tilde{H}|X \times \{1\}$. Then $\tilde{h}$ is a lifting for the map $h|X$. We will show that h is onto by showing that $\tilde{h}$ is onto. Once we have shown this Claim 1 will follow and the proof of Theorem 1.2 will be complete.

Claim 2. *The map* $\tilde{h}: X \to R^n$ *is onto.*

Proof of Claim 2. Suppose that $x \in R^n$ with $x \notin \tilde{h}(X)$. Then by Lemma 1.1 there is an open set U in R^n with $x \in U$ and $\bar{U}$ compact such that if $y \in X$ and $\tilde{H}(y,t) = x$, then $\tilde{H}(\{y\} \times I) \subset U$. Let N be an integer such that $N \cdot B^n$ contains U. Then let Σ^{n-1} be the boundary of $N \cdot B^n$ and

$D_N = N \cdot g_N^{-1}(\Sigma^{n-1}) \subset L_N$. Now since $g_N: L_N \to B^n$ was an essential

map, $f|D_N = N \cdot g_N|D_N$ cannot be extended to a map of L_N taking

values in Σ^{n-1}. We will now get a contradiction to this.

Then Claim 2 will follow.

Claim 3. There must be an extension of $f|D_N$ *to* L_N *taking*

values in Σ^{n-1}.

Proof of Claim 3. Now x is in the interior of $N \cdot B^n$. Let

$r: R^n - \{x\} \to \Sigma^{n-1}$ be the projection along the rays emanating

from x. Let $s: L_N \to \Sigma^{n-1}$ be defined by $s = r \circ h$. Then s is

defined and continuous since $\tilde{h}(y) \neq x$ for all $y \in X$. Consider

the homotopy $\tilde{H}|D_N \times I$. Note that if $x \in \tilde{H}(D_N \times I)$, then

$\tilde{H}(y,t) = x$ for some $y \in D_N$ and $t \in I$. But for that y,

$\tilde{H}(y,0) \in \Sigma^{n-1}$ since $\tilde{H}(y,0) = g(y)$. This contradicts the

choice of U and N, since by Lemma 1.1 $\tilde{H}(\{y\} \times I) \subset U \subset$

int $N \cdot B^n$. Thus $x \notin \tilde{H}(D_N \times I)$. Thus $r \circ \tilde{h}|D_N \times I: D_N \times I \to \Sigma^{n-1}$

is defined and continuous and a homotopy joining the map

$f|D_N$ to the map $s|D_N$. However, $s|D_N$ has a continuous exten-

sion to all of L_N having values in Σ^{n-1}, namely $r \circ \tilde{h}$. By the

Borsuk Extension Theorem $f|D_N$ must also have an extension to

L_N with values in Σ^{n-1}. This proves Claim 3.

Claim 3 is a contradiction of the fact that $f|L_N = N \cdot g_N$

was an essential map. This contradiction shows that $\tilde{h}$ must

be onto and completes the proof of Claim 2. The fact that $\tilde{h}$

is onto shows that $h: \beta X \to M^n$ must be onto. Thus f is homo-

topically onto and the proof of the theorem is complete.

1.3. *Theorem. Let* M^n *be a closed n-manifold whose*

covering space is homeomorphic to R^n. *Suppose that X is*

locally compact and σ-compact such that for every compact

set K ⊂ X, there is a compact set L ⊂ X-K such that dim

$L \geq n$. *Then there is a map* f: βX-X → M^n *which is homotopi-*

cally onto.

   Proof.   This follows from the proof of Theorem 1.2.

Let f: βX → M^n be the map constructed in the proof of Theorem

1.2. We claim that f|βX-X is homotopically onto M^n. Suppose

not. Then let h: βX-X → M^n be a map homotopic to f which is

not onto M^n. We may assume that h(βX-X) ⊂ M^n-O where O is

an open n-ball in M^n with M^n-O a manifold with boundary.

Now M^n-O is an ANR. Thus there must be an extension of h to

a neighborhood V of βX-X in βX taking values in M^n-O. Call

this extension h. Then h(V) ⊂ M^n-O is also not onto M^n. We

can also assume that h and f|V are homotopic as maps into

M^n. Since V is a neighborhood of βX-X, X-V = K must be com-

pact. Let N be such that for i ≥ N, L_i ∩ K = ∅. Then

L_i ⊂ V for all i ≥ N. Let U be an open set containing

K ∪ ($\cup_{i=1}^{N-1} L_i$) such that $\overline{U}$ is compact with $\cup_{i=N}^{\infty} L_i$ ⊂ X-U. Then

let X' = X-U. Then one can repeat the proof of Theorem 1.2

to show that f|βX' must be homotopically onto M^n. However,

X' is a closed subset of X which is normal and thus

$cl_{βX}$X' = βX'. This gives us a contradiction since h|$cl_{βX}$X'

is homotopic to f|$cl_{βX}$X' and h|$cl_{βX}$X' is not onto. This

contradiction shows that f|βX-X must be homotopically onto.

This proves Theorem 1.3.

   1.4.   *Corollary.*   *Suppose that* X *is any normal space*

which contains a closed discrete set of compact sets $\{L_i\}_{i=1}^{\infty}$

such that dim L_i ≥ n *for all* i. *Let* M^n *be a closed n-manifold*

whose universal covering space is homeomorphic to R^n. *Then
there is a map* f: $\beta X \to M^n$ *which is homotopically onto and such
that* f$|\beta X$-X *is also homotopically onto.*

   1.5. *Remark.* There are other extensions of Theorems
1.2 and 1.3 along the lines of Corollary 1.4, but there is
not space here to include them.

   1.6. *Theorem. Let* n $\geq$ 1 *be an integer. Let* X *be a
locally compact and* σ-*compact space such that for every
compact set* K $\subset$ X *there is a compact set* L $\subset$ X-K *such that
dim* L $\geq$ n. *Then* Sd $\beta X \geq$ n *and* Sd(βX-X) $\geq$ n.

   Proof. Let $T^n = M^n$ in Theorems 1.2 and 1.3. Then there
is a map f: $\beta X \to T^n$ which is homotopically onto and such that
f$|\beta X$-X is also homotopically onto. Now if Sd $\beta X <$ n, then
by Theorem 0.4 there would be an (n-1)-dimensional compact
polyhedron Q and maps g: $\beta X \to$ Q and r: Q $\to T^n$ such that r∘g
is homotopic to f. But we can take r to be simplicial and
then dim r(Q) $\leq$ n-1. Thus r∘g(βX) would not be all of T^n.
This is a contradiction. Thus Sd(βX) $\geq$ n. Similarly
Sd(βX-X) $\geq$ n also.

   Actually we will show a much stronger result about the
shape dimension of compacta in βX-X. In Corollary 1.9 we
will show that for X Lindelöf and K a compactum contained in
βX-X, dim K = Sd K. To prove this we will need Theorem 1.8.
First we need an easy lemma.

   1.7. *Lemma. Let* X *be Lindelöf and* A $\subset \beta X$-X *be such
that* A *is Lindelöf and* A *is closed in* X $\cup$ A. *Then* $cl_{\beta X}A = \beta A$.

   Proof. We need to show that every continuous map

$f: A \rightarrow [0,1]$ extends to a map $F: cl_{\beta X}A \rightarrow [0,1]$. Let

$f: A \rightarrow [0,1]$. Then $X \cup A$ is Lindelöf and A is closed in

$X \cup A$. Thus there is an extension of f to $X \cup A$. Call this

extension f again. Then f has an extension βf to βX. Then

let $F = \beta f | cl_{\beta X}A$. Clearly F is an extension of our original

map $f: A \rightarrow [0,1]$.

   1.8. *Theorem. Let K be a compactum contained in $\beta X - X$*

where X is a Lindelöf space. Let M^n be as in Theorem 1.2.

Then if dim $K \geq n$, then there is a map $f: K \rightarrow M^n$ which is

homotopically onto.

   Proof. Actually we will construct a map $f: \beta X \rightarrow M^n$

such that $f|K$ is homotopically onto. First we will need a

sequence of compact sets $L_i \subset K$ such that $L_i \cap L_j = \emptyset$ for

$i \neq j$ and with dim $L_i \geq n$ for all i.

   Claim 1. There is a sequence of compacta $L_i \subset K$ such

that $\{L_i\}_{i=1}^{\infty}$ is a disjoint collection with dim $L_i \geq n$ for

all i.

   Proof of Claim 1. Suppose that for each $x \in K$, there is

a set U_x open in K containing x such that dim $\overline{U}_x < n$. Then

one could cover K by a finite number of such open sets,

$\{U_1, \cdots, U_k\}$ with dim $\overline{U}_i < n$. But then by the Sum Theorem

for dimension (Theorem 0.2), dim $K < n$, a contradiction. Thus

there must be an $x_1 \in K$ such that for every open set U con-

taining x_1, dim $\overline{U} \geq n$. Now there must be a set U_1 open in

K and containing x_1 such that dim $K-U_1 \geq n$ (by [4, Corollary

7, p. 80]). Then let V_1 be a set containing x_1 and open in

K such that $\overline{V}_1 \subset U_1$. Then dim $\overline{V}_1 \geq n$ and dim$(K-U_1) \geq n$ with

$\overline{V}_1 \cap (K-U_1) = \emptyset$. Now let $x_2 \in K-U_1$ be such that every set U

containing x_2 and open in $K-U_1$ has the property that dim

$\overline{U} \geq n$. Then let U_2 be a set open in $K-U_1$ containing x_2 such

that $\dim(K-(U_1 \cup U_2)) \geq n$. Then let V_2 contain x_2 with V_2

open in $K-U_1$ such that $\overline{V}_2 \subset U_2$. Then $\overline{V}_1 \cap \overline{V}_2 = \emptyset$ and dim

$\overline{V}_2 \geq n$ and $\dim(K-(U_1 \cup U_2)) \geq n$. Continuing this process one

gets a sequence of sets $\{\overline{V}_i\}_{i=1}^{\infty}$ such that $\overline{V}_i \cap \overline{V}_j = \emptyset$ for

$i \neq j$ and $\dim \overline{V}_i \geq n$ for all i. Then $L_i = \overline{V}_i$ are the required

compacta in K. This proves Claim 1.

We now resume the proof of Theorem 1.8. Let B^n be the

closed unit ball in R^n and let $g_i: L_i \rightarrow B^n$ be an essential

map. Let $g: \cup_{i=1}^{\infty}L_i \rightarrow R^n$ be defined by $g(x) = k \cdot g_k(x)$ for

$x \in L_k$. Now $\cup_{i=1}^{\infty}L_i$ is Lindelöf. Also $K \cap (X \cup (\cup_{i=1}^{\infty}L_i)) =$

$\cup_{i=1}^{\infty}L_i$ and thus $\cup_{i=1}^{\infty}L_i$ is closed in $X \cup (\cup_{i=1}^{\infty}L_i)$. Thus there

is an extension $g: X \cup (\cup_{i=1}^{\infty}L_i) \rightarrow R^n$. Let $c: R^n \rightarrow M^n$ be the

covering map and $f: \beta X \rightarrow M^n$ be the Čech extension of the map

$c \circ g: X \rightarrow M^n$. Then we claim that $f|K$ is homotopically onto.

Claim 2. The map $f|K$ is homotopically onto M^n.

Proof of Claim 2. Let $D = cl_{\beta X}(\cup_{i=1}^{\infty}L_i)$. Then $D \subset K$ and

actually $f|D$ is homotopically onto. The proof proceeds

exactly as in the proof of Theorem 1.2. The only observation

that needs to be made is that $cl_{\beta X}(\cup_{i=1}^{\infty}L_i)$ is equivalent to

$\beta(\cup_{i=1}^{\infty}L_i)$ so that we may make use of Lemma 1.1. However,

Lemma 1.7 shows that $cl_{\beta X}(\cup_{i=1}^{\infty}L_i) = \beta(\cup_{i=1}^{\infty}L_i)$. The proof of

Claim 2 is now clear. Claim 2 completes the proof of Theorem

1.8.

1.9. *Corollary. Let X be a Lindelöf space and let K be

a compactum contained in $\beta X-X$. Then $\dim K = Sd\ K$.*

Proof. This is clear if $\dim K = 0$ or -1. If

dim $K \geq n \geq 1$, then by Theorem 1.8 there is a continuous map
$f(K) = T^n$ which is homotopically onto. Thus Sd $K \geq n$ also.
Thus Sd $K \geq$ dim K. However, Sd $K \leq$ dim K by definition. Thus
Sd K = dim K.

   1.10. *Example.* The above results imply that the shape
dimension of βR^n and $\beta R^n - R^n$ is n and thus that the dimen-
sion of these spaces is n also. By the results of Calder and
Siegel [1] we have that for $k \geq 2$, $H^k(\beta R^n) = 0 = H^k(R^n)$.
Thus for each $n \geq 2$ there is a continuum $X_n = \beta R^n$ such that
Sd $X_n = n$ with $H^k(X_n) = 0$ for all $k \geq 2$. In the next sec-
tion we will show that this implies the existence of a metric
continuum X_n having these properties. This will provide a
counterexample to a question raised by Nowak in [11].

   1.11. *Example.* Let L be the long line and $X = L \times I^n$.
Then $\beta(X \times I^n) = (L \cup \{\omega_1\}) \times I^n$. The space βX has trivial
shape as does $\beta X - X = \{\omega_1\} \times I^n$. Thus Sd $\beta X = 0 = $ Sd$(\beta X - X)$.
However, dim $\beta X = $ dim$(\beta X - X) = n$. Of course X is not σ-compact
or Lindelöf. This shows that without some assumption on X
one cannot expect any of the theorems in this section to
hold.

2. Metric Continua and Polyhedra

   In this section we apply the results of the first sec-
tion to show the existence of metric continua and polyhedra
which have surprising properties. One of the examples
answers a question of Nowak [11]. We proceed with the ex-
amples.

   2.1. *Theorem. For each $n \geq 2$ there is a finite*

polyhedron P_n *of dimension* n *such that there is a map*
f: $P_n \to T^n$ *with* f *homotopically onto and with* f*: $H^k(T^n) \to$
$H^k(P_n)$ *the zero-homomorphism for all* k $\geq$ 2.

    Proof. Let X = βR^n. Then there is a map g: X $\to$ T^n
which is homotopically onto. Now $H^k(\beta R^n)$ = 0 for all k $\geq$ 2
by [1]. Thus g*: $H^k(T^n) \to H^k(X)$ is the zero-homomorphism.
Now dim βR^n = n and thus there is a cofinal set of finite
open covers of X having order at most n+1. Let $\{\mathcal{U}_\alpha : \alpha \in A\}$ be
this set. Now H*(X) = $\varinjlim \{H*(N(\mathcal{U}_\alpha))\}$ where $N(\mathcal{U}_\alpha)$ is the nerve
of $\mathcal{U}_\alpha$ and the bonding homomorphisms are induced by the pro-
jection maps $\pi_{\alpha\beta}$: $N(\mathcal{U}_\beta) \to N(\mathcal{U}_\alpha)$ for $\mathcal{U}_\beta$ a refinement of $\mathcal{U}_\alpha$.
Now for a cofinal collection of the α's, there are maps
g_α: X $\to N(\mathcal{U}_\alpha)$ and r_α: $N(\mathcal{U}_\alpha) \to T^n$ such that $r_\alpha \circ g_\alpha$ is homotopic
to g where g_α: X $\to N(\mathcal{U}_\alpha)$ is a barycentric map for the cover
$\mathcal{U}_\alpha$. Note that r_α: $N(\mathcal{U}_\alpha) \to T^n$ must be homotopically onto
since g is homotopically onto. Now $H^k(X) = \varinjlim \{H^k(N(\mathcal{U}_\alpha))\}$
and thus g* = $\varinjlim \{r_\alpha$: $H^k(T^n) \to H^k(N(\mathcal{U}_\alpha))\}$. Since $H^k(T^n)$ is
finitely generated for all k and g* = 0 for all k $\geq$ 2, there
must be a β such that r_α^*: $H^k(T^n) \to H^k(N(\mathcal{U}_\alpha))$ is the zero-
homomorphism for all $\alpha \geq \beta$ and all k $\geq$ 2. Then let P_n =
$N(\mathcal{U}_\beta)$ and f = r_α. This proves Theorem 2.1.

The above example has the property that for any map
g: $T^n \to S^n$, g$\circ$f: $P_n \to S^n$ is null-homotopic. This is by
Theorem 0.5 since dim P_n = n. One would tend to think that
the map f: $P_n \to T^n$ could not be degree zero and still be
homotopically onto. However, this is in fact the case and
the result is somewhat surprising.

    2.2. *Theorem. Let* n $\geq$ 2. *Then there is a metric*

continuum X_n *such that dim* X_n = Sd X_n = n *and such that*
$H^k(X_n)$ = 0 *for all* k $\geq$ 2.

    Proof. Let Y_n = βR^n. Let g: $Y_n \rightarrow T^n$ be a map which
is homotopically onto. Then let $\{ \mathcal{U}_\alpha : \alpha \in A \}$ be a cofinal
set of open covers of Y_n such that the order of $\mathcal{U}_\alpha$ is at
most n+1 for all $\alpha \in A$. There must be a $\beta \in A$ such that there
are maps r_β: $N(\mathcal{U}_\beta) \rightarrow T^n$ and a barycentric map g_β: $Y_n \rightarrow N(\mathcal{U}_\beta)$
such that $r_\beta \circ g_\beta$ is homotopic to g: $Y_n \rightarrow T^n$. Then for all
$\alpha \geq \beta$ the maps g_α: $Y_n \rightarrow N(\mathcal{U}_\alpha)$ and $\pi_{\beta\alpha}$: $N(\mathcal{U}_\alpha) \rightarrow N(\mathcal{U}_\beta)$ have
the property that $r_\beta \circ \pi_{\beta\alpha} \circ g_\alpha$ is homotopic to g. Thus $r_\beta \circ \pi_{\beta\alpha}$:
$N(\mathcal{U}_\alpha) \rightarrow T^n$ must be homotopically onto since g is homotopically
onto. Now $H^*(Y_n) = \lim_{\rightarrow}\{H^*(N(\mathcal{U}_\alpha)): \alpha \geq \beta\}$. By an inductive
process one can construct a sequence $\beta = \alpha_1 < \alpha_2 < \cdots$ in A
such that $\lim_{\rightarrow}\{H^k(N(\mathcal{U}_{\alpha_i}))\}$ = 0 for all k $\geq$ 2. One uses the
fact that $H^k(Y_n$ = 0 for all k $\geq$ 2 together with the fact that
$H^k(N(\mathcal{U}_{\alpha_i}))$ is finitely generated for all i. Now let
π_i: $N(\mathcal{U}_{\alpha_{i+1}}) \rightarrow N(\mathcal{U}_{\alpha_i})$ be a projection map for each i. Then
let $X_n = \lim_{\leftarrow}\{N(\mathcal{U}_{\alpha_i}); \pi_i\}$. Then X_n will be a metric continuum.
the reason is that each of the $N(\mathcal{U}_\alpha)$'s will be connected
since βR^n is connected. Also dim $X_n \leq$ n since dim $N(\mathcal{U}_{\alpha_i}) \leq$ n
for all i. Also $H^k(X) = \lim_{\rightarrow}\{H^k(\mathcal{U}_{\alpha_i}); \pi_i^*\}$ = 0 for all k $\geq$ 2.
Now suppose that h_i: $X_n \rightarrow N(\mathcal{U}_{\alpha_i})$ are the maps making X_n the
inverse limit of the inverse system $\{N(\mathcal{U}_{\alpha_i})\}$. Now consider
$g_\beta \circ h_1$: $X_n \rightarrow T^n$. We claim that $g_\beta \circ h_1$ is homotopically onto.

    Claim. *The map* $g_\beta \circ h_1$ *is homotopically onto.*

    Proof of Claim. Suppose not. Then there must be a k
such that $g_\beta \circ \pi_1 \circ \cdots \circ \pi_k$: $N(\mathcal{U}_{\alpha_k}) \rightarrow T^n$ is not homotopically
onto. However, this implies that $g_\beta \circ \pi_{\beta\alpha_{k+1}}$ is not

homotopically onto since $\pi_1 \circ \cdots \circ \pi_k$ is a projection map $\pi_{\beta\alpha_{k+1}}$ from $N(\mathcal{U}_{\alpha_{k+1}})$ to $N(\mathcal{U}_\beta)$. This is a contradiction since we have already remarked that $g_\beta \circ \pi_{\beta\alpha}$ is homotopically onto for all $\alpha \geq \beta$. Thus $g_\beta \circ h_1$ must be homotopically onto as asserted in the claim.

Now by the Claim, $g_\beta \circ h_1 : X_n \to T^n$ is homotopically onto. Thus $Sd(X_n) = Fd(X_n) \geq n$. Since $\dim X \leq n$, we must have $\dim X_n = Sd\, X_n = n$. We have already shown that $H^k(X_n) = 0$ for all $k \geq 2$. Thus X_n has the desired properties and Theorem 2.2 is proved.

2.3. *Remark.* Theorem 2.2 solves Problem 6.7 in [11]. J. Hollingsworth also has an example which solves this problem.

2.4. *Corollary.* *Let* $n \geq 2$. *Then there exists a metric continuum* X_n *such that* $Sd\, X_n = dim\, X_n = n$ *with* $Sd(\Sigma X_n) = 2$ *where* ΣX_n *is the suspension of* X_n.

Proof. Let X_n be the example in Theorem 2.2. Then $Sd(\Sigma X_n) = 2$ by [11, Theorem 4.4].

3. Maps onto $S^n \vee S^1$

In this section we show that there are maps of βX which are homotopically onto other finite polyhedra P besides manifold $K(\pi,1)$'s. In particular we show that this is true for $P = S^n \vee S^1$. First we make the following observations about maps onto wedges of manifold $K(\pi,1)$'s. The proofs are only indicated since they are a straight forward modification of the proofs in section one.

3.1. *Theorem.* *Let* $n \geq 1$. *Let* $n_1, \cdots, n_k$ *be integers*

such that $n_i \leq n$ *for all i and suppose that* M_i *is a closed*
n_i-*manifold whose covering space is* R^{n_i} *for* $i = 1, \cdots, k$.
Suppose that X is a locally compact σ-compact space such that
for every compact set $K \subset X$ *there is a compact set* $L \subset X-K$
such that dim $L \geq n$. *Then there is a map* $f \colon \beta X \to \vee_{i=1}^{k} M_i$
which is homotopically onto.

3.2. *Theorem. Let* $n \geq 1$. *Let* $n_1, \cdots, n_k$ *be integers*
such that $n_i \leq n$ *for all i and suppose that* M_i *is a closed*
n_i-*manifold whose covering space is* R^{n_i} *for* $i = 1, \cdots, k$.
Suppose that X is Lindelöf and that K is a compactum contained
in $\beta X-X$ *with dim* $K \geq n$. *Then there is a map* $f \colon K \to \vee_{i=1}^{k} M_i$
which is homotopically onto.

Indication of Proof of 3.1. Let $\{L_i\}_{i=1}^{\infty}$ be disjoint
compact subsets of X with dim $L_i \geq n$ as in the proof of
Theorem 1.2. Then break this up into k infinite collections:
$\{L_i(1)\}_{i=1}^{\infty}, \cdots, \{L_i(k)\}_{i=1}^{\infty}$. Then let $g_i(j) \colon L_i(j) \to B^{nj}$ be an
essential mapping where B^{nj} is the closed unit ball in R^{nj}.
Then define a map g: $X \to \vee_{j=1}^{k} R^{nj}$ by
$$g|L_i(j) = i \cdot g_i(j) \colon L_i(j) \to R^{nj}$$
This defines g on $\cup_{j=1}^{k} \cup_{i=1}^{\infty} L_i(j)$ and then extend in any
fashion to all of X. Then let $c_j \colon R^{nj} \to M_j$ be the covering
map for $j = 1, \cdots, k$ and let $\vee_{j=1}^{k} c_j \colon \vee_{j=1}^{k} R^{nj} \to \vee_{j=1}^{k} M_j$ be the
wedge of the c_j's. Then let f: $\beta X \to \vee_{j=1}^{k} M_j$ be the Čech ex-
tension of the map
$$(\vee_{j=1}^{k} c_j) \circ g \colon X \to \vee_{j=1}^{k} M_j.$$
Then one can show that f is homotopically onto in a manner
similar to the proof of Theorem 1.2.

The proof of Theorem 3.2 is a similar modification of

the proof of Theorem 1.8. We now proceed to maps which are homotopically onto $S^n \vee S^1$. The following proposition follows from Theorem 10.3 of [16]. We include a simple proof for completeness.

3.3. *Proposition.* *Let* X *be a paracompact space of dimension* n. *Then there is a closed set* $C \subset X$ *and a map* $f: (X,C) \to (B^n, S^{n-1})$ *such that* f *is not null-homotopic as a map of pairs and if* $e: (B^n, S^{n-1}) \to (S^n, p)$ *is the map which takes all of* S^{n-1} *to* p, *then* $e \circ f: (X,C) \to (S^n, p)$ *is also not null-homotopic as a map of pairs.*

Proof. Let C be a closed subset of X so chosen that there is a map $f: C \to S^{n-1}$ which cannot be extended to all of X with values in S^{n-1}. Let $i: C \to X$ be the inclusion map. Let $f: X \to B^n$ be an extension of our original map $f: C \to S^{n-1}$. Let $e: B^n \to S^n$ be the quotient map taking S^{n-1} to $p \in S^n$. Then the following diagram commutes and $e*$ is an isomorphism.

$$
\begin{array}{ccccc}
H^n(S^n,p) & \xrightarrow{\ e*\ } & H^n(B^n,S^{n-1}) & \xrightarrow{\ f*\ } & H^n(X,C) \\
 & & \Big\uparrow{\scriptstyle \delta_1} & & \Big\uparrow{\scriptstyle \delta_2} \\
 & & H^{n-1}(S^{n-1}) & \xrightarrow{\ f*\ } & H^{n-1}(C) \\
 & & & & \Big\uparrow{\scriptstyle i*} \\
 & & & & H^{n-1}(X)
\end{array}
$$

Now by the Hopf Extension Theorem (Theorem 0.3), there must be an $h \in H^{n-1}(S^{n-1})$ with $f*(h) \notin i*H^{n-1}(X)$ or f would have an extension to X with values in S^{n-1}. Thus $\delta_2 f*(h) \neq 0$ in $H^n(X,C)$. Thus $f*\delta_1(h) \neq 0$ and $f*: H^n(B^n, S^{n-1}) \to H^n(X,C)$ is not the zero homomorphism. Thus $f: (X,C) \to (B^n, S^{n-1})$ cannot be null-homotopic. Also, since $e*$ is an isomorphism,

$e \circ f\colon (X,C) \to (S^n,p)$ cannot be null-homotopic either.

   3.4. *Theorem. Let $n \geq 2$. Suppose that X is a locally compact σ-compact space such that for every compact subset K of X there is a compact set $L \subset X$ with dim $L = n$ and with $L \cap K = \emptyset$. Then there is a map f of βX onto $S^n \vee S^1$ which is homotopically onto.*

   Proof. Let $\{L_i\}_{i=1}^{\infty}$ be a sequence of compact subsets of X having the following properties: (1) dim $L_i = n$, (2) $L_i \cap L_j = \emptyset$ for $i \neq j$, and (3) if K is a compact subset of X, then there is an i such that $L_i \cap K = \emptyset$. The construction of such a sequence of L_i's has been carried out in the proof of Theorem 1.2. Now let C_i be a closed subset of L_i and $f_i\colon (L_i,C_i) \to (B^n,S^{n-1})$ be a map of pairs which is not null-homotopic such that $e \circ f_i$ is also not null-homotopic where $e\colon (B^n,S^{n-1}) \to (S^n,p)$. Such maps f_i exist by Proposition 3.3. Now let A be the universal covering space for $S^n \vee S^1$ and $c\colon A \to S^n \vee S^1$ the covering map. We can think of A as the countable union of n-spheres attached to the real line R at the integer points. Now let B be the countable union of closed unit n-balls, B^n, attached at the integer points of R.
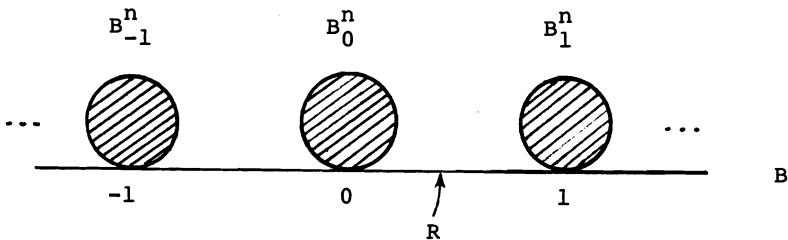
Now define a map $g: X \to B$ by $g|L_i = f_i$ for each i.
Then let g be any extension to all of X. Such an extension
exists since B is an absolute extensor. Let B_i^n be the n-ball
attached at the integer point i in B. Let S_i^n be the n-sphere
attached at the integer point i in A. Let $i \geq 1$ and let
$h_i: B_i \to A$ be defined in the following manner. Let D_i be a
closed collar for the boundary S_i^{n-1} of B_i^n and let Σ_i^{n-1} be
the interior component of the boundary of D_i. Then let h_i
take Σ_i^{n-1} to the point $i \in A$. Then let $h_i(S_i^{n-1})$ map to the
point $2i \in A$ and let $h_i|D_i$ map onto the arc $[i,2i] \subset R \subset A$.
Then let h_i map the interior of Σ_i^{n-1} homeomorphically onto
$S_i^n - \{i\}$.

$$i = h_i(\Sigma_i^{n-1}) \qquad\qquad 2i = h_i(S_i^{n-1})$$
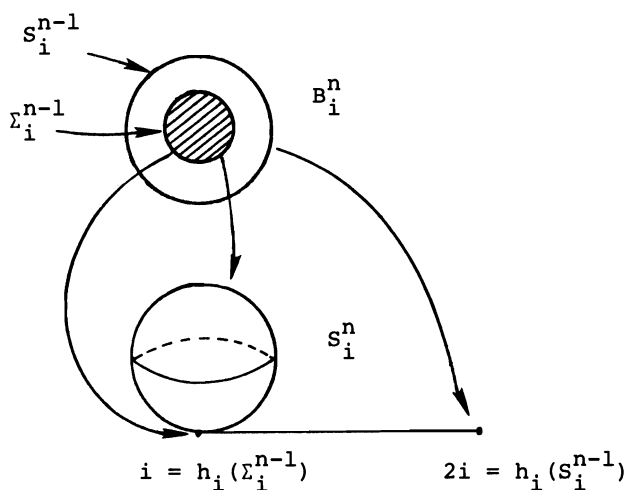
Then define $k_i: L_i \to A$ by $k_i = h_i \circ f_i = h_i \circ g$. Then we make
the following observation.

    Claim 1. *If* $m_i: L_i \to A$ *is homotopic to* k_i *by a homotopy*
H *such that at each stage t of the homotopy* $H(C_i,t) \subset A - S_i^n$,
then $m_i(L_i^n)$ *contains* S_i^n.

Proof of Claim 1. We can convert the homotopy H to one
which maps to S_i^n by projecting all of $A - S_i^n$ to the point i.
Let H' be this homotopy and let m_i' and k_i' be the corresponding
ends of the homotopy. Then the homotopy is actually a homo-
topy of pairs H': $(L_i, C_i) \times I \to (S_i^n, i)$ from k_i' to m_i'. Now
k_i' is homotopic as a map of pairs to $e \circ f_i$ (or to the map
$e \circ f_i$ followed by an orientation reversal of S_i^n). Thus the
map m_i' is also homotopic to f_i as a map of pairs. This implies
that $m_i'(L_i) = S_i^n$ and thus that $m_i(L_i)$ contains S_i^n. This proves
Claim 1.

We now proceed with the proof of Theorem 3.4. Let
h: B → A be any map such that $h|B_i = h_i$ for i = 1,2,···. We
have already defined a map g: X → B and the covering map
c: A → $S^n \vee S^1$. Then let q: X → $S^n \vee S^1$ be defined by q = c∘h∘g.
Then let f = βq: βX → $S^n \vee S^1$. We will now show that this f is
homotopically onto.

Claim 2. *If* p: βX → $S^n \vee S^1$ *is homotopic to f, then* S^n *is
in the image of* p.

Proof of Claim 2. Suppose that p is homotopic to f.
Then f|X has a lift to A, namely the map h∘g: X → A. Call
this map $\tilde{f}$. Let H be the homotopy from f to p and let $\tilde{H}$ be
the lift of this homotopy starting at $\tilde{f}$ and ending at $\tilde{p}$.
Consider the point 0 ∈ R ⊂ A. By Lemma 1.1 there is an open
set U in A with $\overline{U}$ compact with 0 ∈ U such that if x ∈ X and
h: {x} × I → A is any lift of H|{x} × I, then if 0 ∈ h({x} ×
I), then h({x} × I) ⊂ U. Now let N be an integer such that
if r is the projection of A onto R taking S_i^n to i for each
i, then the projection of U is contained in (-N,N). Then for

any $x \in X$, if the homotopy $\tilde{H}$ has the property that $\tilde{H}(\{x\} \times I)$ contains $i \in R \subset A$, then $r \circ \tilde{H}(\{x\} \times I) \subset (i-N, i+N)$. Now let $M > N$. Then $h \circ g | L_M$ maps onto $S_M^n \cup [M, 2M] \subset A$ with $h \circ g(C_M) = \{2M\}$. Now $\tilde{H} | L_M \times I$ must have the property that $r \circ \tilde{H}(C_M, t)$ does not contain the point M for all $t \in I$ since $2M \in \tilde{H}(C_M \times I)$ and $2M - M = M > N$. Thus $\tilde{H}(C_M \times I) \subset A - S_M^n$. Thus the map $\tilde{p} | L_M$ maps onto S_M^n by Claim 1 (since k_M in Claim 1 is just $h \circ g | L_M = \tilde{f} | L_M$). Thus $p | L_M$ maps onto S^n. Thus p contains S^n in its image. This proves Claim 2.

Claim 3. If $p: \beta X \to S^n \vee S^1$ *is homotopic to* f, *then* p *contains* S^1 *in its image.*

Proof of Claim 3. Let H be the homotopy joining f to p. Let $\tilde{f} = h \circ g: X \to A$ and let $\tilde{H}: X \times I \to A$ be the lift of $H | X \times I$ joining $\tilde{f}$ to $\tilde{p}$ as in Claim 2. Let N be a positive integer such that if $x \in X$ and $0 \in h(\{x\} \times I)$ for some lift of $H | \{x\} \times I$, then $r \circ h(\{x\} \times I) \subset (-N, N)$ where $r: A \to R$ is the retraction of A to R which takes each S_i^n to the point i. Let M be a positive integer with $M > 2N+1$. Then noting that $\tilde{f}(C_M) = \{2M\} \subset R$ and $\tilde{f}(y) - M$ for some $y \in L_M$ we must have that for all $x \in C_M$, $r \circ \tilde{p}(x) > 2M-N$ and for y $r \circ \tilde{p}(y) < M+N$. Thus for all $x \in C_M$, $r \circ \tilde{p}(x) - r \circ \tilde{p}(y) > 2M - N - (M+N) = M - 2N > 1$. Now by the proof of Claim 2, $\tilde{p}(L_M) \supset S_M^n$. Now we want to claim that $\tilde{p}(L_M)$ contains all of the interval $[M + N, 2M - N]$. If it does, then this interval maps onto S^1 by $c: A \to S^n \vee S^1$. Thus p will contain S^1 in its image and Claim 3 will be proved. Thus Claim 3 will follow from Claim 4.

Claim 4. The map $\tilde{p} | L_M$ *contains* $[M + N, 2M - N]$ *in its*

image.

Proof of Claim 4. Suppose not and suppose that $z \in$ [M + N, 2M - N] with $z \notin \tilde{p}(L_M)$. We may assume z is not an integer point since the points in [M + N, 2M - N] not in the image of $\tilde{p}|L_M$ is an open set. Then let $L_M = C \cup D$ where $C = (r \circ \tilde{p}|L_M)^{-1}(-\infty, z)$ and $D = (r \circ \tilde{p}|L_M)^{-1}(z, +\infty)$. This is a separation of L_M with $C_M \subset D$. Now since $\tilde{p}(L_M)$ contains S_M^n, it must be that $\tilde{p}(C) \supseteq S_M^n$ since $\tilde{p}(D) \cap S_M^n = \emptyset$. However, $\tilde{p}|C$ is homotopic to $\tilde{f}|C$ in A and $\tilde{f}|C: C \to A$ factors through B_M^n. Thus $\tilde{f}|C$ is null-homotopic and $\tilde{p}|C$ is null-homotopic. Thus $\tilde{p}|L_M$ is homotopic to a map v by a homotopy H' with the property that $v|C$ is constant and $H'|D \times I \equiv \tilde{p}|D$. But then $\tilde{f}|L_M$ is homotopic to the map v by a homotopy H" which joins the homotopies H and H'. The homotopy H" has the property that $H"(C_M \times I) \subset A - S_M^n$ and the map v does not contain S_M^n in its image. This contradicts Claim 1. This contradiction shows that $\tilde{p}(L_M)$ contains all of [M + N, 2M - N] and Claim 4 is proved.

3.5. *Theorem. Let X be locally compact and σ-compact and n ≥ 2 be an integer. Suppose that X has the property that for every compact set K ⊂ X there is a compact set L ⊂ X-K with dim L = n. Then there is a map f: βX-X → $S^n \vee S^1$ which is homotopically onto.*

Proof. The proof is a modification of the proof of Theorem 1.3.

3.6. *Theorem. Let K be a compactum contained in βX-X where X is a Lindelöf space. Let n ≥ 2. Then if dim K = n, then there is a map f: K → $S^n \vee S^1$ which is homotopically onto.*

Proof. The proof is similar to the proof of Theorem
1.8.

3.7. *Question*. Is it true that if X is any paracompact
space with dim $X \geq n$, then there is a closed set $C \subset X$ and a
map f: $(X,C) \rightarrow (B^n, S^{n-1})$ which is not null-homotopic such
that if e: $(B^n, S^{n-1}) \rightarrow (S^n, p)$ collapses S^{n-1} to p, then
$e \circ f$: $(X,C) \rightarrow (S^n, p)$ is also not null-homotopic?

3.8. *Example*. One problem in answering Question 3.7
is that we do not have a nice relationship between maps into
n-spheres and cohomology for infinite-dimensional spaces.
For instance for each odd prime p Kahn [6] has given an exam-
ple of an infinite-dimensional metric continuum X_p such that
$H^k(X_p) = 0$ for all $k > 0$, but with X_p having essential maps
onto $S^{3+i(2p-2)}$ for all $i \geq 0$.

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