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by

W. F. Pfeffer

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
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SOME REMARKS ON GENERALIZED BOREL MEASURES IN TOPOLOGICAL SPACES

W. F. Pfeffer

Given a set A, we shall denote by $|A|$ the cardinality of A and by exp A the family of all subsets of A. Throughout, α will denote an *uncountable* cardinal and β, γ, will denote *infinite* cardinals. Whenever convenient, we shall identify a cardinal with its initial ordinal. As usual, we shall denote by ω and Ω the first infinite and the first uncountable cardinals, respectively.

Definition 1. Let M be a set. A family $\mathcal{M} \subset \exp M$ is called an α-*algebra* in M if

(i) $(\mathcal{A} \subset \mathcal{M},\ |\mathcal{A}| < \alpha) \Rightarrow \cup\mathcal{A} \in \mathcal{M}$;

(ii) $A \in \mathcal{M} \Rightarrow M - A \in \mathcal{M}$.

It follows from (i) that $\emptyset \in \mathcal{M}$; thus by (ii), also $M \in \mathcal{M}$.

Definition 2. Let $\mathcal{M}$ be an α-algebra in a set M. A function $\mu\colon \mathcal{M} \to [0,+\infty]$ is called an α-*measure* on $\mathcal{M}$ if $\mu(\emptyset) = 0$ and

$$\mu(\cup\mathcal{A}) = \Sigma\{\mu(A) : A \in \mathcal{A}\}$$

for each disjoint family $\mathcal{A} \subset \mathcal{M}$ with $|\mathcal{A}| < \alpha$.

The triple $(M,\mathcal{M},\mu)$ is called an α-*measure space*.

Definition 3. Let $(M,\mathcal{M},\mu)$ be an α-measure space. The α-measure μ is called β-*finite* if there is an $\mathcal{A} \subset \mathcal{M}$ such that $|\mathcal{A}| \le \beta$, $\cup\mathcal{A} = M$, and $\mu(A) < +\infty$ for each $A \in \mathcal{A}$.

If $\alpha = \Omega$ and $\beta = \omega$, then the previous definitions reduce to the usual definitions of a σ-algebra, measure, and a σ-finite measure. To illustrate the situation when $\alpha > \Omega$ and $\beta > \omega$, we shall present a few examples.

Example 1. Let M be a set and let $f: M \to [0,+\infty]$. For $A \subset M$ set

$$\mu(A) = \Sigma\{f(x): x \in A\}.$$

Then μ is an α-measure on exp M for each α. If $\mu(M) = +\infty$, then μ is β-finite if and only if

$$|\{x \in M: f(x) > 0\}| \leq \beta.$$

Example 2. Let $\kappa \geq \Omega$ be a regular ordinal and let W be the set of all ordinals less than κ equipped with the order topology. Denote by $\mathcal{H}$ the family of all closed cofinal subsets of W and let $\mathcal{M}$ consist of all sets $A \subset W$ for which either A or W - A contain a set $F \in \mathcal{H}$. For $A \in \mathcal{M}$, let $\mu(A) = 1$ if A contains a set $F \in \mathcal{H}$, and $\mu(A) = 0$ otherwise. Clearly, $\mathcal{M}$ contains all open subsets of W. To show that $(W,\mathcal{M},\mu)$ is a κ-measure space, it suffices to prove the following claim.

Claim. If $\mathcal{H}_0 \subset \mathcal{H}$ and $|\mathcal{H}_0| < \kappa$, then $\cap\mathcal{H}_0 \in \mathcal{H}$.

Proof. Using the interlacing lemma (see [8], chpt. 4, prbl. E, (a), p. 131) in W, it is easy to prove the claim if $|\mathcal{H}_0| = 2$. By induction, the claim is correct if $|\mathcal{H}_0| < \omega$. Let $\omega \leq \xi < \kappa$ and suppose that the claim holds whenever $|\mathcal{H}_0| < \xi$. Let $\mathcal{H}_0 = \{F_\rho: \rho < \xi\}$. Replacing F_ρ by $\cap\{F_\tau: \tau \leq \rho\}$, we may assume that $F_\tau \subset F_\rho$ for each $\tau \leq \rho < \xi$. Choose an $\eta < \kappa$. Since κ is regular, there are $\eta_\tau \in F_\tau$ such that $\eta < \eta_\tau < \eta_\rho$ for each $\tau < \rho < \xi$. If

$$\zeta = \sup\{\eta_\rho : \rho < \xi\},$$

then $\zeta \in \cap\mathcal{H}_0$; for by the regularity of κ, $\zeta < \kappa$. It follows that $\cap\mathcal{H}_0$ is a cofinal subset of W and the claim is proved.

Example 3. Let $|M| = \kappa$ where κ is the first measurable cardinal (see Definition 6), and let μ be a σ-additive measure on exp M such that $\mu(M) = 1$ and $\mu(\{x\}) = 0$ for each $x \in M$. Then it follows from [2], Lemma 0.4.12 that μ is a κ-measure on exp M.

Example 4. Karel Hrbacek kindly pointed out to me the following fact proved in [10], sec. 4, coroll. 1. If α is a regular cardinal, then there is a model for the Zermelo-Fraenkel set theory with the axiom of choice in which

(i) $2^\omega = \alpha$ and Martin's axiom A holds;

(ii) The family $\mathcal{L}$ of all Lebesgue measurable subsets of reals is an α-algebra and the Lebesgue measure is an α-measure on $\mathcal{L}$.

Throughout, X will be a T_1 space and $\mathcal{G}$ will denote the family of all open subsets of X. The intersection of all α-algebras in X containing $\mathcal{G}$ is again an α-algebra in X containing $\mathcal{G}$; it is denoted by $\mathcal{B}_\alpha$ and called the *Borel α-algebra* in X. Clearly, $\mathcal{B}_\Omega$ is then the usual Borel σ-algebra in X. An α-measure μ on $\mathcal{B}_\alpha$ is called a *Borel α-measure* in X if it is *locally finite*, e.g., if each $x \in X$ has a neighborhood $U \in \mathcal{B}_\alpha$ with $\mu(U) < +\infty$.

A set $A \subset X$ is called *γ-Lindelöf* if each open cover of A contains a subcover of cardinality less than γ. Thus Ω-Lindelöf sets are Lindelöf, and ω-Lindelöf sets are compact.

The family of all *closed* γ-Lindelöf subsets of X is denoted by $\mathcal{F}_\gamma$. Clearly, $\mathcal{F}_\gamma$ is the family of all closed subsets of X whenever $\gamma > |X|$.

Definition 4. A Borel α-measure in X is called

(i) *diffused* if $\mu(\{x\}) = 0$ for each $x \in X$;

(ii) β-*moderated* if there is an $\mathcal{A} \subset \mathcal{G}$ such that $|\mathcal{A}| \leq \beta$, $\cup\mathcal{A} = X$, and $\mu(A) < +\infty$ for each $A \in \mathcal{A}$;

(iii) γ-*Radon* if

$$\mu(A) = \sup\{\mu(F): F \in \mathcal{F}_\gamma, F \subset A, \mu(F) < +\infty\}$$

for each $A \in \mathcal{B}_\alpha$;

(iv) γ-*regular* if it is γ-Radon and

$$\mu(A) = \inf\{\mu(G): G \in \mathcal{G}, A \subset G\}$$

for each $A \in \mathcal{B}_\alpha$.

The Borel α-measures which are ω-moderated, or ω-Radon, or ω-regular are usually called *moderated*, or *Radon*, or *regular*, respectively.

Discussion of results. It is clear that a β-moderated α-measure is β-finite, and that a β-finite, γ-regular α-measure is β-moderated. On the other hand, a σ-finite Radon measure is generally not moderated (see [4], ex. 7). It is easy to see that a moderated γ-Radon α-measure is γ-regular, yet the converse is false (e.g., giving M the discrete topology, the measure μ from Example 1 is always regular, but not necessarily moderated). We shall show that for a large family of spaces each diffused γ-regular α-measure with $\alpha \geq \gamma$ is moderated and hence σ-finite. Thus typically a non-σ-finite, γ-Radon α-measure is not γ-regular. We shall prove, however, that a β-finite, γ-Radon α-measure with $\alpha > \beta$ and $\alpha \geq \gamma$ is β-moderated whenever the space X is meta-β-Lindelöf (see Definition 5, (i)). Under a mild set theoretic restriction, we shall also prove that in a metacompact space each β-finite, Borel α-measure with $\alpha > \beta$ is β-moderated.

Sometimes many properties of a Borel α-measure can be deduced from the well-known facts about the usual σ-additive Borel measures. To this end, we shall show that each moderated, γ-Radon α-measure is a restriction of a complete Borel measure. We shall also prove that each γ-Radon α-measure with $\alpha > \gamma^{\omega}$ degenerates to a measure from Example 1.

Proposition 1. *Let* $\alpha \geq \gamma$ *and let* μ *be a* γ*-Radon* α*-measure in* X. *Then there is a disjoint family* $\mathcal{C} \subset \mathcal{F}_\gamma$ *such that*

(i) If $C \in \mathcal{C}$, *then* $C \neq \emptyset$, $\mu(C) < +\infty$, *and* $\mu(C \cap G) > 0$ *for each* $G \in \mathcal{G}$ *with* $C \cap G \neq \emptyset$;

(ii) If $B \in \mathcal{B}_\alpha$, *then*

$$\mu(B) = \Sigma\{\mu(B \cap C) : C \in \mathcal{C}\}.$$

Proof. By Zorn's lemma there is a maximal disjoint family $\mathcal{C} \subset \mathcal{F}_\gamma$ satisfying condition (i). Let $B \in \mathcal{B}_\alpha$. Since $\mathcal{C}$ is disjoint,

$$\mu(B) \geq \Sigma\{\mu(B \cap C) : C \in \mathcal{C}\}.$$

We shall prove the reverse inequality in three steps.

(a) Let $B \cap C = \emptyset$ for each $C \in \mathcal{C}$ and suppose that $\mu(B) > 0$. Then there is an $F \in \mathcal{F}_\gamma$ with $F \subset B$ and $0 < \mu(F) < +\infty$. If

$$H = \cup\{F \cap G : G \in \mathcal{G}, \mu(F \cap G) = 0\},$$

then H is open in F and $\mu(E) = 0$ for each γ-Lindelöf set

$E \subset H$. Since μ is γ-Radon, $\mu(H) = 0$. Letting $C_o = F - H$, we have $C_o \in \mathcal{J}_\gamma$ and $\mu(C_o) = \mu(F)$. In particular, $C_o \neq \emptyset$ and $\mu(C_o) < +\infty$. By the definition of H, $\mu(C_o \cap G) > 0$ for each $G \in \mathcal{G}$ with $C_o \cap G \neq \emptyset$. It follows that $\mathcal{C} \cup \{C_o\}$ is a disjoint subfamily of $\mathcal{J}_\gamma$ satisfying condition (i); a contradiction.

(b) Let $B \in \mathcal{J}_\gamma$. By the local finiteness of μ, there is an open cover $\mathcal{U}$ of B such that $|\mathcal{U}| < \gamma$ and $\mu(U) < +\infty$ for each $U \in \mathcal{U}$. Since $\mathcal{C}$ is disjoint, and since $\mu(C \cap U) > 0$ whenever $C \in \mathcal{C}$, $U \in \mathcal{U}$, and $C \cap U \neq \emptyset$, the family $\{C \in \mathcal{C}: C \cap U \neq \emptyset\}$ is countable for each $U \in \mathcal{U}$. Thus

$$|\{C \in \mathcal{C}: B \cap C \neq \emptyset\}| < \gamma\cdot\omega = \gamma \leq \alpha,$$

and by (a),

$$\mu(B) = \Sigma\{\mu(B \cap C): C \in \mathcal{C}\}.$$

(c) Let $B \in \mathcal{B}_\alpha$. By (b), for each $F \in \mathcal{J}_\gamma$ with $F \subset B$,

$$\mu(F) = \Sigma\{\mu(F \cap C): C \in \mathcal{C}\} \leq \Sigma\{\mu(B \cap C): C \in \mathcal{C}\}.$$

Since μ is γ-Radon, also

$$\mu(B) \leq \Sigma\{\mu(B \cap C): C \in \mathcal{C}\}.$$

In case of $\alpha = \Omega$ and $\gamma = \omega$, a version of Proposition 1 was proved by N. Bourbaki and R. Godement, who called the family $\mathcal{C}$ a μ-*concassage* (see [13], p. 46).

A set $A \subset X$ is called *locally countable* if each $x \in X$ has a neighborhood U with $|A \cap U| \leq \omega$.

Theorem 1. *Let* $\alpha \geq \gamma$ *and let each uncountable locally countable set* $A \subset X$ *contain an uncountable subset* $B \in \mathcal{B}_\alpha$. *If* μ *is a diffused,* γ*-regular* α*-measure in* X, *then* μ *is moderated.*

Proof. Let $\mathcal{C}$ be the family from Proposition 1. If $\mathcal{C}$

is countable, then μ is σ-finite and thus by γ-regularity, also moderated. Hence assume that $\mathcal{C}$ is uncountable, for each $C \in \mathcal{C}$ choose $x_C \in C$, and let $A = \{x_C: C \in \mathcal{C}\}$. It follows from Proposition 1 and the local finiteness of μ that A is locally countable. According to our assumptions, we can find an uncountable set $B \subset A$ with $B \in \mathcal{B}_\alpha$. Let $F \in \mathcal{F}_\gamma$ and $F \subset B$. Since F can be covered by less that γ open sets of finite measures, it follows from Proposition 1 that $|\{C \in \mathcal{C}: C \cap F \neq \emptyset\}| < \gamma \cdot \omega = \gamma$. Consequently, $|F| < \gamma \leq \alpha$. Since μ is a diffused α-measure, $\mu(F) = 0$. By the γ-regularity of μ, $\mu(B) = 0$ and there is a $G \in \mathcal{G}$ such that $B \subset G$ and $\mu(G) < +\infty$. For each $x_C \in B$, we have $\mu(C \cap G) > 0$. Since B is uncountable and $\mathcal{C}$ is disjoint, this implies that $\mu(G) = +\infty$; a contradiction.

Next we shall show that the condition from Theorem 1 is satisfied by a large collection of familiar spaces.

Let M be a set, and let $\mathcal{A} \subset \exp M$. For $x \in M$, set

$$o(x,\mathcal{A}) = |\{A \in \mathcal{A}: x \in A\}|$$

and let $o(\mathcal{A})$ be the least cardinal such that

$$o(x,\mathcal{A}) < o(\mathcal{A})$$

for each $x \in M$. The cardinal $o(\mathcal{A})$ is called the *order* of $\mathcal{A}$.

Definition 5. A space X is called

(i) *meta-*β*-Lindelöf* if each open cover of X has an open refinement $\mathcal{V}$ with $o(\mathcal{V}) \leq \beta$;

(ii) α*-weakly* θ*-refinable* if each open cover of X has an open refinement $\mathcal{V} = \cup\{\mathcal{V}_t: t \in T\}$ such that $|T| < \alpha$ and for each $x \in X$ there is a $t_x \in T$ with $1 \leq o(x,\mathcal{V}_{t_x}) < \omega$.

Clearly, meta-ω-Lindelöf and meta-Ω-Lindelöf spaces are, respectively, metacompact and meta-Lindelöf. Similarly, Ω-weakly θ-refinable spaces are weakly θ-refinable in the sense of [1].

Proposition 2. *Let* X *be* α-*weakly* θ-*refinable and let* $A \subset X$. *If there is a* $\beta < \alpha$ *such that each* $x \in X$ *has a neighborhood* U *with* $|A \cap U| \leq \beta$, *then* $A \in \mathcal{B}_\alpha$. *In particular, if* A *is locally countable, then* $A \in \mathcal{B}_\alpha$.

Proof. Suppose that there is a $\beta < \alpha$ such that each $x \in X$ has an open neighborhood U_x with $|A \cap U_x| \leq \beta$. Let $\mathcal{V} = \cup\{\mathcal{V}_t: t \in T\}$ be an open refinement of $\{U_x: x \in X\}$ such that $|T| < \alpha$ and for each $x \in X$ there is a $t_x \in T$ with

$$1 \leq o(x, \mathcal{V}_{t_x}) < \omega.$$

Since the sets $\{x \in X: o(x,\mathcal{V}_t) \geq n\}$, $t \in T$, $n = 1,2,\cdots$, are open, the sets

$$X_{t,n} = \{x \in X: o(x,\mathcal{V}_t) = n\}$$

are Borel. Clearly,

$$X = \cup\{X_{t,n}: t \in T,\ n = 1,2,\cdots\}.$$

Let $\mathcal{W}_{t,n}$ consist of all sets $X_{t,n} \cap V_1 \cap \cdots \cap V_n$ where $V_1,\cdots,V_n$ are distinct elements of $\mathcal{V}_t$. Then $\mathcal{W}_{t,n}$ is a disjoint family of open (in $X_{t,n}$) subsets of $X_{t,n}$ and $X_{t,n} = \cup\mathcal{W}_{t,n}$. Moreover,

$$A \cap W = \{x_W^\rho: \rho < \kappa_W\}$$

where $\kappa_W \leq \beta$ for each $W \in \mathcal{W}_{t,n}$. Thus the sets

$$A_{t,n,\rho} = \{x_W^\rho: W \in \mathcal{W}_{t,n},\ \kappa_W > \rho\},$$

$t \in T$, $n = 1,2,\cdots$, $\rho < \beta$, are closed in $X_{t,n}$, and therefore Borel. We have

$$\begin{aligned} A &= \cup\{A \cap X_{t,n}: t \in T,\ n = 1,2,\cdots\} \\ &= \cup_{t\in T} \cup_{n=1}^{\infty} \cup\{A \cap W: W \in \mathcal{W}_{t,n}\} \\ &= \cup_{t\in T} \cup_{n=1}^{\infty} \cup_{\rho<\beta} A_{t,n,\rho}\ . \end{aligned}$$

Since $|T| < \alpha$, $\omega < \alpha$, and $\beta < \alpha$, it follows that $A \in \mathcal{B}_\alpha$.

The following corollary is a direct consequence of Theorem 1 and Proposition 2.

Corollary 1. *Let* $\alpha \geq \gamma$ *and let* X *be* α*-weakly* θ*-refinable. If* μ *is a diffused,* γ*-regular* α*-measure in* X*, then* μ *is moderated.*

Next two examples show that the assumptions of Proposition 2 are essential.

Example 5. For a regular ordinal $\kappa \geq \Omega$, let $(W_\kappa, \mathcal{M}_\kappa, \mu_\kappa)$ be the κ-measure space $(W, \mathcal{M}, \mu)$ from Example 2.

Claim. Let $\kappa \geq \Omega$ be a regular ordinal. Then there is a set $A_\kappa \subset W_\kappa$ such that $A_\kappa \cap W_\rho \in \mathcal{M}_\rho$ for no regular ordinal $\rho \in [\Omega, \kappa]$.

Proof. Since Ω is not a measurable cardinal (see [14], thm. (A)), there is a set $A_\Omega \subset W_\Omega$ for which $A_\Omega \notin \mathcal{M}_\Omega$. If $\rho \in (\Omega, \kappa]$ is a regular ordinal, then each closed cofinal subset of W_ρ contains ordinals cofinal with both ω and Ω. Thus it suffices to let A_κ be the union of A_Ω and the set of all ordinals $\zeta \in W_\kappa$ cofinal with Ω.

If the cardinal κ is an immediate successor of a cardinal β, then $|A_\kappa \cap [0,\zeta]| \leq \beta$ for each $\zeta \in W_\kappa$ and yet $A_\kappa \notin \mathcal{B}_\kappa$; for $\mathcal{B}_\kappa \subset \mathcal{M}_\kappa$. However, since each $A \subset W_\kappa$ contains a discrete subset B with $|B| = |A|$, Theorem 1 can be still applied to W_κ.

Example 6. Let κ be a weakly inaccessible ordinal, i.e., κ is a regular ordinal and $\kappa = \omega_\tau$ for some limit ordinal τ (see [9], chpt. IX, sec. 1, p. 309). With the notation from Example 5, let

$$X = \{\zeta,\eta) \in W_\kappa \times W_\kappa^*: \zeta \leq \eta\}$$

where W_κ^* is the set W_κ with the discrete topology. Clearly, X is paracompact. If

$$A = \{(\zeta,\eta) \in X: \zeta \in A_\kappa\},$$

then for each ordinal $\rho \in [\Omega,\kappa)$,

$$A \cap (W_{\rho+1} \times \{\rho\}) = (A_\kappa \cap W_{\rho+1}) \times \{\rho\}$$

and consequently

$$|A \cap (W_{\rho+1} \times \{\rho\})| < \kappa.$$

By the claim in Example 5, $A \in \mathcal{B}_\alpha$ for no regular cardinal $\alpha < \kappa$. It follows easily from the weak inaccessibility of κ that $\cup_{\alpha<\kappa} \mathcal{B}_\alpha$ is a κ-algebra in X containing all open subsets of X. Therefore, $\cup_{\alpha<\kappa} \mathcal{B}_\alpha = \mathcal{B}_\kappa$ and $A \notin \mathcal{B}_\kappa$.

As indicated by Example 5, the condition from Corollary 1 is not necessary. In fact, the following question seems open at this time.

Question. For $\alpha \geq \gamma$, *does there exist a diffused,* γ-*regular* α-*measure which is not moderated?*

Lemma 1. *Let* $(M,\mathcal{M},\mu)$ *be an* α-*measure space with* $\mu(M) < +\infty$, *and let* $\mathcal{A} \subset \mathcal{M}$. *If* $o(\mathcal{A}) < \alpha$ *and* $\varepsilon > 0$, *then*

$$|\{A \in \mathcal{A}: \mu(A) \geq \varepsilon\}| < \max(o(\mathcal{A}),\Omega).$$

Proof. (a) Let $\beta = o(\mathcal{A})$, $\beta < \alpha$, and suppose that there is an $\varepsilon > 0$ such that the set

$$\mathcal{A}_+ = \{A \in \mathcal{A}: \mu(A) \geq \varepsilon\},$$

has cardinality larger than or equal to $\max(\beta,\Omega)$. First we shall show that there is a family $\mathcal{C} \subset \mathcal{A}_+$ such that for each countable collection $\mathcal{D} \subset \mathcal{C}$ we can find a $C \in \mathcal{C}$ with $\mu(C - \cup\mathcal{D}) > 0$.

(b) If no such family exists, then for each $\mathcal{C} \subset \mathcal{A}_+$ there is a countable $\mathcal{C}_o \subset \mathcal{C}$ such that $\mu(C - \cup\mathcal{C}_o) = 0$ for each $C \in \mathcal{C}$. Let $\mathcal{C}_1 = \mathcal{A}_+$ and define inductively $\mathcal{C}_\tau$, $\tau < \beta$, by setting

$$\mathcal{C}_\tau = \mathcal{A}_+ - \cup_{\rho<\tau} \mathcal{C}_{\rho o}$$

where $\mathcal{C}_{\rho o} = (\mathcal{C}_\rho)_o$. Since $|\cup_{\rho<\tau} \mathcal{C}_{\rho o}| \le \tau\cdot\omega < \max(\beta,\Omega)$, the families $\mathcal{C}_\tau$, and consequently $\mathcal{C}_{\tau o}$, are nonempty for each $\tau < \beta$. If $C_\tau = \cup\mathcal{C}_{\tau o}$, then $\mu(C_\tau) \ge \varepsilon$ and $\mu(C-C_\tau) = 0$ for each $C \in \mathcal{C}_\rho$ with $\tau \le \rho < \beta$. Thus $\mu(C_\rho - C_\tau) = 0$ whenever $\tau \le \rho < \beta$, and we obtain

$$\mu(\cap_{\tau\le\rho} C_\tau) = \mu(C_\rho) - \mu(C_\rho - \cap_{\tau\le\rho} C_\tau)$$
$$= \mu(C_\rho) - \mu[\cup_{\tau\le\rho}(C_\rho - C_\tau)] = \mu(C_\rho) \ge \varepsilon$$

for each $\rho < \beta$. Since $\beta < \alpha$,

$$\mu(\cap_{\tau<\beta} C_\tau) = \inf_{\rho<\beta} \mu(\cap_{\tau\le\rho} C_\tau) \ge \varepsilon.$$

In particular, $\cap_{\tau<\beta} C_\tau \ne \emptyset$. If $\mathcal{C}_{\tau o} = \{A_\tau^k\colon k = 1,2,\cdots\}$, then

$$\cap_{\tau<\beta} C_\tau = \cap_{\tau<\beta} \cup_k A_\tau^k = \cup_{\{k(\tau)\}} \cap_{\tau<\beta} A_\tau^{k(\tau)}$$

where the last union is taken over all transfinite sequences $\{k(\tau)\}_{\tau<\beta}$ of positive integers. Thus there are positive integers $k(\tau)$, $\tau < \beta$, and an $x \in M$ with $x \in \cap_{\tau<\beta} A_\tau^{k(\tau)}$. Because the families $\mathcal{C}_{\tau o}$ are mutually disjoint, $A_\rho^{k(\rho)} \ne A_\tau^{k(\tau)}$ whenever $\rho \ne \tau$. Consequently $o(x,\mathcal{A}) = \beta$, and this contradiction establishes the existence of the family C from (a).

(c) Choose $E_0 \in \mathcal{C}$ and suppose that for each $\rho < \tau < \Omega$ we have chosen $E_\rho \in \mathcal{C}$ so that

$$\mu(E_\rho - \cup_{\lambda<\rho} E_\lambda) > 0.$$

Since $\{E_\rho: \rho < \tau\}$ is a countable subfamily of $\mathcal{C}$, there is an $E_\tau \in \mathcal{C}$ such that

$$\mu(E_\tau - \cup_{\rho<\tau} E_\tau) > 0.$$

Letting $F_\tau = E_\tau - \cup_{\rho<\tau} E_\rho$, we obtain an uncountable disjoint family $\{F_\tau: \tau < \Omega\} \subset \mathcal{M}$ of sets with positive measures. It follows that $\mu(M) = +\infty$; a contradiction.

Corollary 2. *Let* $(M,\mathcal{M},\mu)$ *be an* α*-measure space with* $\mu(M) < +\infty$*, and let* $\mathcal{A} \subset \mathcal{M}$*. If* $o(\mathcal{A}) < \alpha$ *and* $\varepsilon > 0$*, then*

$$|\{A \in \mathcal{A}: \mu(A) \geq \varepsilon\}| < \max(o(\mathcal{A}),\omega).$$

Proof. In view of Lemma 1, it suffices to consider the case of $o(\mathcal{A}) \leq \omega$. This implies that the set

$$\mathcal{A}_+ = \{A \in \mathcal{A}: \mu(A) \geq \varepsilon\}$$

is countable. If χ_A is the characteristic function of a set $A \subset M$, then

$$\Sigma\{\chi_A(x): A \in \mathcal{A}_+\} = o(x,\mathcal{A}_+)$$

for each $x \in M$. Since $\mathcal{A}_+$ is countable, the function $x \to o(x,\mathcal{A}_+)$ is measurable, and so are the sets

$$M_n = \{x \in M: o(x,\mathcal{A}_+) \leq n\},$$

$n = 1,2,\cdots$. The sequence $\{M_n\}$ is increasing and $\cup_{n=1}^{\infty} M_n = M$; for if $x \in M$, then

$$o(x,\mathcal{A}_+) < o(\mathcal{A}) \leq \omega.$$

Thus there is an integer $p \geq 1$ such that $\mu(M-M_p) < \frac{\varepsilon}{2}$. We have

$$\Sigma\{\mu(A \cap M_p): A \in \mathcal{A}_+\} = \Sigma\{\textstyle\int_{M_p}\chi_A\, d\mu: A \in \mathcal{A}_+\} \leq \textstyle\int_{M_p} p\, d\mu$$
$$= p\mu(M_p) < +\infty.$$

Because $\mu(A \cap M_p) \geq \frac{\varepsilon}{2}$ for each $A \in \mathcal{A}_+$, it follows that $|\mathcal{A}_+| < \omega$.

Example 7. Let $\kappa \geq \Omega$ be a regular ordinal and let $(W, \mathcal{M}, \mu)$ be the κ-measure space from Example 2. If $\mathcal{A} = \{[\rho,\kappa): \rho < \kappa\}$, then $\mu(A) = 1$ for each $A \in \mathcal{A}$ and $o(\mathcal{A}) = |\mathcal{A}| = \kappa$.

Proposition 3. *Let* $\alpha > \beta$ *and let* $(M, \mathcal{M}, \mu)$ *be an* α-*measure space with* μ β-*finite. If* $\mathcal{A} \subset \mathcal{M}$ *and* $o(\mathcal{A}) \leq \beta$, *then*

$$|\{A \in \mathcal{A}: \mu(A) > 0\}| \leq \beta.$$

Proof. Let $M = \cup\mathcal{C}$ where $\mathcal{C} \subset \mathcal{M}$, $|\mathcal{C}| \leq \beta$, and $\mu(C) < +\infty$ for each $C \in \mathcal{C}$. For $C \in \mathcal{C}$ and $A \in \mathcal{M}$, let $\mu_C(A) = \mu(A \cap C)$. Clearly, μ_C is a finite α-measure on $\mathcal{M}$ and

$$\mu(A) = \Sigma\{\mu_C(A): C \in \mathcal{C}\}$$

for every $A \in \mathcal{M}$. If

$$\mathcal{A}_C = \{A \in \mathcal{A}: \mu_C(A) > 0\},$$

then

$$\mathcal{A}_C = \cup_{n=1}^{\infty}\{A \in \mathcal{A}: \mu_C(A) \geq \tfrac{1}{n}\}$$

and so by Corollary 2, $|\mathcal{A}_C| \leq \omega\cdot\beta = \beta$. Since

$$\{A \in \mathcal{A}: \mu(A) > 0\} = \cup\{\mathcal{A}_C: C \in \mathcal{C}\},$$

the proposition follows.

Letting $\alpha = \Omega$ in Proposition 3, we obtain the following corollary.

Corollary 3. *Let* $(M, \mathcal{M}, \mu)$ *be a measure space with a* σ-*finite measure* μ. *If* $\mathcal{A} \subset \mathcal{M}$ *is a point-finite family, then* $\mu(A) = 0$ *for all but countably many* $A \in \mathcal{A}$.

Note. Without proof, Corollary 3 was first communicated to me by Heikki Junnila. Although quite analogous, his proof of Corollary 3 (see [7]) and my proof of Lemma 1 were obtained independently. There is a simple direct proof of Corollary 3 (see [12], chpt. 18, ex. (18-18)), which was found jointly by Don Chakerian and myself.

Theorem 2. *Let* $\alpha > \beta$, $\alpha \geq \gamma$, *and let* μ *be a* β-*finite,* γ-*Radon* α-*measure in* X. *If* X *is meta-*β-*Lindelöf, then* μ *is* β-*moderated.*

Proof. If X is meta-β-Lindelöf, then there is an open cover $\mathcal{A}$ of X such that $o(\mathcal{A}) \leq \beta$ and $\mu(U) < +\infty$ for each $U \in \mathcal{A}$. By Proposition 3,

$$|\{U \in \mathcal{A}: \mu(U) > 0\}| \leq \beta.$$

Thus if $\mathcal{A}_o = \{U \in \mathcal{A}: \mu(U) = 0\}$ and $G = \cup\mathcal{A}_o$, it suffices to show that $\mu(G) < +\infty$. Let $F \in \mathcal{F}_\gamma$ and $F \subset G$. There is $\mathcal{V} \subset \mathcal{A}_o$ such that $|\mathcal{V}| < \gamma$ and $F \subset \cup\mathcal{V}$. It follows that $\mu(F) = 0$ and since μ is γ-Radon, also $\mu(G) = 0$.

Definition 6. A cardinal κ is called *measurable* if there is a discrete space Y of cardinality κ and a diffused, Borel κ-measure μ in Y with $\mu(Y) = 1$.

The basic properties of measurable cardinals which do not involve axiomatic set theory are proved in [14]; more recent results can be found, e.g., in [2], chpt. 0, sec. 4.

The next lemma is proved by a modified technique of Haydon (see [6], Prop. 3.2).

Lemma 2. *Let* $\alpha > \beta$ *and let* μ *be a* β-*finite, Borel* α-*measure in* X. *Let* $\mathcal{A} \subset \mathcal{G}$ *be a point-finite family such that* $\mu(U) = 0$ *for each* $U \in \mathcal{A}$. *If* X *contains no discrete subspace of measurable cardinality, then* $\mu(\cup\mathcal{A}) = 0$.

Proof. Let $X = \cup\mathcal{C}$ where $\mathcal{C} \subset \mathcal{B}_\alpha$, $|\mathcal{C}| \leq \beta$, and $\mu(C) < +\infty$ for each $C \in \mathcal{C}$. Because the sets $\{x \in X: o(x,\mathcal{A}) \geq n\}$,

$n = 1,2,\cdots$, are open, the sets

$$X_n = \{x \in X: o(x,\mathcal{A}) = n\}$$

are Borel. Clearly $\cup\mathcal{A} = \cup_{n=1}^{\infty} X_n$, and so it suffices to show that $\mu(C \cap X_n) = 0$ for each $C \in \mathcal{C}$ and $n = 1,2,\cdots$. Fix a $C \in \mathcal{C}$ and an integer $n \geq 1$, and suppose that $\mu(C \cap X_n) > 0$. Consider the family $\mathcal{V}$ of all nonempty sets

$$V = C \cap X_n \cap U_1 \cap \cdots \cap U_n$$

where $U_1,\cdots,U_n$ are distinct elements of $\mathcal{A}$. Since $\mathcal{V}$ is a disjoint open (in $C \cap X_n$) cover of $C \cap X_n$, we can define an α-measure ν on $\exp \mathcal{V}$ by letting

$$\nu(\mathcal{V}') = \frac{\mu(\cup\mathcal{V}')}{\mu(C \cap X_n)}$$

for each $\mathcal{V}' \subset \mathcal{V}$. It follows from [2], lemma 0.4.12 that $\mathcal{V}$ contains a family $\mathcal{V}_o$ of measurable cardinality. Choosing an $x_V \in V$ for each $V \in \mathcal{V}_o$ we obtain a discrete subspace $X_o = \{x_V: V \in \mathcal{V}_o\}$ with $|X_o| = |\mathcal{V}_o|$; a contradiction.

Theorem 3. *Let* $\alpha > \beta$ *and let* μ *be a* β*-finite, Borel* α*-measure in* X. *If* X *is metacompact and contains no discrete subspace of measurable cardinality, then* μ *is* β*-moderated.*

Proof. Choose a point-finite open cover $\mathcal{A}$ of X such that $\mu(U) < +\infty$ for each $U \in \mathcal{A}$. By Proposition 3,

$$|\{U \in \mathcal{A}: \mu(U) > 0\}| \leq \beta,$$

and by Lemma 2,

$$\mu(\cup\{U \in \mathcal{A}: \mu(U) = 0\}) = 0.$$

The theorem follows.

We do not know whether Theorem 3 remains correct if the assumption "X contains no discrete subspace of measurable cardinality" is relaxed to "X contains no *closed* discrete subspace of measurable cardinality." However, using techniques of Moran (see [11], prop. 4.2) one can show easily that in Theorem 3, instead of assuming that X contains no discrete subspace of measurable cardinality, we may assume that all closed discrete subspaces of X have semi-reducible cardinality (for the definition and basic properties of semi-reducible cardinals see [11], sec. 3).

Note. Using a slightly different technique, Theorems 2 and 3 were proved in [4] for $\alpha = \Omega$ (see [4], Lemma 3 and Remark 2).

Let μ be a Borel α-measure. Since the Borel σ-algebra $\mathcal{B} = \mathcal{B}_\Omega$ is contained in $\mathcal{B}_\alpha$, we can define a measure space $(X,\overline{\mathcal{B}},\overline{\mu})$ as the usual completion of the measure space $(X,\mathcal{B},\mu)$ (see [5], sec. 13, p. 55). The measure space $(X,\overline{\mathcal{B}},\overline{\mu})$ is said to be *associated* with the Borel α-measure μ.

Proposition 4. *Let μ be a Borel α-measure in X and let $(X,\overline{\mathcal{B}},\overline{\mu})$ be the measure space associated with μ. If μ is moderated and γ-Radon with an arbitrary γ, then $\mathcal{B}_\alpha \subset \overline{\mathcal{B}}$ and $\overline{\mu}(A) = \mu(A)$ for each $A \in \mathcal{B}_\alpha$.*

Proof. (a) Let $A \in \mathcal{B}_\alpha$ and suppose that $A \subset H$ for some $H \in \mathcal{G}$ with $\mu(H) < +\infty$. Then

$$\begin{aligned}\mu(A) &= \sup\{\mu(F) : F \in \mathcal{F}_\gamma,\ F \subset A\}\\ &= \inf\{\mu(G) : G \in \mathcal{G},\ A \subset G\}.\end{aligned}$$

Thus there is an F_σ set F_o and a G_δ set G_o such that $F_o \subset A \subset G_o$ and $\mu(F_o) = \mu(A) = \mu(G_o) < +\infty$. It follows that $A = F_o \cup (A-F_o)$ belongs to $\overline{\mathcal{B}}$ and that

$$\overline{\mu}(A) = \overline{\mu}(F_o) = \mu(F_o) = \mu(A).$$

(b) Let $A \in \mathcal{B}_\alpha$ be arbitrary. Since μ is moderated, $X = \cup_{n=1}^{\infty} H_n$ where $H_n \in \mathcal{G}$ and $\mu(H_n) < +\infty$, $n = 1,2,\cdots$. By (a), $A \cap H_n \in \overline{\mathcal{B}}$ and $\overline{\mu}(A \cap H_n) = \mu(A \cap H_n)$, $n = 1,2,\cdots$. Consequently, $A \in \overline{\mathcal{B}}$ and $\overline{\mu}(A) = \mu(A)$.

The following theorem generalizes an unpublished result of Gary Gruenhage.

Theorem **4**. *Let* μ *be a* γ*-Radon* α*-measure in* X. *If* $\alpha > \Pi_{n=1}^{\infty} \gamma_n$ *whenever* $\gamma_n < \gamma$, $n = 1,2,\cdots$, *then*

$$\mu(B) = \Sigma\{\mu(\{x\}): x \in B\}$$

for each $B \in \mathcal{B}_\alpha$.

Proof. (a) Let X be γ-Lindelöf and let $\mu(X) < +\infty$. For each $x \in X$ and $n = 1,2,\cdots$, choose an $F_{x,n} \in \mathcal{F}_\gamma$ such that $F_{x,n} \subset X - \{x\}$ and

$$\mu(F_{x,n}) > \mu(X-\{x\}) - \frac{1}{n} .$$

Each open cover $\{X-F_{x,n}: x \in X\}$ of X has a subcover $\mathcal{U}_n$ with $|\mathcal{U}_n| < \gamma$, $n = 1,2,\cdots$. Since $\mu(X) < +\infty$, the set $A = \{x \in X: \mu(\{x\}) > 0\}$ is countable, and since

$$\mu(X-F_{x,n}) < \mu(\{x\}) + \frac{1}{n}$$

for each $x \in X$, $\mu(U-A) < \frac{1}{n}$ for each $U \in \mathcal{U}_n$. Thus

$$\mu[\cap_{n=1}^{\infty}(U_n-A)] = 0$$

for every sequence $\{U_n\}$ with $U_n \in \mathcal{U}_n$, $n = 1,2,\cdots$. We have

$$X - A = \cap_{n=1}^{\infty} \cup\{U-A: U \in \mathcal{U}_n\}$$
$$= \cup_{\{U_n\}} \cap_{n=1}^{\infty}(U_n-A)$$

where the last union is taken over all sequences $\{U_n\}$ with $U_n \in \mathcal{U}_n$, $n = 1,2,\cdots$. Because $|\mathcal{U}_n| < \gamma$, the collection of all these sequences has the cardinality less than α. Consequently, $\mu(X-A) = 0$ and

$$\mu(B) = \mu(A \cap B) = \Sigma\{\mu(\{x\}): x \in A \cap B\}$$
$$= \Sigma\{\mu(\{x\}): x \in B\}$$

for each $B \in \mathcal{B}_\alpha$.

(b) Let X and μ be arbitrary, and let $B \in \mathcal{B}_\alpha$. There are $F_n \in \mathcal{F}_\gamma$ such that $F_n \subset F_{n+1} \subset B$, $\mu(F_n) < +\infty$, $n = 1,2,\cdots$, and $\lim \mu(F_n) = \mu(B)$. Using (a), we obtain

$$\mu(B) = \lim \Sigma\{\mu(\{x\}): x \in F_n\}$$
$$= \Sigma\{\mu(\{x\}): x \in \cup_{n=1}^{\infty} F_n\} \leq \Sigma\{\mu(\{x\}): x \in B\}.$$

The equality holds trivially when $\mu(B) = +\infty$. If $\mu(B) < +\infty$, then $\mu(B - \cup_{n=1}^{\infty} F_n) = 0$ and the equality holds again.

Remark. The cardinality assumption in Theorem 4 is clearly satisfied when $\alpha > \gamma^\omega$. However, this later condition is generally stronger. For example, choose infinite cardinals κ_ρ so that $2^{\kappa_\rho} < 2^{\kappa_\tau}$ for each $\rho < \tau < \Omega$, and let $\gamma = \sup\{2^{\kappa_\rho}: \rho < \Omega\}$. If $\gamma_n < \gamma$, $n = 1,2,\cdots$, then $\gamma_n \leq 2^{\kappa_\rho}$ for some $\rho < \Omega$, and consequently

$$\Pi_{n=1}^{\infty} \gamma_n \leq (2^{\kappa_\rho})^\omega = 2^{\omega\cdot\kappa_\rho} = 2^{\kappa_\rho} < \gamma \leq \gamma^\omega.$$

We shall close this paper by stating two theorems about α-measures which for $\alpha = \Omega$ were proved previously by Gardner, Gruenhage, and the author (see [3], corollary to Theorem 6.1, and [4], Theorem 2). Recall that a space X is:

(i) *hereditarily* α-*weakly* θ-*refinable* if each subspace of X is α-weakly θ-refinable;

(ii) *locally* γ-*Lindelöf* if each $x \in X$ has a γ-Lindelöf neighborhood.

Theorem 5. *Suppose that* X *is a regular, hereditarily* α-*weakly* θ-*refinable space which contains no discrete subspace* *of measurable cardinality. Let* $\alpha > \beta$ *and let* μ *be a* β-*finite, Borel* α-*measure in* X. *Then* μ *is* γ-*Radon for each* $\gamma > |X|$.

Theorem 6. *Suppose that* X *is a regular,* α-*weakly* θ-*refinable, locally* γ-*Lindelöf space which contains no discrete subspace of measurable cardinality. Let* $\alpha > \beta$ *and let* μ *be a* β-*finite, Borel* α-*measure in* X. *Then* μ *is* γ-*Radon if and only if it is* δ-*Radon for some* $\delta \geq \omega$.

Modulo the obvious adjustments, for a finite α-measure μ the proofs of Theorems 5 and 6 are the same as those of Theorem (18.31) in [12] and Theorem 2 in [4], respectively. Since each β-finite α-measure with $\alpha > \beta$ is a sum of finite α-measures (see the proof of Proposition 3), it suffices to observe that a sum of γ-Radon α-measures is also γ-Radon.

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University of California
Davis, CA 95616