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0. Introduction

Considering the relationships amongst cardinal invariants of a T_1 space as diagrammed in Figure 1, one can scarcely fail to be struck by the evident symmetry. This paper is the result of an attempt to derive that symmetry from the duality between τX, the lattice of open sets of X, and σX, the lattice of closed sets in X. Since both lattices are complete, with 0 and 1, and distributive,[1] we make the blanket assumption: *all lattices are complete, distributive lattices with* 0 *and* 1. We further assume that *all spaces are* T_1.

Notation for topological cardinal invariants follows Juhász [1], save that $\Delta(X) = \sup\{|D|: D \subseteq X \text{ is closed \& discrete}\}$ is Hodel's discreteness character. Lower-case Greek letters denote ordinals (which are thought of as sets of smaller ordinals), except that κ, λ, and μ denote cardinals, i.e., initial ordinals.

1. Basic Definitions

We begin by introducing some basic terminology. Let L be a lattice, let $S \subseteq L$, and let $x \in L$. S is a G-cover of x iff $x \le \bigvee S$. S is an F-cover of x iff $y \in L$ and $y \wedge x \ne 0$ imply $y \wedge s \ne 0$ for some $s \in S$. If S is both an F- and a G-cover of

[1] 'Distributivity' is finite distributivity: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. A distributive lattice, even if it is complete, need not be *completely distributive*; i.e., it need not be true in such a lattice that $x \wedge \bigvee A = \bigvee\{x \wedge a: a \in A\}$ and $x \wedge \bigvee A = \bigwedge\{x \vee a: a \in A\}$. In the lattice σX, for example, $\bigvee\mathcal{F} = \mathrm{cl}_X \cup\mathcal{F}$, so, if $x \in \bigvee\mathcal{F} \setminus \cup\mathcal{F}$, then $\{x\} \wedge \bigvee\mathcal{F} = \{x\} \ne \emptyset = \bigvee\{\{x\} \cap F: F \in \mathcal{F}\}$.

x, we say that S *covers* x. If S is an F-cover (a G-cover) [a cover] of 1, we say simply that S is an F-cover (a G-cover) [a cover] of L. S is a *base* for L iff for each $x \in L$ there is an $S(x) \subseteq S$ such that S(x) covers x and $x = \bigvee S(x)$. S is *irreducible* iff $x \not\leq \bigvee(S\backslash\{x\})$ for each $x \in S$. Finally, a sequence $\langle x_\xi : \xi < \alpha \rangle$ of elements of L is an H-sequence iff $x_\xi < x_\eta$ whenever $\xi < \eta < \alpha$. Our basic cardinal invariants of L are now defined as follows.

$$w(L) = \omega + \inf\{|S| : S \text{ is a base for } L\}$$

$$h(L) = \omega + \sup\{|\alpha| : L \text{ has an H-sequence of length } \alpha\}$$

$$c(L) = \omega + \inf\{\kappa : \forall\, S \subseteq L\ (S \text{ covers } L \to \exists T \subseteq S(|T| \leq \kappa\ \&\ T \text{ G-covers } L))\}$$

$$i(L) = \omega + \sup\{|S| : S \subseteq L \text{ is irreducible}\}$$

$$d(L) = \omega + \sup\{|S| : S \text{ is an irreducible cover of } L\}$$

These are, respectively, the *weight*, *height*, *covering degree*, *irreducibility degree*, and *discreteness degree* of L.

We shall also be interested in $\phi(L^*)$, where ϕ is some cardinal invariant, and L* is the dual lattice of L, obtained by interchanging $\leq$ and $\geq$, $\wedge$ and $\vee$, $\bigwedge$ and $\bigvee$, and 0 and 1.[2] For notational convenience, and to avoid having too many lattices floating about, we shall write $\phi^*(L)$ instead of $\phi(L^*)$.

1.0. *Proposition.* $d(L) \leq c(L) \leq h(L) \leq w(L) \leq |L|$, *and* $d(L) \leq i(L) \leq h(L)$.

Proof. That $d(L) \leq i(L)$ and $w(L) \leq |L|$ is obvious.

$d(L) \leq c(L)$: Let S be an irreducible cover of L, and

[2] The lattice σX is dual to τX. If $\mathcal{V} \subseteq \tau X$, and $\mathcal{F} \subseteq \sigma X$, $\bigvee\mathcal{V} = \cup\mathcal{V}$, $\bigwedge\mathcal{V} = \mathrm{Int}_X \cap \mathcal{V}$, $\bigvee\mathcal{F} = \mathrm{cl}_X \cup \mathcal{F}$, and $\bigwedge\mathcal{F} = \cap\mathcal{F}$.

let $x \in S$ be arbitrary. If $S\setminus\{x\}$ were a G-cover of L, then we should have $x \leq 1 = \bigvee(S\setminus\{x\})$, a contradiction; thus, $|S| \leq c(L)$, and the result follows.

$c(L) \leq h(L)$: Let $\lambda = c(L)$, $\kappa = h(L)$, and suppose that $\kappa < \lambda$. Then there is a cover, S, of L with no G-subcover of power κ. Enumerate $S = \{x_\xi: \xi < \mu\}$ (for some $\mu > \kappa$). Let $y_0 = x_0$. Given $T_\eta = \{y_\xi: \xi < \eta\} \subseteq S$ for some $\eta < \kappa^+$, observe that $|T_\eta| \leq \kappa$, so $\bigvee T_\eta < 1$, and there is a $\rho < \mu$ such that $x_\rho \not\leq \bigvee T_\eta$; let $y_\eta = x_\rho$. Now, for $\eta < \kappa^+$, let $z_\eta = \bigvee T_\eta < 1$. Obviously $\langle z_\eta: \eta < \kappa^+\rangle$ is an H-sequence, which is impossible.

$i(L) \leq h(L)$: Let $S = \{x_\xi: \xi < \kappa\}$ be irreducible. For $\eta < \kappa$, let $y_\eta = \bigvee\{x_\xi: \xi < \eta\}$, and let $T = \{y_\eta: \eta < \kappa\}$. For any $\eta < \kappa$, $y_\eta \leq \bigvee(S\setminus\{x_\eta\}) \not\geq x_\eta$, so $x_\eta \not\leq y_\eta$, and hence $y_\eta < x_\eta \vee y_\eta = y_{\eta+1}$. Thus, $\langle y_\eta: \eta < \kappa\rangle$ is an H-sequence, and $h(L) \geq \kappa$.

$h(L) \leq w(L)$: Let $\langle x_\xi: \xi < \alpha\rangle$ be an H-sequence, and let B be a base for L. For $\eta < \alpha$, let $B_\eta \subseteq B$ be such that $x_\eta = \bigvee B_\eta$. Then $B_{\eta+1}\setminus\bigcup\{B_\xi: \xi < \eta\} \neq 0$ whenever $\xi + 1 < \alpha$, since $\bigvee\bigcup\{B_\xi: \xi < \eta\} = \bigvee\{x_\xi: \xi < \eta\} \leq x_\eta < x_{\eta+1}$. Thus, if $\alpha \geq \omega$, $|B| \geq |\alpha|$. (The case $\alpha < \omega$ requires only trivial modifications and will henceforth be ignored.)

1.1. *Proposition.* $i(L) = i^*(L)$.

Proof. Let $S = \{x_\xi: \xi < \kappa\}$ be irreducible in L. For $\eta < \kappa$, let $y_\eta = \bigvee(S\setminus\{x_\eta\})$, and let $T = \{y_\eta: \eta < \kappa\}$. If $\xi < \eta < \kappa$, then $x_\eta \leq y_\xi$, but $x_\eta \not\leq y_\eta$, so $y_\xi \neq y_\eta$, and $|T| = \kappa$. Moreover, if $\eta < \kappa$, $x_\eta \leq \bigwedge(T\setminus\{y_\eta\})$, so $\bigwedge(T\setminus\{y_\eta\}) \not\leq y_\eta$, and T is therefore irreducible in L*.

1.2. *Proposition.* $h^*(L) \leq w(L)$ *(and hence, of course,* $h(L) \leq w^*(L)$*).*

Proof. Let $\langle x_\xi : \xi < \alpha \rangle$ be an inverted H-sequence; i.e., $\xi < \eta < \alpha$ implies $x_\xi > x_\eta$. Let B be a base for L. If $\xi + 1 < \alpha$, then $x_{\xi+1} < x_\xi$, so there must be a $b_\xi \in B$ such that $b_\xi \leq x_\xi$ but $b_\xi \not\leq x_{\xi+1}$. Let $B_0 = \{b_\xi : \xi + 1 < \alpha\}$; clearly the members of B_0 are distinct, so (for $\alpha \geq \omega$) $|B| \geq |B_0| = |\alpha|$.

The relationships demonstrated so far are shown schematically in Figure 2; the analogy with Figure 1 is clear. To complete it, however, we need a notion for lattices corresponding to that of "subspace" for topological spaces. The notion we take is that of a "nice" homomorphic image, or quotient lattice, of L.

Fix $S \subseteq L$, and define an equivalence relation, E_S, on L as follows: $x\, E_S\, y$ iff $x \wedge s = y \wedge s$ for all $s \in S$. E_S is then a lattice congruence, so we may form the quotient lattice $L(S) = L/E_S$. Dually, we have $L'(S) = L/E'_S$, where $x\, E'_S\, y$ iff $x \vee s = y \vee s$ for all $s \in S$. $L(S)$ and $L'(S)$ are obviously distributive. For $x \in L$, let $[x]$ denote the E_S-equivalence class of x, so that $[\cdot]$ may be regarded as the canonical homomorphism from L onto $L(S)$. Define ϕ: $L(S) \to L$: $[x] \mapsto \bigwedge [x]$, and, for $x \in L$, let $\hat{x} = \phi([x])$. Since $\hat{x} \wedge s = (\bigwedge [x]) \wedge s = \bigwedge \{y \wedge s : y \in [x]\} = x \wedge s$ for any $x \in L$ and $s \in S$, it is clear that $\hat{x} \in [x]$. Indeed, it is easy to see that in general $\hat{x} = \bigvee \{x \wedge s : s \in S\}$, (and therefore $s = \hat{s}$ for $s \in S$). The importance of the map ϕ stems from the following result.

1.3. *Proposition.* ϕ *is an order-preserving injection.*

Proof. Suppose that $[x] < [y]$; we must show that $\hat{x} < \hat{y}$. Now $[\hat{x} \wedge \hat{y}] = [\hat{x}] \wedge [\hat{y}] = [x] \wedge [y] = [x] = [\hat{x}]$, so $\hat{x} \wedge \hat{y} \in [\hat{x}]$, and hence $\hat{x} \wedge \hat{y} = \hat{x}$, i.e., $\hat{x} \leq \hat{y}$. But if $\hat{x} = \hat{y}$, then $[x] = [\hat{x}] = [\hat{y}] = [y]$, so in fact $\hat{x} < \hat{y}$.

Indeed, it is clear now that $\hat{x} < \hat{y}$ iff $[x] < [y]$.

1.4. *Proposition.* *(i)* $[\cdot]$ *preserves arbitrary infima, and, in fact, (ii)* $L(S)$ *is complete.*

Proof. Fix $A \subseteq L$, and let $x = \bigwedge A$. Clearly $[x]$ is a lower bound for $[A]$ $(= \{[a]: a \in A\})$. Suppose that $[y]$ is a lower bound for $[A]$, with $[x] \leq [y]$. Then $\hat{y} \leq \hat{a} \leq a$ for all $a \in A$, whence $\hat{y} \leq x$, and therefore $[y] = [\hat{y}] \leq [x]$. It follows that $[y] = [x] = \bigwedge[A]$, which establishes (i).

Now let $x = \bigvee\{\hat{a}: a \in A\}$. Clearly $[a] = [\hat{a}] \leq [x]$ for $a \in A$, so $[x]$ is an upper bound for $[A]$. Suppose that $[y]$ is also an upper bound for $[A]$, with $[y] \leq [x]$. Then clearly $\hat{a} \leq \hat{y}$ for each $a \in A$, and thus $x \leq \hat{y} \leq y$, and it follows that $[x] = [y] = \bigvee[A]$.

1.5. *Proposition.* $[S]$ *covers* $L(S)$*; moreover, if* $T \subseteq S$ *with* $\bigvee T < \bigvee S$*, then* $[T]$ *is not a G-cover of* $L(S)$.

Proof. By the proof of 1.4, $\bigvee[S] = [\bigvee\{\hat{s}: s \in S\}] = [\bigvee S] = [1]$, so $[S]$ is a G-cover of $L(S)$. If $[x] \neq [0]$, then there is an $s \in S$ such that $x \wedge s \neq 0$, so that $[x] \wedge [s] \neq [0]$; $[S]$ is therefore also an F-cover of $L(S)$. Finally, if $T \subseteq S$ and $\bigvee T < \bigvee S$, then $s \not\leq \bigvee T$ for some $s \in S$, whence $s \wedge \bigvee T \neq s = s \wedge 1$, and $[\bigvee T] \neq [1]$, so that $[T]$ is not a G-cover of $L(S)$.

In the case of $L'(S)$ we define the map $[\cdot]$ analogously. For $x \in L$, $\hat{x}$ is now $\bigvee[x]$; as before, $\hat{x} \in [x]$, and the map $[x] \mapsto \hat{x}$ is an order-preserving injection. Also, $[\cdot]$ preserves arbitrary suprema, and $L'(S)$ is complete. Details are left to the reader, together with the observation that $L'(S)$ is isomorphic to $(L^*(S))^*$.

If $S = \{s\}$, we write $L(s)$ instead of $L(\{s\})$. The quotients $L(s)$ and $L'(s)$ (for $s \in L$) are particularly well-behaved: $L(s)$, for example, can be identified in an obvious way with $\{x \in L: x \leq s\} = \{\hat{x}: x \in L\}$, an observation which makes the next result almost obvious.

1.6. *Proposition. For any* $s \in L$, $c^*(L(s)) \leq c^*(L)$.

Proof. Let A be a cover of $(L(s))^*$; that is, by abuse of notation, $A \subseteq L$, $a \leq s$ for each $a \in A$, $\bigwedge A = 0$, and, if $x < s$, then $x \vee a < s$ for some $a \in A$. Let $x \in L$ with $x < 1$; if $x \vee a = 1$ for all $a \in A$, then $s = s \wedge (x \vee a) = (s \wedge x) \vee (s \wedge a) = (s \wedge x) \vee a$ for all $a \in A$, and it follows from the choice of A that $s \wedge x = s$. But then $s \leq x$, so that $x \vee a = x < 1$ for all $a \in A$. This contradiction shows that A is in fact a cover of L^* and has a G-subcover, C, (in L^*), of power at most $c^*(L)$. Since C can obviously be viewed as a G-subcover of A in $(L(s))^*$, the result follows.

We now define the "hereditary" version of our cardinal invariants: for any invariant ψ, $\overline{\psi}(L)$ is the supremum of all $\psi(L(S))$ and $\psi(L'(S))$ as S ranges over subsets of L.

1.7. *Proposition.* $\overline{h}(L) = h(L)$, *and* $\overline{i}(L) = i(L)$.

Proof. It suffices to show that for any $S \subseteq L$,

$h(L(S)) \leq h(L)$, $h(L'(S)) \leq h(L)$, $i(L(S)) \leq i(L)$, and $i(L'(S)) \leq i(L)$. We prove only the first of these; the rest are proved similarly.

If $\langle [x_\xi]: \xi < \alpha \rangle$ is an H-sequence in $L(S)$, $\langle \hat{x}_\xi: \xi < \alpha \rangle$ is an H-sequence in L (by the remark following 1.3).

1.8. *Proposition.* *(i)* $h(L) = \bar{c}(L)$. *(ii)* $i(L) = \bar{d}(L)$.

Proof. (i) for any $S \subseteq L$, $c(L(S)) \leq h(L(S)) \leq h(L)$, and similarly for $L'(S)$, so it suffices to show that $h(L) \leq \bar{c}(L)$. Let $\kappa = h(L)$, and let $\lambda \leq \kappa$ be regular. Then there is an $S = \{s_\xi: \xi < \lambda\}$ in L such that $\langle s_\xi: \xi < \lambda \rangle$ is an H-sequence. If $T \subseteq S$ with $|T| < \lambda$, then obviously $\bigvee T < \bigvee S$, so $[T]$ is not a G-cover of $L(S)$ (by 1.5). Thus, $c(L(S)) \geq \lambda$, and (i) follows.

(ii) If $S \subseteq L$ is irreducible, then $[S]$ is an irreducible cover of $L(S)$, since, for any $s \in S$, $\bigvee(S \setminus \{s\}) < \bigvee S$. The result now follows as above for (i).

As may be seen from Figure 3, we now have relationships which mimic exactly those of Figure 1. In the next section we show that the similarity is anything but accidental.

2. The Connection with Topology

Our first result is a catch-all theorem specifying the relationship between Figures 1 and 3.

2.0. *Theorem. Let X be a T_1 space, and let $L = \tau X$. Then:*

(i) $o(X) = \|L\|$;	*(ii)* $w(X) = w(L)$;
(iii) $h(X) = h(L)$;	*(iv)* $L(X) = c(L)$;
(v) $\Delta(X) = d(L)$;	*(vi)* $s(X) = i(L)$;

(vii) $|X| = w^*(L)$; *(viii)* $z(X) = h^*(L)$;

(ix) $d(X) = c^*(L)$; *(x)* $c(X) = d^*(L)$.

Proof. Parts (i), (ii), (iii), (vii), and (viii) are obvious. To see (iv), note that $A \subseteq L$ is a G-cover of L iff A covers X (as a family of open sets) iff A covers L. Part (ix) follows from the observation that $A \subseteq L^*$ G-covers L^* iff $\mathrm{cl} \bigcup A = X$ (thinking of A as a family of closed sets) iff $\bigcup A$ is dense in X, and A F-covers L^* iff $\bigcup A = X$. Thus if A is a cover of L^*, and $D \subseteq X$ is dense, choose, for $x \in D$, an $a(x) \in A$ with $x \in a(x)$; then $\{a(x): x \in D\}$ is a G-subcover of A in L^* of power at most $|D|$, and therefore $c^*(L) \leq d(X)$. Since $\{\{x\}: x \in X\}$ is a cover of L^* having no G-subcover of power less than $d(X)$, (ix) is established.

If, now, $D \subseteq L$ is irreducible, then clearly D is irreducible as a family of open sets in X, and thus $s(X) \geq |D|$, i.e., $s(X) \geq i(L)$. Conversely, if $D \subseteq X$ is discrete, there is a $V \subseteq L$ and a bijection $f: D \leftrightarrow V$ such that $D \cap f(x) = \{x\}$ for each $x \in D$; since V is obviously irreducible in L, we have proved (vi). If D is also closed, V may be chosen to cover X (by $f(x) = (X\backslash D) \cup \{x\}$), so $\Delta(X) \leq d(L)$, and the reverse inequality is easily obtained by "reversing" the construction.

It remains only to verify (x). But $A \subseteq L^*$ is an irreducible cover of L^* iff A covers X and $\forall a \in A$ $(a \nsubseteq \mathrm{cl} \bigcup(A\backslash\{a\}))$ iff $\{X\backslash \mathrm{cl} \bigcup(A\backslash\{a\}): a \in A\}$ is a pairwise disjoint family of non-empty, open sets in X, and the proof is complete.

The results shown in Figure 1 are therefore deducible from those of Figure 3, and the "closed-versus-open" duality is seen to be anything but accidental.

We close this section by stating without proof the result which justifies our rather odd-seeming definition of $\overline{\psi}$ for ψ a cardinal-valued lattice invariant.

2.1. *Proposition. Let* X *be a* T_1 *space, and let* $L = \tau X$ *(resp.* $L = \sigma X$*). Then* $\{L(S): S \subseteq L\} \cup \{L'(S): S \subseteq L\}$ *is, up to isomorphism,* $\{\tau Y: Y \subseteq X\}$ *(resp.,* $\{\sigma Y: Y \subseteq X\}$*). Specifically, if* $\rho \in \{\tau,\sigma\}$, $L = \rho X$, *and* $S \subseteq L$, *then* $L(S) \cong \rho(\bigcup S)$, *and* $L'(S) \cong \rho(X \setminus \bigcap S)$, *the isomorphisms being* $x \mapsto x \cap \bigcup S$ *and* $x \mapsto x \setminus \bigcap S$, *respectively.*

3. More Ambitious Lattice-Theoretic Results

In this section we attempt to mimic, for lattices, some more sophisticated topological results.

3.0. *Proposition.* $|L| \leq 2^{w(L)}$.

3.1. *Proposition.* $|L| \leq w(L)^{h(L)}$.

Proof. Fix a base, B, for L of power w(L), and let $x \in L$. There is a $B(x) \subseteq B$ which covers x exactly. But, if $[\cdot]: L \to L(x)$ is as in Section 1, $[B(x)]$ is a cover of $L(x)$ and has a G-subcover, $[B_0(x)]$, of power at most $c(L(x)) \leq h(L)$. If, now, $x \neq y$, clearly $B_0(x) \neq B_0(y)$; and since B has only $w(L)^{h(L)}$ subsets of power $\leq h(L)$, we are done.

Recall that for any set X, $[X]^{\leq\kappa} = \{A \subseteq X: |A| \leq \kappa\}$.

3.2. *Proposition. Let* $\kappa = i(L)$. *Suppose that* X, $Y \subseteq L$ *are such that*

(a) $\forall x \in X \exists y \in Y$ $(x \wedge y = 0)$, *and*

(b) whenever $X_0 \in [X]^{\leq\kappa}$, $Y_0 \in [Y]^{\leq\kappa}$, *and* $\bigwedge Y_0 \not\leq \bigvee X_0$,

there is an $x \in X$ *such that* $x \leq \bigwedge Y_0$ *and* $x \not\leq \bigvee X_0$.

Then there are $X_0 \in [X]^{\leq\kappa}$ *and* $Y_0 \in [Y]^{\leq\kappa}$ *such that* $\bigwedge Y_0 \leq \bigvee X_0$.

Proof. Suppose not. A simple recursion suffices to construct families $D = \{x_\xi : \xi < \kappa^+\}$ and $E = \{y_\xi : \xi < \kappa^+\}$ in L such that

(i) $D \subseteq X$ and $E \subseteq Y$,

(ii) $\forall\xi < \kappa^+ \ (x_\xi \wedge y_\xi = 0)$, and

(iii) $\forall\eta < \kappa^+ \ (x_\eta \leq \bigwedge\{y_\xi : \xi < \eta\} \ \& \ x_\eta \not\leq \bigvee\{x_\xi : \xi < \eta\})$.

But then D is irreducible, which is impossible. For, if $\eta < \kappa^+$, and $x_\eta \leq \bigvee(D \setminus \{x_\eta\})$, then $x_\eta \leq \bigvee\{x_\xi : \xi < \eta\} \vee \bigvee\{x_\xi : \eta < \xi < \kappa^+\} \leq \bigvee\{x_\xi : \xi < \eta\} \vee y_\eta$ (by (iii)), and hence $x_\eta = x_\eta \wedge x_\eta \leq x_\eta \wedge (\bigvee\{x_\xi : \xi < \eta\} \vee y_\eta) = x_\eta \wedge \bigvee\{x_\xi : \xi < \eta\} < x_\eta$ (by (iii)), which is absurd.

3.3. *Theorem*. *Let* $\kappa = i(L)$. *Suppose that* $X, Y \subseteq L$ *satisfy (a) of* 3.2, *and suppose further that* X *is a base for* L. *Then there are* $X_0 \in [X]^{\leq\kappa}$ *and* $Y_0 \in [Y]^{\leq\kappa}$ *such that* $\bigwedge Y_0 \leq \bigvee X_0$.

Proof. We check that X and Y satisfy (b) of 3.2. Suppose that $X_0 \in [X]^{\leq\kappa}$, $Y_0 \in [Y]^{\leq\kappa}$, and $\bigwedge Y_0 \not\leq \bigvee X_0$. Since X is a base for L, there is an $A \subseteq X$ such that $\bigwedge Y_0 = \bigvee A$. If $a \leq \bigvee X_0$ for each $a \in A$, then clearly $\bigwedge Y_0 = \bigvee A \leq \bigvee X_0$, so there is an $a \in A$ such that $a \not\leq \bigvee X_0$.

To proceed further along these lines we introduce another invariant. Let B be a base for L, and let $y \in L$; a family $X \subseteq L$ is a B-*base at* y iff $\forall x \in X \ (y \wedge x = 0)$ and $\forall b \in B \ (y \wedge b = 0 \rightarrow \exists x \in X \ (b \leq x))$. The B-*character of* y, $\chi(y,B)$, is defined to be the smallest cardinal of a B-base at y. The B-character of L, $\chi(L,B)$, is $\sup\{\chi(b,B): b \in B\}$; and the *character of* L, $\chi(L)$, is the least B-character of L for B a base for L. Finally, the B-*weight character* of L, $\omega(L,B)$, is $\sup\{\chi(b,L): b \in B\}$, and the *weight character* of L, $\omega(L)$, is the least B-weight character of L (B a base for L).

These rather odd definitions are motivated by the following observations. Suppose that X is a space, and let $\mathcal{J}$ be a base for the lattice σX; then $\mathcal{J} \supseteq \mathcal{J}_0 = \{\{x\}: x \in X\}$, and therefore trivially $\chi(\sigma X) = \chi(\sigma X, \mathcal{J}_0)$, and $\omega(\sigma X) = \omega(\sigma X, \mathcal{J}_0)$. If $\{x\} \in \mathcal{J}_0$, an $\mathcal{J}_0$-base at $\{x\}$ is a family $\mathcal{B} \subseteq \sigma X$ such that $x \notin \cup\mathcal{B}$, and such that whenever $x \notin H \in \sigma X$, there is a $B \in \mathcal{B}$ containing H. Let $\mathcal{V} = \{X \backslash B: B \in \mathcal{B}\}$; clearly $\mathcal{B}$ is an $\mathcal{J}_0$-base at $\{x\}$ iff $\mathcal{V}$ is a pseudobase for x in X. Moreover, $\mathcal{B}$ is a σX-base at $\{x\}$ iff $\mathcal{V}$ is a base at x. Thus, $\chi(\sigma X) = \psi(X)$, and $\omega(\sigma X) = \chi(X)$. (This result is later recorded as Proposition 4.1.) The foregoing definitions are the result of an attempt to capture the notions of character and pseudocharacter of a space without reference to the atomicity of σX.

3.4. *Lemma. Let* B *be a base for* L, *and let* κ, X, *and* Y *be as in* 3.2. *Assume further that* $X \subseteq B$ *and that* Y *contains a* B-*base at each* $x \in X$. *Then for each* $b \in B$ *such that* $b \wedge x = 0$ *for all* $x \in X$, *there is an* $X(b) \in [X]^{\leq\kappa}$ *such that* $b \leq \bigvee X(b)$.

Proof. Fix such a b. For each $x \in X$, find $y(x) \in Y$ such that $x \wedge y(x) = 0$ and $b \leq y(x)$. Let $\tilde{Y} = \{y(x): x \in X\}$; then X and $\tilde{Y}$ satisfy the hypothesis of 3.2, so $\bigwedge Y_0 \leq \bigvee X(b)$ for some $Y_0 \in [\tilde{Y}]^{\leq\kappa}$ and $X(b) \in [X]^{\leq\kappa}$. Since $b \leq \bigwedge Y_0$, we are done.

3.5. *Theorem. Let* $\kappa = i(L)$, *and let* λ *be such that* $\lambda^{\kappa} = \lambda$; *e.g.,* $\lambda = 2^{\kappa}$. *If* X *is a base for* L *such that* $\chi(L,X) \leq \lambda$, *then there is an* $\tilde{X} \in [X]^{\leq\lambda}$ *such that if* $x \in X$ *is disjoint from each member of* $\tilde{X}$, *then* $x \leq \bigvee\tilde{X}(x)$ *for some* $\tilde{X}(x) \in [\tilde{X}]^{\leq\kappa}$.

Proof. Suppose that $\eta < \kappa^{+}$ and that $X_{\xi} \in [X]^{\leq\lambda}$ and $Y_{\xi} \in [L]^{\leq\lambda}$ have been defined for each $\xi < \eta$ so that Y_{ξ} contains an X-base for each $x \in X_{\xi}$. Put $X'_{\eta} = \bigcup\{X_{\xi}: \xi < \eta\}$ and $Y'_{\eta} = \bigcup\{Y_{\xi}: \xi < \eta\}$, and, for each $A \in [X'_{\eta}]^{\leq\kappa}$ and $C \in [Y'_{\eta}]^{\leq\kappa}$, if $\bigwedge C \not\leq \bigvee A$, choose $x(A,C) \in X$ so that $x(A,C) \leq \bigwedge C$ and $x(A,C) \not\leq \bigvee A$. (See the proof of 3.3 for details.) Let $X_{\eta} = \{x(A,C): A \in [X'_{\eta}]^{\leq\kappa}, C \in [Y'_{\eta}]^{\leq\kappa}, \text{ and } \bigwedge C \not\leq \bigvee A\}$, and let Y_{η} be the union of X-bases $B(x)$ for all $x \in X_{\eta}$, with $|B(x)| \leq \lambda$ for all $x \in X_{\eta}$. Continue.

Now let $\tilde{X} = \bigcup\{X_{\eta}: \eta < \kappa^{+}\}$ and $\tilde{Y} = \bigcup\{Y_{\eta}: \eta < \kappa^{+}\}$; clearly $|\tilde{X}| \leq \kappa^{+} \cdot \lambda = \lambda$, and $\tilde{X}$, $\tilde{Y}$, and X satisfy the hypotheses of 3.4 in place of X, Y, and B, respectively.

3.6. *Proposition. If* X *is a base for* L, κ *is a cardinal, and* $Y \subseteq X$ *is such that whenever* $x \in X$ *is disjoint from each* $y \in Y$, *there is a* $Y(x) \in [Y]^{\leq\kappa}$ *such that* $x \leq \bigvee Y(x)$, *then* $Y^{\kappa} = \{\bigvee A: A \in [Y]^{\leq\kappa}\}$ *is an F-cover of* L.

Proof. If $x \in X$ and $x \not\leq a$ for any $a \in Y^{\kappa}$, then x meets some $y \in Y$, whence x meets some $a \in Y^{\kappa}$. Since X is a base for L, every $z \in L$ meets some $a \in Y^{\kappa}$.

3.7. *Corollary. If* $\kappa = i(L)$, $\lambda^{\kappa} = \lambda$, *and* B *is a base for* L *with* $\chi(L,B) \leq \lambda$, *then there is an* $X \in [B]^{\leq\lambda}$ *such that* X^{κ} *F-covers* L.

3.8. *Problem.* Find a "reasonable" condition on L to guarantee that the hypothesis of 3.7 holds with $\lambda = 2^{\kappa}$, say.

We next turn to some seemingly unrelated results whose discovery was, however, motivated by attempts to solve the above problem. We begin by considering a stronger order on L: for $x, y \in L$, we write $x << y$ iff $x < y$ and there is a $z \in L$ such that $x \wedge z = 0$ and $y \vee z = 1$. A sequence $\langle x_\xi: \xi < \alpha \rangle$ is a *K-sequence* iff $x_\xi << x_\eta$ whenever $\xi < \eta < \alpha$; equivalently, $\langle x_\xi: \xi < \alpha \rangle$ is a K-sequence iff there is a sequence $\langle x'_\xi: \xi + 1 < \alpha \rangle$, called a *co-K-sequence* for $\langle x_\xi: \xi < \alpha \rangle$, such that $x'_\xi \wedge x_\xi = 0$ and $x'_\xi \vee x_{\xi+1} = 1$ for each ξ with $\xi+1 < \alpha$. We define the *depth* of L, k(L), to be $\omega + \sup\{|\alpha|: \text{L has a K-sequence of length } \alpha\}$.

3.9. *Lemma. Let* $\langle x_\xi: \xi < \alpha \rangle$ *be a K-sequence, and let* $\langle x'_\xi: \xi + 1 < \alpha \rangle$ *be a corresponding co-K-sequence. Then whenever* $\xi+1 < \alpha$, *either* $x'_\xi = x'_{\xi+1}$, *or* $x'_\xi >> x'_{\xi+1}$.

Proof. $x_{\xi+1} \wedge x'_{\xi+1} = 0$ and $x_{\xi+1} \vee x'_\xi = 1$, and moreover $x'_{\xi+1} = x'_{\xi+1} \wedge (x'_\xi \vee x_{\xi+1}) = x'_{\xi+1} \wedge x'_\xi$, so that $x'_{\xi+1} \leq x'_\xi$.

3.10. *Lemma. If* $\langle x_n: n < 4 \rangle$ *is a K-sequence with co-K-sequence* $\langle x'_n: n < 3 \rangle$, *then* $x'_2 \wedge x << x'_0 \wedge x$ *whenever* $x_3 \leq x$.

Proof. By the proof of 3.9, it suffices to show that $x'_2 \wedge x < x'_0 \wedge x$. If not, $x'_2 \wedge x = x'_0 \wedge x$, so that $x_2 \wedge x'_0 = x_2 \wedge x'_0 \wedge x = 0$. But then $x_2 = x_2 \wedge (x_1 \vee x'_0) = x_2 \wedge x_1$, whence $x_2 \leq x_1$, which is impossible.

3.11. *Corollary.* $k(L) = k^*(L)$.

A slightly weaker notion is that of a *free sequence* in

L: $\langle x_\xi : \xi < \alpha \rangle$ is a free sequence iff $\mathbb{V}\{x_\xi : \xi < \eta\} \wedge \mathbb{V}\{x_\xi : \eta \leq \xi < \alpha\} = 0$ for each $\eta < \alpha$.

3.12. *Lemma. Let* $\langle x_\xi : \xi < \kappa \rangle$ *be a* K-*sequence with co*-K-*sequence* $\langle x'_\xi : \xi < \kappa \rangle$. *For each* $\xi < \kappa$, *let* $y_\xi = x'_\xi \wedge x_{\xi+2}$; *then* $\langle y_\eta : \eta < \kappa$ & η *is even*$\rangle$ *is free.*

Proof. For any even $\eta < \kappa$, $\mathbb{V}\{y_\xi : \xi < \eta$ & ξ is even$\} \leq x_\eta$, $\mathbb{V}\{y_\xi : \eta \leq \xi < \kappa$ & ξ is even$\} \leq x'_\eta$, and $x_\eta \wedge x'_\eta = 0$; it suffices to show that each $y_\eta \neq 0$, which follows from the computation $x_{\eta+1} \vee y_\eta = x_{\eta+1} \vee (x'_\eta \wedge x_{\eta+2}) = x_{\eta+1} \vee x_{\eta+2} = x_{\eta+2} > x_{\eta+1}$.

3.13. *Theorem. If* $\langle x_\xi : \xi < \alpha \rangle$ *is a free sequence in* L, *then* $\alpha < \kappa^+$ *where* $\kappa = \omega(L) \cdot c^*(L)$.

Proof. Suppose that there is a free sequence $\langle x_\xi : \xi < \kappa^+ \rangle$ in L. For $\eta < \kappa^+$, let $y_\eta = \mathbb{V}\{x_\xi : \xi < \eta\}$, let $y'_\eta = \mathbb{V}\{x_\xi : \eta \leq \xi < \kappa^+\}$, and let $y = \mathbb{V}\{y_\eta : \eta < \kappa^+\}$. Finally, let B be a base for L with $\omega(L,B) \leq \kappa$.

Suppose that $b \in B$ with $b \leq \mathbb{A}\{y'_\eta : \eta < \kappa^+\} \leq y$, and choose $W \in [L]^{\leq\kappa}$ so that whenever $z \wedge b' = 0$, there is a $w \in W$ such that $w \wedge b = 0$ and $z \leq w$. Since clearly $b \wedge y_\eta = 0$ for all $\eta < \kappa^+$, there is, for each $\eta < \kappa^+$, a $w_\eta \in W$ such that $b \wedge w_\eta = 0$ and $y_\eta \leq w_\eta$. But $|W| \leq \kappa$, so there is a $w \in W$ and an $A \in [\kappa^+]^{\kappa^+}$ such that $w_\eta = w$ for all $\eta \in A$. Then $y = \mathbb{V}\{y_\eta : \eta \in A\} \leq w$, so $b \wedge y = 0$, and it follows that $\mathbb{A}\{y'_\eta : \eta < \kappa^+\} = 0$.

Moreover, for any $\eta < \kappa^+$, $x_\eta \leq y'_\eta$, but $x_\eta \wedge y'_{\eta+1} = 0$, so $y'_{\eta+1} < y'_\eta$, and $\{y'_\eta : \eta < \kappa^+\}$ has no G-subcover *of* L* of power less than κ^+. But $c^*(L) \leq \kappa$, so $\{y'_\eta : \eta < \kappa^+\}$ must not F-cover L*, i.e., there must be a $z < 1$ such that

$z \vee y'_\eta = 1$ for all $\eta < \kappa^+$.

Fix $\eta < \kappa^+$ arbitrarily. Then $y_\eta = y_\eta \wedge (z \vee y'_\eta) = y_\eta \wedge z$, so $y_\eta \leq z$; it follows that $y \leq z$, so that $z \vee y'_\eta = z < 1$ for all $\eta < \kappa^+$, which is the final contradiction.

3.14. *Corollary.* $k(L) \leq \omega(L) \cdot c^*(L)$.

4. More Topological Consequences

4.0. *Proposition. If* $L = \tau X$, $\chi(L) = \omega(L) = 1$.

Proof. Fix $V \in \tau X$. If $W \in \tau X$ is disjoint from V, then $W \subseteq X \backslash \mathrm{cl}\, V \subseteq X \backslash V$.

(More generally, $\chi(L) = \omega(L) = 1$ whenever L has the following property: for each $x \in L$, $x \wedge \bigvee\{y \in L\colon y \wedge x = 0\} = 0$.)

4.1. *Proposition. If* $L = \sigma X$, *then* $\chi(L) = \psi(X)$, *and* $\omega(L) = \chi(X)$.

Proof. In each case consider the base of singletons.

4.2. *Proposition. If* $L = \sigma X$, *then* $k(L) = k(X)$.

Proof. We show that $k(L) \leq k(X)$; the reverse inequality is proved even more easily.

Let $\langle x_\xi\colon \xi < \kappa\rangle$ be a K-sequence in L with co-K-sequence $\langle x'_\xi\colon \xi < \kappa\rangle$. In X, then, $x_\xi \cap x'_\xi = \emptyset$, $x_{\xi+1} \cup x'_\xi = X$, and $x_\xi \subseteq \mathrm{Int}\, x_{\xi+1}$ for each $\xi < \kappa$. Define $V_\xi = X \backslash x_\xi$ (for $\xi < \kappa$); then $V_{\xi+1} \subseteq \mathrm{cl}\, V_{\xi+1} \subseteq x'_\xi \subseteq V_\xi$, so $\langle V_\xi\colon \xi < \kappa\rangle$ is a depth sequence in X.

We now give a number of topological cardinal relationships which follow from the results of Section 3 and the "translation theorems" above and in Section 2. As usual, X is a T_1 space.

4.3. *Proposition.* $o(X) \le min\{2^{w(X)}, 2^{|X|}, w(X)^{h(X)}, |X|^{z(X)}\}$.

(It is interesting that the relationship $o(X) \le |X|^{z(X)}$ is more obvious as a consequence of duality than it is purely topologically.)

4.4. *Theorem.* (Hodel's version of a lemma due to Šapirovskiĭ.) *Let* $\kappa = s(X)$, *and let* $\mathcal{V}$ *be an open cover of* X. *Then there are* $D \in [X]^{\leq\kappa}$ *and* $\mathcal{W} \in [\mathcal{V}]^{\leq\kappa}$ *such that* $\cup\mathcal{W} \cup \mathrm{cl}\ D = X$.

(4.4 follows from 3.3: L, X, and Y are to be replaced by σX, $\{\{x\}: x \in X\}$, and $\{X\backslash V: V \in \mathcal{V}\}$, respectively.)

4.5. *Theorem.* (Essentially due to Šapirovskiĭ.) *Let* $\kappa = s(X)$, *let* $\lambda^{\kappa} = \lambda$, *and suppose that* $\psi(X) \le \lambda$; *then there is an* $A \in [X]^{\leq\lambda}$ *such that* $X = \cup\{\mathrm{cl}\ S: S \in [A]^{\leq\kappa}\}$.

By applying 3.3 to τX we get the rather uninteresting result that if $\kappa = s(X)$ and $\mathcal{V}$ is an open cover of X, there is a $\mathcal{W} \in [\mathcal{V}]^{\leq\kappa}$ such that $\cup\mathcal{W}$ is dense in X. (Fix a base, $\mathcal{B}$, that refines $\mathcal{V}$, and take the collections $\mathcal{B}$ and $\{X\backslash\mathrm{cl}\ B: B \in \mathcal{B}\}$ for the X and Y, respectively, of 3.3.) In fact, the conclusion holds even if just $c(X) \le \kappa$. Analogously, 4.5 has the even worse dual: if $s(X) = \kappa$ and $\mathcal{V}$ is an open cover of X, then there is a $\mathcal{W} \in [\mathcal{V}]^{\leq 2^{\kappa}}$ whose union is dense in X.

4.6. *Theorem.* *(i) A free sequence in* X *can be no longer than* $\chi(X) \cdot L(X)$. *(ii)* $k(X) \le min\{\chi(X) \cdot L(X), c(X)\}$.

The presence of $c(X)$ in 4.6 (ii) may require comment: a free sequence in τX is a family of pairwise disjoint, non-empty, open sets, and the result follows from 3.12.

5. Remarks

The lattice-theoretic approach, and duality, break down badly when we consider cardinal inequalities which hold only in the presence of separation axioms. A good example is the relationship $|X| \leq 2^{h(X)}$ for Hausdorff X; a look at its proof shows that Hausdorffness is used only to guarantee that if $F \in \sigma X$ is neither $\emptyset$ nor an atom, (i.e., a singleton), then $F = F_0 \cup F_1$ for some $F_0, F_1 \in \sigma X$ with $\emptyset \subset F_i \subset F$ for $i < 2$. The lattice τX *always* has this property (for T_1 X), yet the dual inequality, $w(X) \leq 2^{z(X)}$, is false in general for T_1 spaces: consider the co-finite topology on a set of power $(2^{\omega})^+$. (Question: Is $w(X) \leq 2^{z(X)}$ for Hausdorff X?) What makes the proof 'go' is the atomicity of σX; and since an atomic lattice is always isomorphic to the lattice of closed sets of some topology on its atoms, the result seems to be essentially topological.

There does, however, appear to be a sort of duality at work even here. Suppose that ϕ is a cardinal invariant appearing on the right-hand side of Figure 1, and suppose that $\phi(X) \leq$ "thing" for Hausdorff X; it often happens that $\phi^*(X) \leq$ "dual of thing" when X is T_3. For example, $|X| \leq 2^{h(X)}$, $|X| \leq 2^{2^{s(X)}}$, and $d(X) \leq 2^{s(X)}$ for Hausdorff X, whereas $w(X) \leq 2^{z(X)}$, $w(X) \leq 2^{2^{s(X)}}$, and $L(X) \leq 2^{s(X)}$ for T_3 X. (Of course, $w(X) \leq 2^{d(X)}$ if X is T_3, so some of these inequalities can be improved; still, they are at least *true*.) It would be interesting to know what, if anything, underlies this phenomenon.

References

[1] I. Juhász, *Cardinal functions in topology*, Mathematical Centre Tracts 34, Mathematisch Centrum, Amsterdam, 1971.

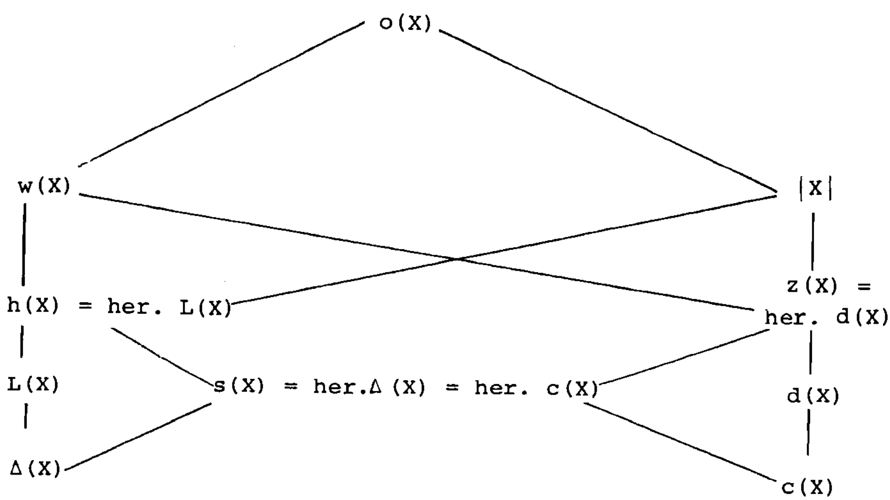

Figure 1.

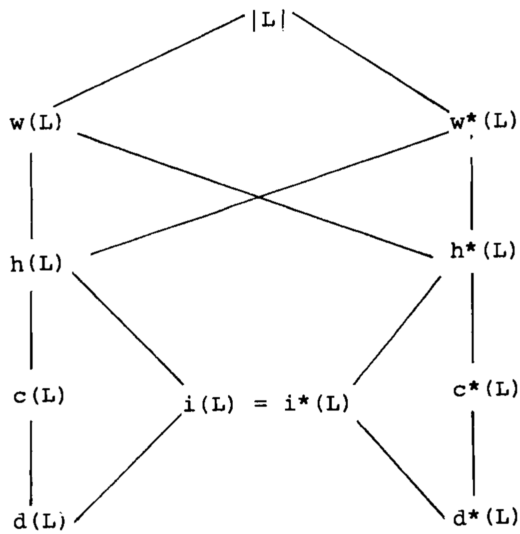

Figure 2.

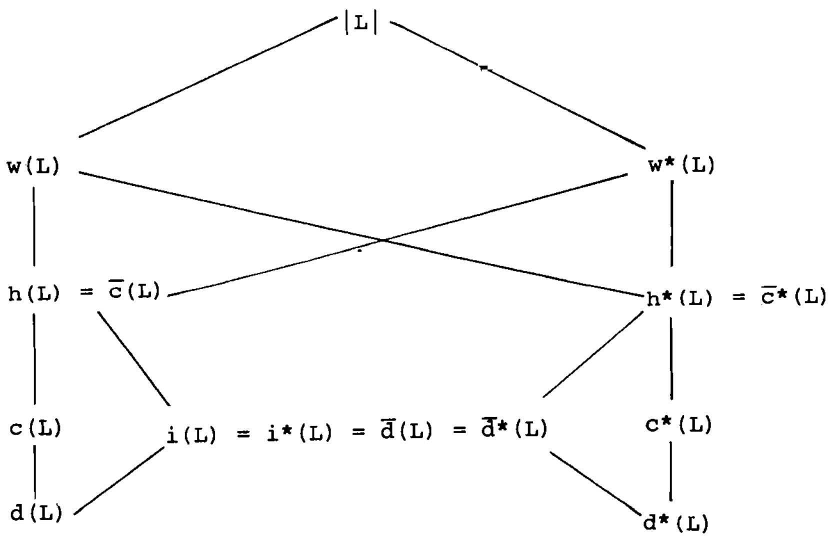

Figure 3.

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