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REMARKS ON λ-COLLECTIONWISE HAUSDORFF SPACES

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The purpose of this note is to answer questions raised by Fleissner in [F]. Explicitly, our results are

Theorem 1. *Let* Σ *be the statement "there is a locally countable, locally compact, normal Moore space which is* $\leq\omega_1$*-collectionwise Hausdorff but not* $\leq\omega_2$*-collectionwise Hausdorff."* Σ *is consistent with* ZFC *(the usual axioms for set theory). Moreover, both* Σ + *not* CH *and* Σ + CH *are consistent with* ZFC.

Theorem 2. *Let* M *be a model of set theory obtained by using Levy forcing to collapse a weakly compact cardinal to* ω_2. *In* M, *let* X *be a locally countable space. Then* X *is* $\leq\omega_2$*-collectionwise Hausdorff if* X *is* $<\omega_2$*-collectionwise Hausdorff.*

There are variations on Theorem 2. We may replace "locally countable" with "first countable and locally of cardinaltiy $\leq\omega_1$." Also, if we collapse a supercompact cardinal (rather than a merely weakly compact cardinal), we may strengthen the conclusion to X is collectionwise Hausdorff.

A subset Y of a topological space X is called closed, discrete if every point of X has a neighborhood containing

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at most one point of Y. A closed discrete set $Y = \{y_i: i \in I\}$ can be screened if there is a family of disjoint open sets $\{U_i: i \in I\}$ such that $U_i \cap Y = \{y_i\}$. A space X is called collectionwise Hausdorff if every closed discrete subset of X can be screened. X is $<\lambda$-collectionwise Hausdorff if every closed discrete subset of cardinality $<\lambda$ can be screened; $\leq\lambda$-collectionwise Hausdorff is defined similarly.

On the situation of Theorem 1 for Σ + GCH, see [S'].

1. Proof of Theorem 1

For concreteness, let us start with a model of V = L. Then by Jensen's work [J], there is a subset E of ω_2 such that

a) $\alpha \in E$ implies cf $\alpha = \omega$

b) E is stationary in ω_2

c) $E \cap \delta$ is not stationary in δ for any $\delta < \omega_2$.

For each $\alpha \in E$, choose $\eta_\alpha: \omega \to \alpha$ to be strictly increasing with range η_α cofinal in α. Set $I = \{\eta_\alpha | m: a \in E\ m \in \omega\}$. The point set of our space is $E \cup I$. (Note that E and I are disjoint). Points of I are isolated; the nth neighborhood of $\alpha \in E$, $B(\alpha,n)$, is $\{\alpha\} \cup \{\eta_\alpha | m: m > n\}$. Where we consider E as a subset of X we will call it Y; it is easy to check that Y is closed, discrete.

From b) and the Pressing Down Lemma it quickly follows that Y can not be screened, and so X is not $\leq\omega_2$-collectionwise Hausdorff. It is straightforward to prove by induction on $\rho < \omega_2$ that $\{\alpha \in Y: \alpha < \rho\}$ can be screened. Thus X is $\leq\omega_1$-collectionwise Hausdorff. Clearly X is locally countable, locally compact Moore space. This space has been described in [F'].

In L X is not normal, so our plan is to extend to a model in which X is normal. Of course, we must check that a), b), and c) are preserved. First all a) and c) assert is that certain sets exist, and so are preserved by any extension. Further we want ω_2 in the ground model to remain ω_2 in the extension so that c) has its intended meaning. This will happen because our two extensions are ccc, and ω_2-cc and ω-Baire, respectively. Finally we note that b) is preserved by an ω_2-cc extension. To see this, note that the α^{th} element of C°, a club set in the extension, is contained in a set in the ground model of cardinality ω_1. Using this fact, we can find point that are limits of elements of C° whatever C° is. Thus we can find for every club set C° in the extension a club set C in the ground model satisfying $C \subset C°$. We conclude that a set stationary in ω_2 in the ground model remains stationary in ω_2 in the extension. Our extensions will preserve a), b), and c), and X will remain $\leq\omega_1$-collectionwise Hausdorff and not $\leq\omega_2$-collectionwise Hausdorff.

The first extension is the Solovay-Tennenbaum extension forcing Martin's Axiom and $c > \omega_2$, a ccc extension [ST]. We aim at showing that X is normal in this model. It is sufficient to show that disjoint subsets H and K of Y can be separated by disjoint open sets. Define P to be the set of pairs (u,v) satisfying

d) $u \cap v = \emptyset$

e) u (respectively, v) is the union of finitely many basic open sets $B(\alpha,n)$ with $\alpha \in H$ (respectively, $\alpha \in K$).

Define $(u,v) \leq (u',v')$ if $u \supseteq u'$ and $v \supseteq v'$. That this partial order has ccc follows quickly from the Delta System Lemma. Since $c > \omega_2 = \text{card } H \cup K$, by Martin's Axiom we can define disjoint open sets U and V separating H and K.

We have shown that Σ + not CH is consistent. To show that Σ + CH is consistent, we need a Martin's Axiom-like extension which adds no subsets of ω. Analogues of Martin's Axiom have been shown consistent and investigated [T], [S], but they are not applicable in this situation because we need a notion of forcing which is not countably closed. Our plan is to make X normal by an extension which is not countably closed, but is "sufficiently" countably closed.

Let H and K be disjoint subsets of Y. We will define a partial order P(H,K) of pairs (u,v) parallel to the order P used above with Martin's Axiom. Requirement d) remains the same, but in order to make P(H,K) sufficiently countably closed, we change e) to

e') u (respectively, v) is the union of countably many basic open sets $B(\alpha,n)$ where $\alpha \in H$ (respectively, $\alpha \in K$).

Now a new problem arises. It can happen that there is a $y \in \text{closure } u \cap K$. If this occurs, (u,v) cannot be extended to (u',v') with $y \in v$, and so the generic filter need not define a separation of H and K. To prevent this we add, defining $s((u,v))$ to be $[\text{closure } (u \cup v)] \cap Y$,

e") $s((u,v)) \subseteq u \cup v$.

We now define P(H,K) to be the set of pairs (u,v) satisfying d), e') and e"). P(H,K) is not countably closed, for the "union" of a countable sequence of elements of P(H,K) will satisfy d) and e'), but might not satisfy e"). A sufficient condition for (u,v) to satisfy e") is that $s((u,v))$ be closed in E (with the topology inherited from ω_2 with the order topology). Lemma 1, below will give us a way to insure that certain countable sequences of elements of $P(H,K)$ will have an infimum. (In our application, the A_α's will be $s((u,v))$'s).

Call a well ordered sequence of sets, $\{A_\alpha: \alpha < \rho\}$, continuous and increasing if

f) $\alpha < \beta$ implies $A_\alpha \subseteq A_\beta$

g) δ a limit ordinal implies $A_\delta = \cup\{A_\alpha: \alpha < \delta\}$

Lemma 1. *Suppose that* E *satisfies* a), b), c); ν *is an ordinal less than* ω_2; *and* $\{A_\alpha: \alpha < \omega_1\}$ *is a continuous increasing sequence of countable sets with* $\cup\{A_\alpha: \alpha < \omega_1\} = E \cap \nu$. *Give* $E \cap \nu$ *the topology inherited from* ω_2 *with the order topology. Then*

$\{\alpha: A_\alpha$ *is closed in* $E \cap \nu\}$ *contains a club set.*

Proof. We prove the lemma by induction on ν. For $\nu < \omega_1$ or ν a successor ordinal, the induction step is trivial.

Case 1: ν is a limit ordinal of cofinality ω. Let ν_n, $n < \omega$, be increasing and cofinal in ν. By induction hypothesis, $\{\alpha: A_\alpha \cap \nu_n$ is closed in $E \cap \nu_n\}$ contains a club set. Then $\{\alpha: A_\alpha$ is closed in $E \cap \nu\} = \cap\{\alpha: A_\alpha \cap \nu_n$ is closed in $E \cap \nu_n\}$ contains a club set.

Case 2: ν is a limit ordinal of cofinality ω_1. By c) we can find $\{\nu_\alpha: \alpha < \omega_1\}$ continuous, increasing, cofinal in ν, and disjoint from E. If $A_\alpha \cap \nu$ is not closed in $E \cap \nu$, define $h(\alpha)$ to be the least ordinal such that $A_\alpha \cap \nu_{h(\alpha)}$ is not closed in $E \cap \nu$. Using the regularity of ω_1, the hypothesis that each A_α is countable, and f), we can find a club set C of limit ordinals such that if $\gamma \in C$ and $\alpha < \gamma$, then $A_\alpha \subseteq \nu_\gamma$ and $h(\alpha)$ is either undefined or less than γ. Using g), for $\gamma \in C$, $A_\gamma \subseteq \nu_\gamma$, hence any limit point of A_γ in E is less than ν_γ (not equal to $\nu_\gamma \notin E$). Hence $h(\gamma)$ is either undefined or $h(\gamma) < \gamma$.

If h presses down on a stationary set, then by the Pressing Down Lemma $h(\alpha) = \beta$ for some β and stationarily many α's. Then the lemma fails for ν_β and $\{A_\alpha \cap \nu_\beta : \alpha < \omega_1\}$, contradicting the inductive hypothesis.

We now define our desired forcing, P_{ω_3}, by inductively defining notions of forcing P_β, $\beta \leq \omega_3$. Simultaneously, we will show that P_β is ω_2-cc and ω-Baire (i.e. adds no ω-sequences of ordinals), so that we may require j) and k) below. Explicitly, by induction on $\beta \leq \omega_3$, we define P_β to be the set of p satisfying

h) p is a function with domain β

i) $p(\alpha) \in P(H_\alpha, K_\alpha)$ where H_α, K_α are terms for disjoint subsets of Y in the forcing language for P_α

j) $\{(H_\alpha, K_\alpha) : \alpha < \omega_3\}$ enumerates all terms for disjoint subsets of Y in the language for P_{ω_3}

k) $p(\alpha) \in L$ (the ground model) (i.e. it is not a term for an element of $P(H_\alpha, K_\alpha)$, it is an element of $P(H_\alpha, K_\alpha)$).

ℓ) $p(\alpha) = (\emptyset, \emptyset)$ for all but countably many α's

m) $p \leq q$ if $p(\alpha) \supseteq q(\alpha)$ for all $\alpha < \beta$.

That P has ω_2-cc follows from the continuum hypothesis, ℓ), and the Delta System Lemma. So j) is possible.

Aiming towards showing that P_β is ω-Baire, let $\{D_n: n \in \omega\}$ be a countable set of dense open subsets of P_β, and p an arbitrary element of P_β. Let $\underline{N}$ be a structure containing everything relevant, e.g. $\underline{N} = \langle V_{\omega_4}, \in P_\beta, \Vdash_{P_\beta}, E, \beta, \{D_n: n \in \omega\}\rangle$. Let $\underline{N}_\rho$, $\rho < \omega_1$, be a continuous increasing sequence of countable elementary submodels of $\underline{N}$ satisfying $\underline{N}_\rho \in N_{\rho+1}$. Set $\omega_2 \cap \bigcup\{N_\rho: \rho < \omega_1\} = \nu$, an ordinal less than ω_2. Applying Lemma 1 to E, ν, $\{E \cap N_\rho: \rho < \omega_1\}$, we can find ρ_n, $n \in \omega$, $\sup\{\rho_n: n \in \omega\} = \rho$, such that

n) $E \cap N_\rho$ is closed in $E \cap \nu$.

We define a sequence $\{p_n: n \in \omega\}$ of forcing conditions satisfying

o) $p_0 = p$, $p_{n+1} < p_n$

p) $p_{n+1} \in D_n \cap N_{\rho_n}$

q) $s(p_{n+1}(\alpha)) \supseteq N_{\rho_n} \cap E$, when $p_{n+1}(\alpha) \neq (\phi,\phi)$

Define q with domain β by $q(\alpha) = \bigcup\{p_n(\alpha); n \in \omega\}$. Clearly q satisfies h), k), and ℓ), and q satisfies i) by n) and q), so $q \in P_\beta$. We have found q, $q < p$ and $q \in \bigcap\{D_n: n \in \omega\}$ and may conclude that P_β, $\beta \leq \omega_3$, is ω-Baire. This completes the simultaneous definition of P_β and verification of ω_2-cc and ω-Baire.

In the extension by P_{ω_3}, X is normal. For it is sufficient to consider disjoint H and K subsets of Y, and by j) there is a generic pair of open sets separating them. The Continuum Hypothesis is preserved by the extension because it is ω-Baire.

2. Proof of Theorem 2

We imitate Baumgartner [B]. Let κ be weakly compact in

M, the ground model, and let $P(\kappa,\omega_2)$ be the Levy forcing collapsing κ to ω_2. Let X° be the name of a locally countable, $\le\omega_1$-collectionwise Hausdorff space with $\{y_\alpha: \alpha < \kappa\}$ a closed discrete subset of X° that can not be screened. We may assume that $X^\circ \subset V_\kappa$, by Π^1_1 indescribability, there is a $\lambda < \kappa$ with the same properties. Explicitly, $X^\circ \cap V_\lambda$ is the name in the language for $P(\lambda,\omega_2)$ of a locally countable, $\le\omega_1$-collectionwise Hausdorff space with $\{y_\alpha: \alpha < \lambda\}$ a closed discrete subset of $X^\circ \cap V_\lambda$ that can not be screened.

Let G be an M-generic ultrafilter on $P(\lambda,\omega_2)$. We will work in $M^1 = M[G]$, where $\omega_2 = \lambda$, X is $\le\omega_1$-collectionwise Hausdorff, and $Y = \{y_\alpha: \alpha < \lambda\}$ witnesses that X is not $\le\omega_2$-collectionwise Hausdorff. For each $\alpha < \kappa$ we choose a countable neighborhood B_α of y_α, fixed throughout this section. Set $W_\beta = \bigcup\{B_\alpha: \alpha < \beta\}$.

Lemma 2. *There are* S, h *such that*

1) S *is a stationary subset of* ω_2

2) $\delta \in S$ *implies that* $\operatorname{cf}\delta = \omega$

3) $h: S \to \omega_2$, $h(\delta) \ge \delta$

4) $y_{h(\delta)} \in$ *closure* W_δ.

Proof. It suffices to find a set S satisfying 1), 2) and

5) for $\delta \in S$, *closure* $W_\delta \cap \{y_\alpha: \delta \le \alpha < \omega_2\} \ne \emptyset$.

Aiming for a contradiction, we assume that there is no such set S. Specifically, we assume that there is a club set C such that for $\delta \in C_0 = \{\delta \in C: \operatorname{cf}\delta = \omega\}$, closure $W_\delta \cap \{y_\alpha: \delta \le \alpha < \omega_2\} = \emptyset$. Let C' be the set of limit points of C_0; C' is a club set. We claim

6) for $\delta \in C'$, *closure* $W_\delta \cap \{y_\alpha: \delta \le \alpha < \omega_2\} = \emptyset$.

There are two cases. First, if $\delta \in C'$, cf $\delta = \omega$, 6) holds because $\delta \in C_0$. Second, if $\delta \in C'$, cf $\delta > \omega$ we show 6) using the fact that X is locally countable. If there were $y \in$ closure $W_\delta \cap \{y_\alpha: \delta \leq \alpha < \omega_2\}$, then $y \in$ closure W_γ for cofinally many γ in δ, in particular for some $\gamma \in C_0$, contradiction.

Let $\langle \gamma(\nu): \nu < \omega_2 \rangle$ be the natural, monotone increasing enumeration of C'. Define $U_\nu = W_{\gamma(\nu+1)}$-closure $W_{\gamma(\nu)}$; set $\mathcal{U} = \{U_\nu: \nu < \omega_2\}$. By definition, $\mathcal{U}$ is a disjoint family of open sets, each containing at most ω_1 points of Y. By 6) U covers Y. Using that X is $\leq\omega_1$-collectionwise Hausdorff, we can improve U to screen Y. This contradiction establishes Lemma 2.

Note that $P(\kappa,\omega_2) = P(\lambda,\omega_2) \oplus P'$, where P' is countably closed. Our goal is to show that P' does not add a screening of Y. Since in the extension $Y = \{y_\alpha: \alpha < \lambda\}$ has cardinality ω_1, we will have shown that X^0 is not $<\omega_2$-collectionwise Hausdorff, a contradiction. Towards this goal, suppose that $p \in P'$ forces that $\{V_\alpha: \alpha < \lambda\}$ screens Y.

Working in M^1, let $\underline{N} = \langle V_{\kappa+\omega}, P', \Vdash_{P'}, p, \{y_\alpha: \alpha < \kappa\}, X°, \{B_\alpha: \alpha < \kappa\}\rangle$. Define a continuous increasing sequence $\underline{N}_\rho$, $\rho < \omega_2$, of elementary submodels of $\underline{N}$ satisfying $\omega_1 \subset N_0$, card $N_\rho = \omega_1$, $W_\rho \subset N_\rho$. Set $\delta_\rho = N_\rho \cap \lambda$. Then $\{\delta \in \lambda: \delta = \delta_\delta\}$ is a club set in λ, so there is such a δ in S. Let $B_{h(\delta)} \cap N_\delta = z_n: n \in \omega\}$.

We define a sequence p_n, $n \in \omega$, of forcing conditions as follows. Set $p_0 = p$; let $p_{n+1} \in N_\delta$ decide z_n--either $z_n \notin \cup\{V_\alpha: \alpha < \lambda\}$ or $z_n \in V_\alpha$ for some specific α. The point is that this specific α must be in N_δ, and thus can not be $h(\delta)$. Set $q = \cup\{p_n: n \in \omega\}$; q might not be in N_δ, but q is in P'. Let $q' \supseteq q$ choose $V_{h(\delta)}$. Because P' is countably closed, $V_{h(\delta)} \cap B_{h(\delta)} \in M'$. By our choice of p_n's $V_{h(\delta)} \cap B_{h(\delta)} \cap N_\delta = \emptyset$. As $W_\delta \subset N_\delta$, $V_{h(\delta)} \cap B_{h(\delta)}$ is an open neighborhood of $y_{h(\delta)}$ demonstrating that $y_{h(\delta)} \notin$ closure W_δ. We chose $\delta \in S$, so this contradicts 4). This contradiction completes the proof of Theorem 2.

The proofs of the variants of Theorem 2 are parallel and so omitted.

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