TOPOLOGY PROCEEDINGS
Volume 2, 1977
Pages 593–619
http://topology.auburn.edu/tp/

FLEXIBLE REGULAR NEIGHBORHOODS FOR COMPLEXES IN E^3

by

Michael Starbird

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

FLEXIBLE REGULAR NEIGHBORHOODS FOR COMPLEXES IN E^3

Michael Starbird[1]

1. Introduction

Let (C,T) be a finite complex with triangulation T linearly embedded in E^3. In this paper we produce for each complex C with no local cut points a regular neighborhood N of C with triangulation T_N so that (C,T) is a subcomplex of (N,T_N) and which is so flexible that any linear isotopy of (C,T) starting at the identity can be extended to (N,T_N). That is, for any linear isotopy f_t: (C,T) $\to E^3(t \in [0,1])$ such that f_o = id, there is a linear isotopy F_t: $(N,T_N) \to E^3(t \in [0,1])$ such that F_o = id and for each t in [0,1], $F_t|C = f_t$ (Theorem 3.1). (See definition of linear isotopy below.)

The main tool used in the proof of Theorem 3.1 is Theorem 2.2 which concerns spherically linear maps of triangulated disks into the 2-sphere. (See definition of spherically linear in Section 2.) This theorem about disks mapped into S^2 is similar to the super triangulation theorem about disks mapped into E^2 which appears in [1, Theorem 3.4]. Theorem 2.2 may have some interest in its own right; however, an affirmative answer to Question 2.1 would make it obsolete.

Definitions. Let C be a complex with triangulation T. A *linear embedding* of C (or (C,T)) into E^n is an embedding

[1]This work was supported by NSF Grant MCS 76-07242.

that is linear on each simplex of T. A *linear isotopy* h_t: $(C,T) \to E^n (t \in [0,1])$ is a continuous family of linear embeddings. A *simple push* is a linear isotopy which is the identity outside of the open star of one vertex. A *push* is a linear isotopy obtained by performing a finite sequence of simple pushes one after the preceding.

2. Linear Isotopies on 2-Spheres

In this section the notion of a spherically linear isotopy is defined and a theorem is proved about maps of triangulated disks into S^2 which is similar to the super triangulation theorem given in [1, Theorem 3.4]. First we give the definition of a super triangulation of a disk found in [1].

Definition. A triangulation T of a disk P is *super* if and only if it has the following three properties.

1. Every linear embedding of Bd P in E^2 can be extended to a linear embedding of (P,T).
2. If f and g are two linear embeddings of (P,T) which agree on Bd P, then there is a linear isotopy h_t: $(P,T) \to E^2 (t \in [0,1])$ such that $h_0 = f$, $h_1 = g$, and for all $t \in [0,1]$, $h_t|\text{Bd } P = f|\text{Bd } P = g|\text{Bd } P$.
3. If h_0 and h_1 are two linear embeddings of (P,T) into E^2 and f_t is a linear isotopy of Bd P into E^2 from $h_0|\text{Bd } P$ to $h_1|\text{Bd } P$, then f_t can be extended to a linear isotopy of P from h_0 to h_1.

It may be noted that Properties 1 and 2 imply Property 3.

The following theorem was proved in [1, Theorem 4.2] and will be used in this section.

Theorem 2.1. *Every triangulation* T *of a disk* P *has a super subdivision which does not subdivide the boundary.*

Definitions. In dealing with a complex C linearly embedded in E^3, it will be convenient to consider the intersection of C with a small round 2-sphere S centered at a vertex v of C. This intersection, $C \cap S$, is a natural embedding of Lk(v) into S. An embedding of a complex into S that can be so obtained is called a *spherically linear embedding.* A continuous family of spherically linear embeddings is called a *spherically linear isotopy.*

Let π be the homeomorphism which takes the open lower hemisphere of a unit 2-sphere centered at (0,0,1) onto the plane (x,y,0) defined by taking each point p of the hemisphere to the point (x,y,0) which lies on the line determined by (0,0,1) and p. Note that π provides a one-to-one correspondence between segments of great circles in the hemisphere and straight line intervals in the plane. The symbol π will denote any such projection map from a hemisphere of a round 2-sphere onto a plane.

The purpose of this section is to prove the following theorem.

Theorem 2.2. *Let* n *be an integer greater than* 2 *and* S^2 *be a round* 2*-sphere. Then there is a triangulation* T_n *of a disk* P *so that* Bd P *has* n 1*-simplexes and for each spherically linear isotopy* h_t: Bd P $\to S^2$ (t $\in$ [0,1]) *and component* D *of*

$S^2-h_o(\text{Bd } P)$ *there is a spherically linear isotopy* $H_t\colon (P,T_n) \to S^2 (t \in [0,1])$ *so that* $H_o(P) = \bar{D}$ *and* H_t *extends* h_t.

Furthermore, if $h_o = h_1$, H_t *can be chosen so that* $H_o = H_1$.

Before beginning the proof we should notice the similarities and differences between the triangulation T_n in this theorem and a super triangulation. The triangulation T_n has the analog in S^2 of Property 1 of a super triangulation but only weakened versions of Properties 2 and 3.

Proof of Theorem 2.2. We obtain the triangulation T_n by first subdividing P into small subdisks by a 1-complex Γ and then giving each subdisk a super triangulation. These subdisks will be so numerous that, given any spherically linear isotopy $h_t (t \in [0,1])$ of Bd P, it will be possible to extend $h_t (t \in [0,1])$ to Γ in such a way that each subdisk into which Γ divides P will lie entirely in an open hemisphere of S^2 at all times. Then using the projection maps π, we will be able to take advantage of the super triangulation of each subdisk to complete the extension.

The proof of Theorem 2.2 is divided into two sections. In the first section the triangulation T_n is constructed. In the second section several lemmas are proved which show that T_n has the desired properties.

The triangulation T_n. (See Figure 2.1.) The triangulation T_n contains n concentric annuli which make up one large annulus A which has Bd P as one of its boundary components.

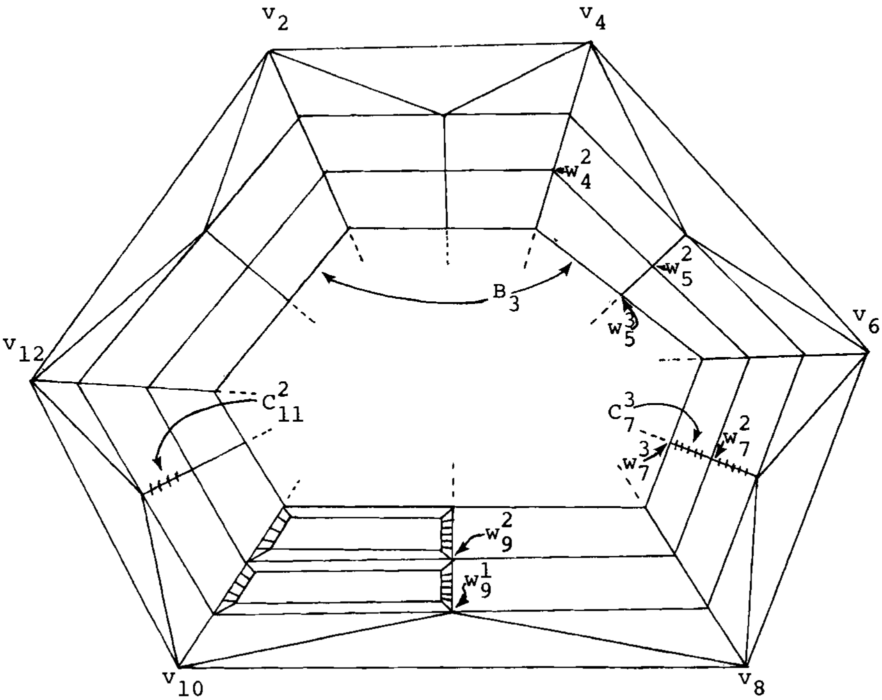


Figure 2.1

Each of these n annuli, except the outermost one, will be divided into 2n subdisks, each of which will be given a super triangulation. The triangulation T_n is completed by giving $Cl(P-A)$ a super triangulation.

We begin by describing the first annulus A_1 which has Bd P as one boundary component. The remaining n-1 annuli will have triangulations identical to each other but different from the triangulation of A_1.

Let $\{v_{2i}\}_{i=1}^{n}$ be the vertices of Bd P in order. The interior boundary component, B_1, of A_1 has 2n vertices $\{w_i^1\}_{i=1}^{2n}$. The 1-simplexes in the triangulation of A_1 are those in Bd P, those on B_1, and all 1-simplexes of the form

$v_{2k}w_j^1$ where $j = 2k$, $j = 2k-1$, or $j = 2k+1$ (counting mod $2n$).

Next we insert additional concentric annuli $A_2, A_3, \cdots, A_n$ with B_{i-1} and B_i the two boundary components of A_i. Each B_i has $2n$ 1-simplexes with vertices $\{w_j^i\}_{j=1}^{2n}$. The triangulations of the A_i's $(i = 2,\cdots,n)$ are identical.

For each i in $\{2,3,\cdots,n\}$, A_i is triangulated as follows. For each j in $\{1,2,\cdots,2n\}$, there is a broken arc C_j^i in the 1-skeleton of the triangulation of A_i so that the endpoints of C_j^i are w_j^{i-1} and w_j^i. For $j \neq k$, $C_j^i \cap C_k^i = \emptyset$. Each C_j^i has $2n$ 1-simplexes. The C_j^i's therefore divide A_i into $2n$ subdisks D_j^i $(j = 1,2,\cdots,2n)$, where D_j^i is bounded by $C_j^i \cup w_j^{i-1}w_{j+1}^{i-1} \cup C_{j+1}^i \cup w_j^i w_{j+1}^i$. For each j in $\{1,2,\cdots,2n\}$, D_j^i is triangulated by first putting a collar on Bd D_j^i and giving the rest of D_j^i a super triangulation. The collar is triangulated as follows. Let $\{s_i\}_{i=1}^m$ and $\{t_i\}_{i=1}^m$ be the boundary vertices of the two components of the collar. Then $s_i t_i$ is a 1-simplex in the triangulation and each quadrilateral into which the collar is now divided is triangulated by coning from an interior point.

There is one further refinement of T_n near each C_j^i. For C_j^i, let I be the arc consisting of the $2n-2$ interior 1-simplexes of C_j^i and let I^+ and I^- be the corresponding arcs on the boundaries of the collar around C_j^i. Let X be the disk whose boundary is $I^+ \cup I \cup$ (the two 1-simplexes which join the corresponding ends of I and I^+). We use Theorem 2.1 to give X a super triangulation which refines the triangulation which X has from the collar and does not subdivide Bd X. For each C_j^i and on each side make this modification. At times in the proof we will need to think of the collar around C_j^i as having its collar triangulation and at others we will need the super triangulation of this part of the collars.

Finally the triangulation T_n is completed by giving $\mathrm{Cl}(P-\cup_{i=1}^{n} A_i)$ a super triangulation similar to those of each subdisk above, that is, it begins with a triangulated collar and then is filled in with a super triangulation obtained from Theorem 2.1.

The proof that T_n *satisfies the conclusions of Theorem* 2.2. Our strategy is first to describe a method of extending a certain kind of spherically linear embedding g of Bd P to the B's and some C's of T_n so that each subdisk into which these B's and C's divide P can then be mapped entirely into the northern or southern hemisphere. This extension of g to these B's and C's, found in Lemma 2.3, will be continuously canonical provided that no vertex of Bd P is mapped to the equator. Thus Lemma 2.3 could be used to finish the proof of Theorem 2.2 in the very restricted case where for no $t \in [0,1]$ does h_t map a vertex of Bd P to the equator. Lemma 2.5 is used to show how the extension of h_t can be accomplished when h_t pushes a vertex across the equator. The idea, then, of the proof is that an essentially canonical extension of h_t is available for most values of t; but at those infrequent moments when h_t pushes a vertex across the equator, we need to bridge the gap between the canonical extension associated with having that vertex on one side of the equator and the canonical extension associated with its being on the other side.

Lemma 2.3. *Let* g: Bd P $\to S^2$ *be a spherically linear* *embedding of* (Bd P, T_n|Bd P) *into* S^2 *so that no vertex of* Bd P *is mapped into* E, *the equator, but* $g(\text{Bd P}) \cap E \neq \emptyset$. *Let* D *be a component of* $S^2 - g(\text{Bd P})$. *Then there is a procedure for choosing an extension* G *of* g *so that*

(1) G *is a spherically linear embedding of* (P, T_n) *into* $\overline{D}$ *extending* g,

(2) if G_o *and* G_1 *are two extensions of* g *chosen by the procedure, then there is a spherically linear isotopy* $G_t\colon (P, T_n) \to S^2 (t \in [0,1])$ *from* G_o *to* G_1 *where for each* $t \in [0,1]$, G_t *is an extension of* g *chosen by the procedure, and*

(3) if $g_t\colon \text{Bd P} \to S^2 (t \in [0,1])$ *is a spherically linear isotopy where for each* t *in* [0,1], g_t *satisfies the hypotheses of this lemma, then there is a spherically linear isotopy* $G_t\colon P \to S^2 (t \in [0,1])$ *so that for each* t *in* [0,1], $G_t|\text{Bd P} = g_t$ *and* G_t *is an extension of* g_t *obtained from the procedure.*

Proof. Let g be given as above. For each simplex σ in Bd P let $\varepsilon_1(\sigma) = d(g(\sigma), \cup\{g(\tau) | \tau$ is a simplex of Bd P not contained in $\text{St}(\sigma)\})$. Let $\varepsilon_1 = \frac{1}{4}\min\{\varepsilon_1(\sigma) | \sigma$ is a simplex of Bd P$\}$. Let $\varepsilon_2 = \frac{1}{4}\min\{d(g(v), E) | v$ is a vertex of Bd P$\}$. For each simplex σ of Bd P let $\varepsilon_3(\sigma) = \max\{\delta |$ the δ-neighborhood of $g(\sigma)$ lies in a hemisphere of $S^2\}$. Let $\varepsilon_3 = 1/3\min\{\varepsilon_3(\sigma) | \sigma$ is a simplex of Bd P$\}$. Let $\varepsilon_4 = \frac{1}{4}\min\{d(p,q) | p \in g(\text{BD P}) \cap E$, $q \in g(\text{Bd P}) \cap E$, and $p \neq q\}$.

Finally let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$. The reader is advised to think of ε as a small number which varies continuously with the embedding g satisfying the hypothesis of the lemma.

Figure 2.2 shows the results of Steps 1, 2, and 3 which follow.

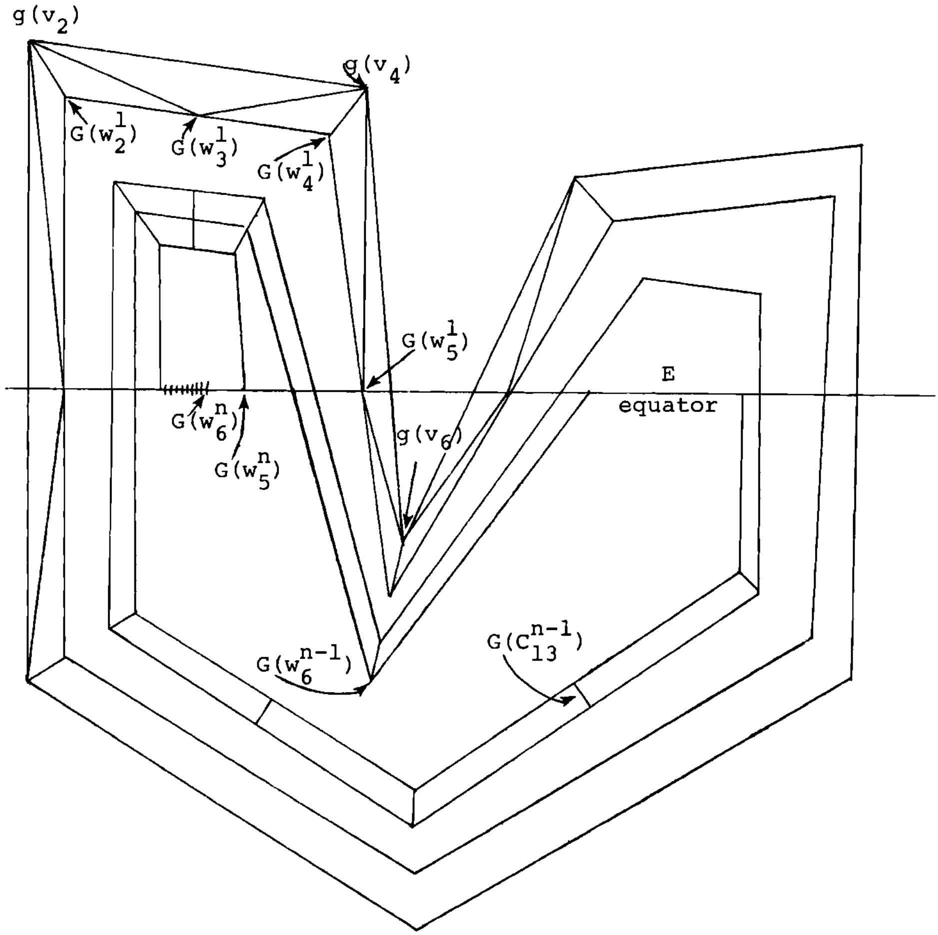


Figure 2.2

Step 1. *Canonically extending* g *to* A_1. For each vertex of B_1 with an even subscript w^1_{2k}, $G(w^1_{2k})$ lies at a distance $\varepsilon/2n$ from $g(v_{2k})$ in D and on the bisector of the angle formed by $g(v_{2(k-1)}v_{2k})$ and $g(v_{2k}v_{2(k+1)})$. Let K be the closure of a component of $D \cap E$. The endpoints x and y of K lie on the interiors of images of 1-simplexes of Bd P say $g(v_{2i}v_{2(i+1)})$

and $g(v_{2j}v_{2(j+1)})$ respectively. Then $G(w^1_{2i+1})$ will lie on K distance $\varepsilon/2n$ from x while $G(w^1_{2j+1})$ will lie on K distance $\varepsilon/2n$ from y.

For any remaining vertex w^1_{2i+1} of B_1, let $G(w^1_{2i+1})$ be the midpoint of the shorter great circle segment between $G(w^1_{2i})$ and $G(w^1_{2(i+1)})$.

This map G on the vertices of B_1 can be extended to a spherically linear embedding of B_1 and then to a spherically linear embedding of A_1.

Step 2. Defining G *canonically on the remaining* B_i*'s.* For each arc component K of $\overline{D} \cap E$, let m(K) be the number of times the equator E must be crossed in going from $g(v_2)$ to K in $\overline{D}$. That is, m(K) is the smallest integer for which there is an arc C in $\overline{D}$ from $g(v_2)$ to K so that Int C $\cap$ E contains m(K) points.

Let K be an arc component of $\overline{D} \cap E$ containing $G(w^1_{2i+1})$ and $G(w^1_{2j+1})$. Let F be the arc on $B_{n-m(K)}$ between $w^{n-m(K)}_{2i+1}$ and $w^{n-m(K)}_{2j+1}$ which does not contain $w^{n-m(K)}_2$. Then G(F) will lie on K with $G(w^{n-m(K)}_{2i+1})$ distance $\varepsilon/2n(n-m(K))$ from $G(w^1_{2i+1})$ and $G(w^{n-m(K)}_{2j+1})$ distance $\varepsilon/2n(n-m(K))$ from $G(w^1_{2j+1})$ with the vertices of F evenly spaced between. The vertices $\{w^k_{2i+1} | 1 < k < n-m(K)\}$ are mapped into K with $G(w^k_{2i+1})$ distance $\varepsilon/2n(k)$ from $G(w^1_{2i+1})$. The map G is defined similarly on the vertices $\{w^k_{2j+1} | 1 < k < n-m(K)\}$. The preceding process is carried on for each arc component K of $\overline{D} \cap E$.

For each unassigned vertex with an even subscript w^k_{2r}, $G(w^k_{2r})$ will be in $D-G(A_1)$ distance $\varepsilon/2n(k)$ from $G(w^1_{2r})$ and on the great circle segment determined by $g(v_{2r})$ and $G(w^1_{2r})$.

Note that if $j < k$, $G(w_{2r}^j)$ is nearer $G(w_{2r}^1)$ than $G(w_{2r}^k)$ is.

Finally, for each unassigned vertex w_{2r+1}^k, $G(w_{2r+1}^k)$ will be the midpoint of the shorter great circle segment determined by $G(w_{2r}^k)$ and $G(w_{2(r+1)}^k)$.

This definition of G on the vertices of the B_i's allows us to extend G to a spherically linear embedding of the B_i's.

Note that the annulus on S^2 bounded by $G(B_k)$ and $G(B_{k+1})$ has thin parts which may wander between the northern and southern hemispheres of S^2; however, the fat parts lie entirely in one of the hemispheres.

Step 3. *Defining* G *canonically on some* C_j^i*'s.* Although G has not been defined on the annuli $A_i (i = 2,3,\cdots,n)$, we know that $G(A_i)$ will be the annulus between $G(B_{i-1})$ and $G(B_i)$. Therefore, we denote the annulus between $G(B_{i-1})$ and $G(B_i)$ by $G(A_i)$ even though G has not yet been defined on Int A_i.

Recall that C_j^i is an arc in A_i between w_j^{i-1} and w_j^i $(i = 2,3,\cdots,n)$ which has 2n 1-simplexes. If $d(G(w_j^i), G(w_j^{i-1})) \leq \varepsilon/n$, define $G(C_j^i)$ to be the short great circle segment between $G(w_j^i)$ and $G(w_j^{i-1})$ with the vertices evenly spaced. Note that $G(C_j^i)$ is in $G(A_i)$.

Step 4. *Defining* G *canonically on the collars of the* B*'s and* C*'s on which* G *is now defined.* (See Figure 2.3.) The collars are mapped in the neatest possible way. Let $\delta = \min\{d(G(v),G(w)) \mid v \text{ and } w \text{ are vertices of } T_n \text{ on which } G \text{ has been defined}\}$. For each vertex w_j^i, (except $i = 1$) there are two, three, or four 1-simplexes of T_n on which G has been defined which contain w_j^i as a vertex. In each case bisect each angle formed by the images of those simplexes and locate

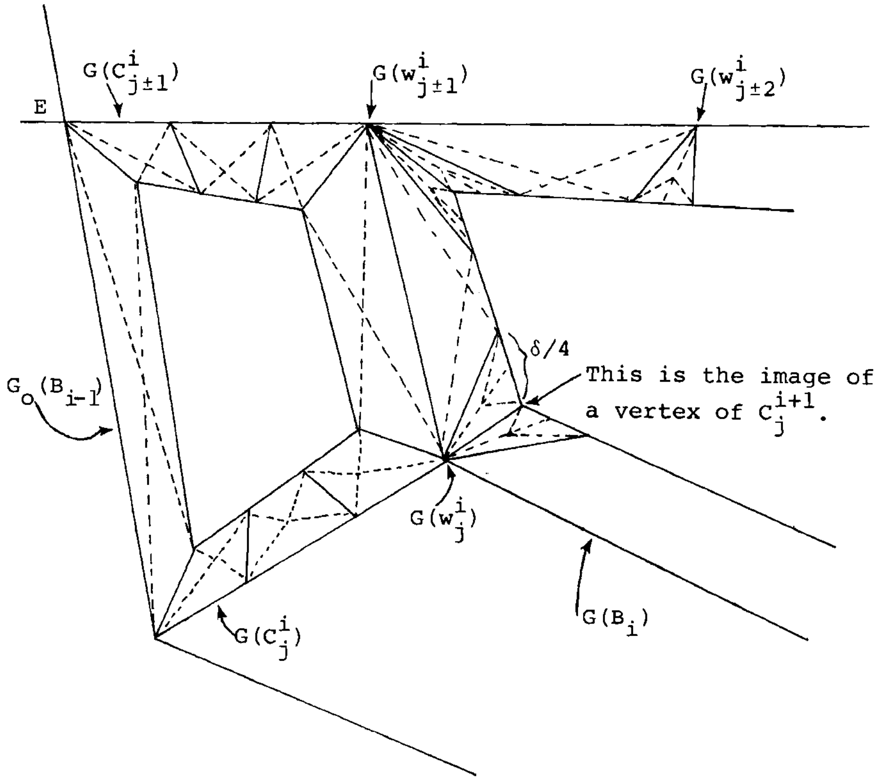


Figure 2.3

points on those bisectors which are distance $\delta/4$ from $G(w^i_j)$. By joining appropriate pairs of these points, the boundaries of the collars are determined. The boundaries of the collars are triangulated as indicated in Figure 2.3. To complete the definition of G on the collars, we need to locate the interior vertex of each of the quadrilaterals into which each collar is divided. There are two cases. If both diagonals can be embedded in a quadrilateral, locate the interior vertex at the point of intersection of the diagonals. If it is not true that both diagonals can be embedded, as is the case around vertex $G(w^i_j)$ in Figure 2.3, locate the interior vertex at the midpoint of the diagonal which can be embedded.

Now G has been defined on a neighborhood of each B_i and C_j^i on which G is defined.

Step 5. *Defining* G *on the rest of* P. This step is the first which is not continuously canonical. Notice that G has now been defined on all of P except some disjoint subdisks. Recall that D is a component of S^2-g(Bd P). Note that if F is a component of D minus the image under G of the part of P on which G is now defined, then $\overline{F}$ is a disk lying entirely in the open northern hemisphere of S^2 or entirely in the open southern hemisphere of S^2. Each such disk $\overline{F}$ corresponds to a disk $\overline{F}^{-1}$ in P, namely the one bounded by $G^{-1}(\text{Bd }\overline{F})$. If $\overline{F}^{-1}$ has a super triangulation, use the map π to extend G to all of $\overline{F}^{-1}$.

Now let $\overline{F}$ be a subdisk of D whose interior is still not in the image of G at this point. Then $\overline{F}$ lies in an annulus $G(A_i)$ and is bounded by the outside boundary of the collars on $G(C_j^i)$ and $G(C_k^i)$ for some j and k together with the outside boundaries of the collars on an arc of $G(B_{i-1})$ and an arc on $G(B_i)$.

Let $\overline{F}^{-1}$ be the subdisk of A_i which needs to be mapped to $\overline{F}$. Note that for each m with $j < m < k$, C_m^i is a spanning arc of $\overline{F}^{-1}$. Note that $\pi(\overline{F})$ has fewer than $2n + 3$ straight sides, although some are subdivided. If the triangulation of $\overline{F}^{-1}$ were super, we would be done. Instead it is necessary first to map in the remaining C_j^i's which span $\overline{F}^{-1}$. For this purpose we use the following sublemma which we state with notation suggestive of its use.

Sublemma 2.4. *Let* $\overline{F}^{-1}$ *be a triangulated disk. Let* $\{C_m\}_{m=1}^{s}$ *be a set of disjoint spanning arcs in the* 1-*skeleton of* $\overline{F}^{-1}$ *so that each subdisk into which the* C_m*'s divide* $\overline{F}^{-1}$ *has a super triangulation. Let* $\pi \circ G$ *be a linear embedding of* $\mathrm{Bd}(\overline{F}^{-1})$ *into* E^2. *For each* m *let* K_m *and* L_m *be the two arcs into which the endpoints of* C_m *divide* $\mathrm{Bd}(\overline{F}^{-1})$. *If for each* m *the number of* 1-*simplexes in* C_m *is greater than or equal to the number of straight segments in* $\pi \circ G(K_m)$ *or* $\pi \circ G(L_m)$, *then* $\pi \circ G$ *can be extended to a linear embedding of* $\overline{F}^{-1}$.

Furthermore, for any two such extensions of $\pi \circ G$ *there is a linear isotopy between them which leaves the map on* $\mathrm{Bd}(\overline{F}^{-1})$ *the same throughout.*

Proof. The proof is to map the C_m's neatly along $(\pi \circ G)(\mathrm{Bd}\ \overline{F}^{-1})$ and then use Property 1 of the super triangulation of each subdisk to complete the extension.

To establish the "furthermore" statement, use [1, Theorem 2.4] which implies that the C_m's can be pushed from any one linear embedding to any other keeping $(\pi \circ G)(\mathrm{Bd}\ \overline{F}^{-1})$ fixed. Since each subdisk has a super triangulation, the push of the C_m's can be extended to each subdisk.

To finish the definition of G on $\overline{F}^{-1}$, then, use the Sublemma to extend $\pi \circ G$ over $\overline{F}^{-1}$ where the C_m's in the Sublemma are the spanning arcs of $\overline{F}^{-1}$ which are C_j^i's and the boundaries of the collars of each spanning C_j^i. Compose the extended $\pi \circ G$ with π^{-1} to produce the desired extension of G.

Step 6. *Seeing that* G *satisfies the conclusion of Lemma*

2.3. Certainly G is a spherically linear embedding of (P,T_n) which extends g. The remainder of the conclusion follows from the facts that Steps 1-4 were done in a continuously canonical way and in Step 5, the "furthermore" statement of Sublemma 2.4 guarantees the necessary flexibility.

Lemma 2.5. *Let* $g\colon \text{Bd } P \to S^2$ *be a spherically linear embedding of* $(\text{Bd } P, T_n|\text{Bd } P)$ *into* S^2 *and let* D *be a component of* $S^2-g(\text{Bd } P)$. *Let* $E_t (t \in [0,1])$ *be a continuous family of equators. Suppose that for each* t *in* $[0,1]$, $E_t \cap g(\text{Bd } P) \neq \emptyset$, *for some* $a \in (0,1)$ *exactly one vertex* $g(v_{2i})$ $(i \neq 1)$ *lies on* E_a, *and no vertex* $g(v_{2k})$ *is on* E_t *for* $t \neq a$.

Then there is a spherically linear isotopy $G_t\colon (P,T_n) \to S^2 (t \in [0,1])$ *so that* $G_o(P) = \overline{D}$, *for all* t *in* $[0,1]$, $G_t|\text{Bd } P = g$, *and* G_o *is an extension of* g *obtained from Lemma* 2.3 *using* E_o *as the equator while* G_1 *is such an extension of* g *using* E_1 *as the equator.*

Indication of proof. We examine five cases. (See Figures 2.4, 2.5, 2.6, and 2.7.) Let G_o be an extension of g given by Lemma 2.3 using E_o as the equator. In each case we indicate how to obtain a spherically linear isotopy on the B_j's which start at G_o restricted to the B_j's and end at G_1 restricted to the B_j's. In each case the extension to the remainder of the disk P is possible after recalling the properties of the remainder of the triangulation T_n.

If the moving equator does not actually cross $g(v_{2i})$, there is no problem, so we assume it does.

Case 1. Suppose that $g(v_{2(i-1)})$ and $g(v_{2(i+1)})$ are separated by E_o. By changing the roles of E_o and E_1 if necessary, we assume that $g(v_{2(i-1)})$ and $g(v_{2i})$ are in the same component of S^2-E_o. For some j, an arc of B_j with endpoint w^j_{2i+1} is mapped into E_o by G_o with $G_o(w^j_{2i+1})$ lying near $E_o \cap g(v_{2i}v_{2(i+1)})$. In Case 1 we suppose that $G_o(w^j_{2i})$ and $G_o(w^j_{2i-1})$ are not on E_o. (See Figure 2.4.)

Let us first notice what the major differences are between the extensions G_o and G_1 on the B_j's. First, the edges $w^j_{2i-1}w^j_{2i}$ and $w^j_{2i}w^j_{2i+1}$ should be mapped into E_1 by G_1. All other vertices w^k_m which are mapped into E_o by G_o (except for $k < j$ and $m = 2i+1$) should be mapped into E_1 by G_1. The vertices w^k_{2i+1} $(k < j)$ should be mapped near the midpoint of $g(v_{2i}v_{2(i+1)})$ by G_1 while the vertices w^k_{2i-1} $(k < j)$ should be mapped into E_1 by G_1. We see that B_j is the curve on which the most radical changes occur. All that needs to be done is to move B_j into its correct position while letting the parts of the B_k's which were mapped into E_o by G_o and must be mapped into E_1 by G_1 just stay on the moving equator. Minor adjustments of the map on the w^k_m's must be made to bring them to the required position as indicated above. These adjustments of the map on the B_j's can be extended to a spherically linear isotopy of T_n which ends by being a desired extension G_1.

Case 2. This case is identical to Case 1 except that $G_o(w^j_{2i})$ and $G_o(w^j_{2i-1})$ are on E_o. (See Figure 2.4.) Note that Case 2 is just like Case 1 if the roles of E_o and E_1 were changed.

Case 3. Suppose that $g(v_{2(i-1)})$ and $g(v_{2(i+1)})$ are in

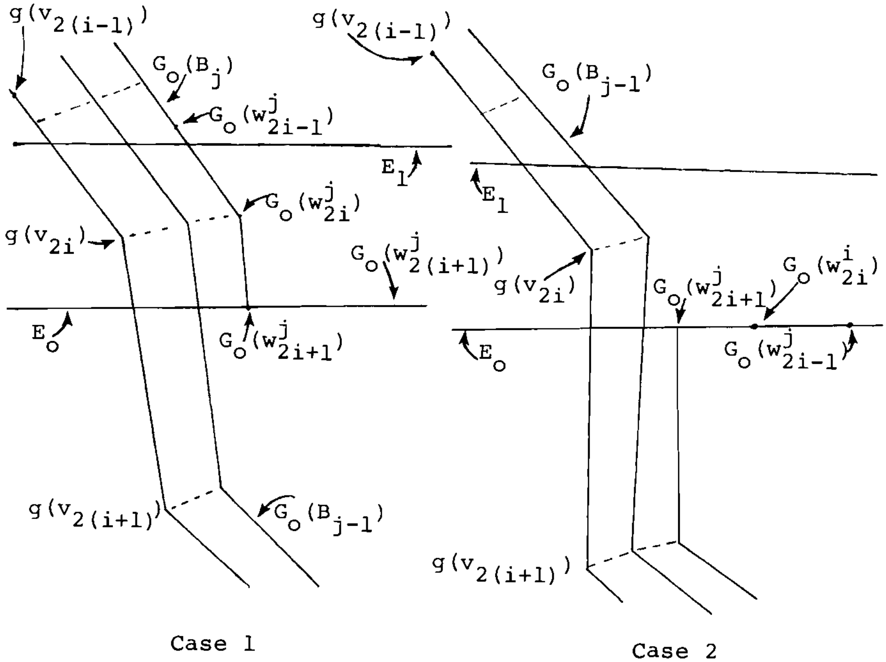
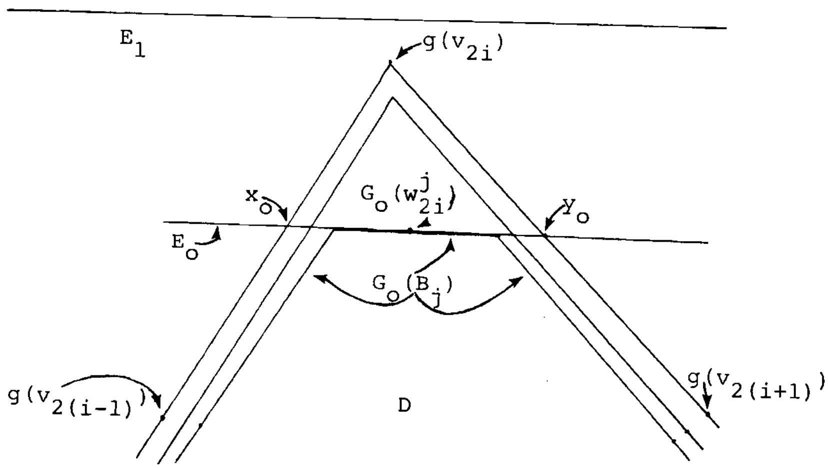


Figure 2.4

the same component of S^2-E_o. By interchanging the roles of E_o and E_1 if necessary, we assume that $g(v_{2i})$ lies in the other component of S^2-E_o than do $g(v_{2(i-1)})$ and $g(v_{2(i+1)})$. Also recall from the hypothesis of Lemma 2.5 that $i \neq 1$. Let $x_t = E_t \cap g(v_{2(i-1)}v_{2i})$ and $y_t = E_t \cap g(v_{2i}v_{2(i+1)})$ for each t in $[0,a)$ where $g(v_{2i}) \in E_a$. Let L_t be the arc on E_t between x_t and y_t which shrinks as t approaches a. Let J be the simple closed curve made up of L_o, the subset of $g(v_{2(i-1)}v_{2i})$ from x_o to $g(v_{2i})$, and the subset of $g(v_{2i}v_{2(i+1)})$ from y_o to $g(v_{2i})$. Let D_o be the component of S^2-J which contains $\text{Int}(L_t)$ for $t \in (0,a)$. Case 3 occurs when D_o is a subset of D. (See Figure 2.5.)

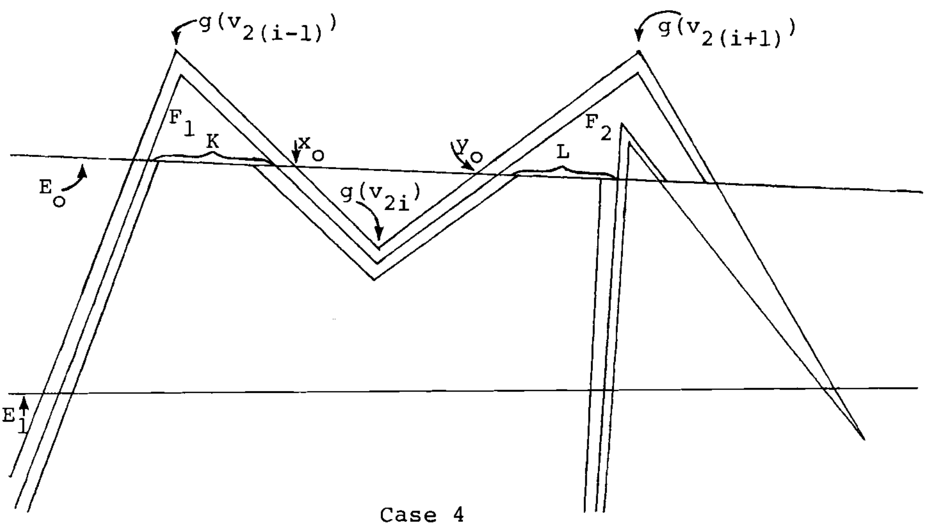


Case 3

Figure 2.5

In this case, for some B_j, the segment from w^j_{2i-1} to w^j_{2i+1} is mapped into L_o from a point near x_o to a point near y_o. All we need to do is to move the image of w^j_{2i} up near $g(v_{2i})$ and then push the vertices w^k_{2i-1} and w^k_{2i+1} $(k \leq j)$ to near the midpoints of $g(v_{2(i-1)}v_{2i})$ and $g(v_{2i}v_{2(i+1)})$ respectively. The rest of the B_j's easily follow the moving E_t.

Case 4. Suppose that $g(v_{2(i-1)})$ and $g(v_{2(i+1)})$ lie in the same component of S^2-E_o and again pick E_o so that $g(v_{2i})$ lies in the other component. Let F_1 and F_2 be the components of $D-E_o$ which contain $g(v_{2(i-1)})$ and $g(v_{2(i+1)})$ in their closures respectively. Let $x_o = g(v_{2(i-1)}v_{2i}) \cap E_o$ and $y_o = g(v_{2i}v_{2(i+1)}) \cap E_o$. Case 4 occurs when the minimal number of times that an arc from $g(v_2)$ to x_o in $\overline{D}$ must cross E_o is equal to that number for y_o. (See Figure 2.6.) Let K be the



Case 4

Figure 2.6

component of $\overline{F}_1 \cap E_o$ which contains x_o and L be the component of $\overline{F}_2 \cap E_o$ which contains y_o. Let j be the integer such that a segment of B_j is mapped along K by G_o. Notice that the hypothesis for Case 4 implies that another segment of B_j is mapped along L by G_o. The only major adjustment necessary in this case is to map w^j_{2i} to the equator E_1 by the map G_1. So as the equator E_o moves, simply let $G_o(w^j_{2i})$ stick to the moving equator and therefore be on it at the end. Only minor adjustments are necessary to move the images of vertices w^k_{2i-1} and w^k_{2i+1} $(k < j)$ to points near the midpoints of $g(v_{2(i-1)}v_{2i})$ and $g(v_{2i}v_{2(i+1)})$ respectively. Otherwise vertices on E_o are just moved along with the moving equator. The map can then easily be adjusted and extended to a desired G_1.

Case 5. Suppose none of the previous cases applies. As in Case 4 we suppose that $g(v_{2i})$ is in the component of S^2-E_o which does not contain $g(v_{2(i-1)})$ and $g(v_{2(i+1)})$. Define x_o, y_o, K, L, F_1 and F_2 as in Case 4. In this case suppose that the number k of times an arc from $g(v_2)$ to x_o in $\overline{D}$ must cross E_o is less than that number for y_o. Note that that number for y is k + 1. (See Figure 2.7.)

Let x_o and w_o be the endpoints of K and y_o and z_o the endpoints of L. Then w_o belongs to $g(v_{2r}v_{2(r+1)})$ for some r and z_o belongs to $g(v_{2s}v_{2(s+1)})$ for some s. Note that $G_o(w_{2s+1}^{n-k-1})$ is near z_o and $G_o(w_{2s+1}^{n-k})$ is mapped into K. As the equator rotates let the map G_t be defined to keep the appropriate parts of the B_j's on the rotating equator except for parts of B_{n-k} and B_{n-k-1}. On those we do the following. When the equator has moved to point E_1, define G_{t_o} so that C_{2s+1}^{n-k} is mapped into E_1, the arc on B_{n-k} from w_{2s+1}^{n-k} to w_{2r+1}^{n-k} is mapped into E_1 while the arc on B_{n-k} from w_{2s+1}^{n-k} to w_{2i-1}^{n-k} is pushed off of E_1 and the arc on B_{n-k-1} from w_{2s+1}^{n-k-1} to w_{2i-1}^{n-k-1} is also mapped into the same component of $D-E_1$. (See Figure 2.7, Stage 2).

Next adjust the map on B_{n-k-1} so that at some later time t_1, the arc on B_{n-k-1} from w_{2s+1}^{n-k-1} to w_{2i-1}^{n-k-1} is mapped parallel to the corresponding arc on B_{n-k-2}. Note that this adjustment is constant to one side of E_1. Then adjust the map on B_{n-k} so that the image of C_{2s+1}^{n-k} is short and so that the arc on B_{n-k} from w_{2s+1}^{n-k} to w_{2i-1}^{n-k} is also parallel to the maps of B_{n-k-1} and B_{n-k-2} on their corresponding arcs. These moves are possible because they take place in one hemisphere. Now there may be some featherbedding where corresponding arcs on B_{n-k-2}, B_{n-k-1}, and B_{n-k} are all mapped

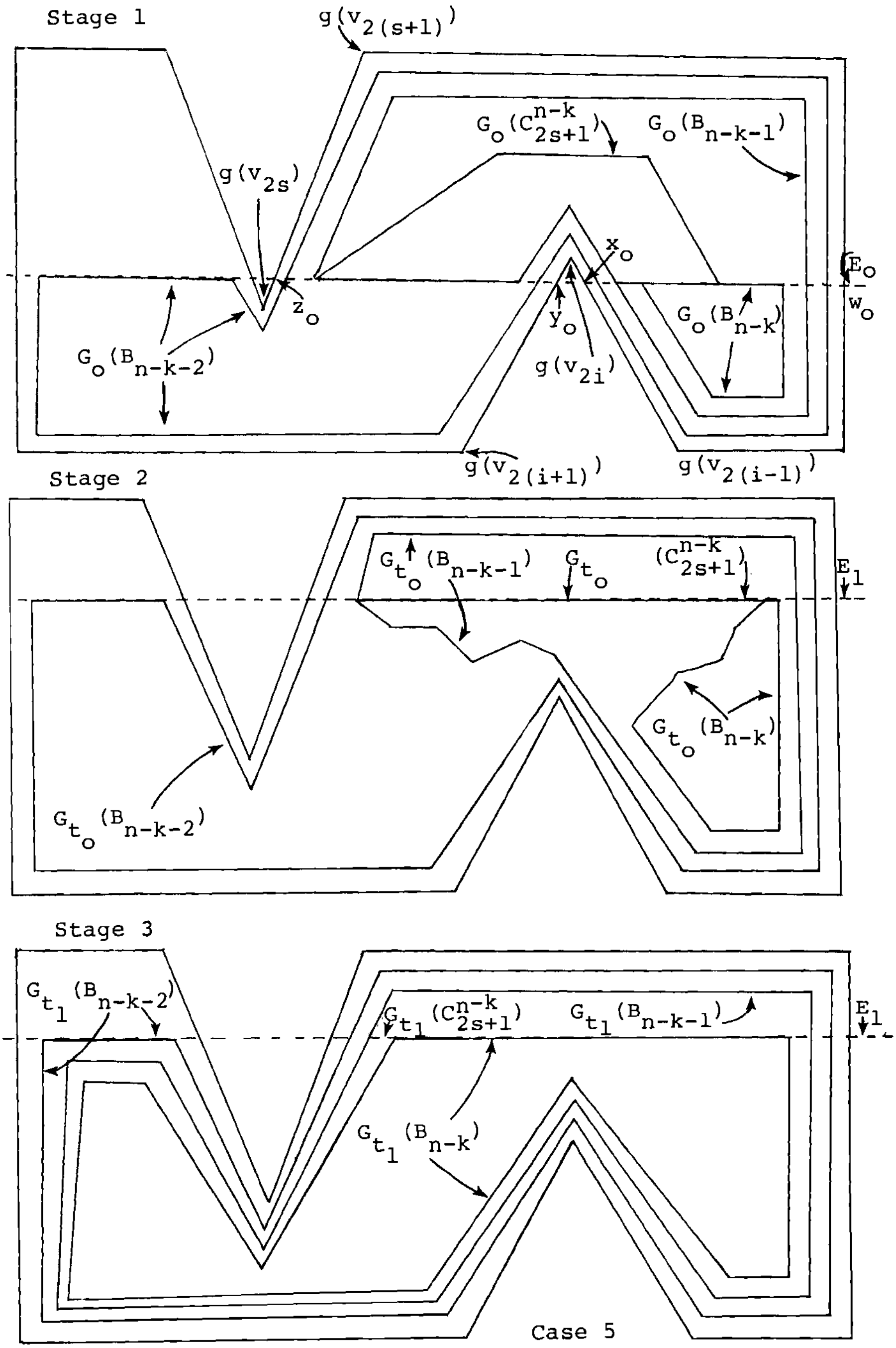

Stage 1
g(v2(s+1))
Go(C n-k 2s+1)
Go(Bn-k-1)
g(v2s)
xo
Eo
zo
wo
Go(Bn-k)
yo
Go(Bn-k-2)
g(v2i)
g(v2(i+1))
g(v2(i-1))
Stage 2
Gto(Bn-k-1)
Gto
(C n-k 2s+1)
E1
Gto(Bn-k)
Gto(Bn-k-2)
Stage 3
Gt1(Bn-k-2)
Gt1(C n-k 2s+1)
Gt1(Bn-k-1)
E1
Gt1(Bn-k)
Case 5

Figure 2.7

parallel to the same arc component of $D \cap E_1$. (See Figure 2.7, Stage 3.) Note that such an arc from B_{n-k-2} is mapped into E_1 with the other corresponding arcs mapped to one side. Now shift everything over the equator so that the arc on B_{n-k} is the one mapped into E_1. Now the image of B_{n-k} is in the correct position. Next move the misplaced parts of B_{n-k-2} and B_{n-k-1} parallel to the corresponding parts of B_{n-k-3}. Again there may be parallel arcs near the equator. If so, continue the process. This process arranges the B_j's correctly with respect to the new equator. The "furthermore" part of Sublemma 2.4 can be used in proving that the long distance moves of the B_j's (which took place in one hemisphere) can be extended over all of P. The crossings of the equator were short, carefully controlled moves, so the extension is possible. This completes the outline of the proof of Lemma 2.5.

In the application it will be necessary to have a slightly stronger version of Lemma 2.5 where more than one vertex may cross the moving equator. The Stronger Lemma 2.5 is precisely the same as Lemma 2.5 except that the assumption in Lemma 2.5 that "for some $a \in (0,1)$ exactly one vertex $g(v_{2i})$ $(i \neq 1)$ lies on E_a" is changed to the weaker assumption that "for some $a \in (0,1)$ some vertices $\{g(v_{2i})\}$ $(i \neq 1)$ lie on E_a." The proof, which is left to the reader, requires one to show that the vertices on E_a can be handled one at a time using the methods developed above.

Finishing the proof of Theorem 2.2. Let $h_t (t \in [0,1])$ be a spherically linear isotopy of (Bd P, $T_n|$Bd P) into S^2.

Let $E^t(t \in [0,1])$ be a continuous family of equators so that for each t in [0,1], $h_t(\mathrm{Bd}\ P) \cap E_t \neq \emptyset$, for no t in [0,1] is $h_t(v_2)$ in E_t, no vertex is mapped into E_o or E_1 by h_o or h_1 respectively, and the set of points t in [0,1] for which there is a k with $h_t(v_{2k})$ on E_t is discrete. By repeated applications of the Stronger Lemma 2.5, the theorem is proved. Note that the "furthermore" statement of Theorem 2.2 follows from Property 2 of Lemma 2.3.

Question 2.1. Is there a spherically super triangulation of a disk? Are the triangulations T_n described in Theorem 2.2 spherically super? If one takes a planar super triangulation and takes the first barycentric subdivision mod the boundary, that is, not subdividing the 1-simplexes on the boundary, does one obtain a spherically super subdivision?

Question 2.2. In general, to what extent can the theory of linear isotopies in E^2 (see [1], [2], [3], and [4]) be duplicated for spherically linear isotopies on S^2?

3. Flexible Neighborhoods for Complexes

Given a complex linearly embedded in E^2, it is easy to construct a triangulated regular neighborhood for it so that any linear isotopy of the complex into E^2 which begins at the identity can be extended to a linear isotopy of the neighborhood. Here we prove the analogous theorem for complexes without local cut points in E^3. The difficult part of the proof of this theorem is already behind us, namely in Theorem 2.2. The proof here of Theorem 3.1 is very similar to the proof in [5, Theorem 3.1].

Theorem 3.1. *Let* (C,T) *be a triangulated finite complex with no local cut points linearly embedded in* E^3. *Then there is a regular neighborhood* N *of* C *with triangulation* T_N *so that* (C,T) *is a subcomplex of* (N,T_N) *and for every linear isotopy* $h_t\colon (C,T) \to E^3 (t \in [0,1])$ *there is a linear isotopy* $H_t\colon (N,T_N) \to E^3 (t \in [0,1])$ *such that for every* t *in* $[0,1]$, $H_t|C = h_t$. *Furthermore, if* $h_o = h_1$, *the linear isotopy* H_t *can be chosen so that* $H_o = H_1$.

Proof. (Simplified by R. H. Bing) Suppose C contains an isolated point v. Then about v we could put a 3-simplex which is triangulated by coning from v to the boundary as part of N. Henceforth, therefore we assume that C contains no isolated points.

Let ε be a positive number such that if σ and σ' are two simplexes in T such that σ has barycenter p and $p \notin \sigma'$, then $d(p,\sigma') > \varepsilon$. (Consider a vertex to be its own barycenter.) Corresponding to this number ε we will describe a linear embedding of N into E^3.

First we build the part of N which is over the accessible 2-simplexes of C. Let σ^2 be a 2-simplex with barycenter p such that p is accessible from E^3-C. It may be that p is accessible from both sides of σ^2 or only one side. On each accessible side of σ^2 find a point v_p which is on the line perpendicular to σ^2 going through p and at distance $\varepsilon/2$ from p. The 3-simplex obtained by coning from v_p to σ^2 is a subset of N although not a 3-simplex in T_N. It requires some subdivision when the part of N which covers the vertices is built.

Next we build the part of N which is over the accessible

1-simplexes of C. Let σ^1 be a 1-simplex with barycenter p such that p is accessible from E^3-C. Let F be a round, flat disk of radius $\varepsilon/2$ which has p at its center and is perpendicular to σ^1. Let G be a component of Bd F - C. Then G is an open arc on Bd F. Let w_G be the midpoint of G. Then w_G is a vertex of N. Let σ^2 be a 2-simplex of C which meets $\overline{G}$. Above the barycenter of σ^2, there is now a vertex v which was described in the last paragraph. (Note: If $\overline{G}$ is a simple closed curve, then vertices v were described on both sides of σ^2.) The 3-simplex $\sigma^1 * v * w_g$ is a subset of N, although again not a 3-simplex in T_N.

Note that the construction so far is completely determined by the embedding of C and ε.

At this point all of the regular neighborhood of C has been canonically constructed except over the accessible vertices. Let v be an accessible vertex of C. Let S be the round 2-sphere of radius $\varepsilon/2$ centered at v. Each component G of S minus the part of N already constructed is an open disk such that Bd G is a spherically linearly embedded simple closed curve. Suppose Bd G has n vertices. (The vertices are the ones naturally induced by intersections of S with 1-simplexes of N previously constructed.) Give $\overline{G}$ the spherically linear triangulation T_n guaranteed in Theorem 2.2 where the spherically linear embedding of $(\overline{G},T_n)$ into S is that given by the procedure described in Lemma 2.3. For each 2-simplex xyz of $(\overline{G},T_n)$, there is a 3-simplex xyzv in the triangulation T_N. We are essentially coning from the triangulation of $\overline{G}$ down to v to produce the part of N around the vertex v.

After doing the above procedures, the set N has been constructed. In its present form, however, it is not triangulated since for two adjacent vertices x_1 and x_2 of Bd G, the 1-simplex x_1x_2 cuts across a 2-face of a 3-simplex τ^3 which was constructed above the 1-simplexes. Also x_1 or x_2 lies on the interior of a 1-simplex belonging to a 3-simplex μ^3 which was constructed over a 2-simplex of C. So the triangulation T_N of N is completed by subdividing τ^3 and μ^3 so that such 1-simplexes x_1x_2 are in the 1-skeleton. This subdivision can be accomplished without adding any new vertices to τ^3 and μ^3 except the vertices x_i.

We need to see that (N,T_N) satisfies the conclusion of Theorem 3.1. First note that the neighborhood N described above was associated with the number ε. If δ is any number satisfying the conditions that ε satisfies, there is linear isotopy of (N,T_N) which starts at the identity, leaves C fixed throughout, and ends with an embedding of (N,T_N) associated with δ. This linear isotopy simply consists of moving the vertices straight to where they need to go.

Now suppose $h_t\colon (C,T) \to E^3 (t \in [0,1])$ is a linear isotopy satisfying the hypothesis of Theorem 3.1. Then there is a $\delta > 0$ such that if σ and σ' are two simplexes in T such that σ has barycenter p and $p \notin \sigma'$, then $d(h_t(p), h_t(\sigma')) > \delta$ for each t in [0,1]. The desired extension $H_t\colon (N,T_N) \to E^3 (t \in [0,1])$ is obtained by first performing the linear isotopy which moves N to an embedding associated with δ rather than ε. This linear isotopy takes place in a very short time. Next the 3-simplexes τ^3 in N which were constructed above the 2-simplexes and 1-simplexes of C are moved so that for each time t in [0,1], $H_t(\tau^3)$ is the canonical embedding associated with $h_t(C)$ and δ. The extension H_t on points of N around the accessible vertices of C are determined by taking advantage of the conclusion of Theorem 2.2. The "furthermore" statement in Theorem 3.1 follows from the "furthermore" statement in Theorem 2.2.

Example 3.1. This example shows the necessity for the "no local cut-point" condition in Theorem 3.1. Let C be a 2-complex consisting of a 2-sphere embedded as the boundary of a tetrahedron, together with one additional 2-simplex σ which intersects the rest of C in 2 vertices. Let T_N be a triangulation of a regular neighborhood N of C. Let h_t be a linear isotopy which spins σ about its attached ends often enough to make extensions to (N,T_N) impossible in the neighborhood of an attached end of σ.

References

1. R. H. Bing and M. Starbird, *Super triangulations*, to appear in Pac. J. Math.
2. ________, *Linear isotopies in* E^2, Trans. Amer. Math. Soc. 237 (1978), 205-222.
3. S. S. Cairns, *Isotopic deformations of geodesic complexes on the 2-sphere and plane*, Ann. of Math. 45 (1944), 207-217.
4. C. W. Ho, *On certain homotopy properties of some spaces of linear and piecewise linear homeomorphisms*, Trans. Amer. Math. Soc. 181 (1973), 213-243.
5. M. Starbird, *Linear isotopies in* E^3, Proc. Amer. Math. Soc. 65 (1977), 342-346.

The University of Texas

Austin, Texas 78712