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PL HOMOLOGY 3-SPHERES AND TRIANGULATIONS OF MANIFOLDS

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One of the least understood but important groups arising in geometric topology is θ_3^H, the abelian group obtained from the set of oriented 3-dimensional PL homology spheres using the operation of connected sum, modulo those which bound acyclic PL 4-manifolds.

In this paper we will show how the group θ_3^H and the following theorem of Rohlin play an important role in triangulating topological manifolds.

V. A. Rohlin Signature Theorem ([4], [8], [12], [15]). *Every closed oriented smooth* 4*-manifold* M^4 *whose second Stiefel-Whitney class vanishes has signature* $\sigma(M) \in Z$ *divisible by* 16.

By classical smoothing theory the same theorem holds for PL 4-manifolds. However, the theorem is an important undecided result for topological 4-manifolds.

Given a PL homology 3-sphere H^3, then H^3 bounds a parallelizable PL 4-manifold W^4. Let $\alpha(H^3) \in Z_2$ be the Kervaire-Milnor-Rohlin invariant given by

$$\alpha(H^3) = \alpha(\overline{W}^4)/8 \pmod 2$$

where $\overline{W}^4$ is the closed (polyhedral) homology 4-manifold $\overline{W}^4 = W^4 \cup_{H^3} cH^3$. That $\sigma(\overline{W}^4)$ is divisible by 8 follows from the fact that the cup-product pairing of $\overline{W}^4$ is an even

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quadratic form (see [14]). Also, an application of Rohlin's theorem and the additivity properties of the signature show that $\alpha(H^3)$ is independent of the bounding parallelizable manifold W^4, and in fact α determines a well-defined homomorphism $\alpha\colon \theta_3^H \to Z_2$.

The classical example of a PL homology 3-sphere is the Poincaré homology 3-sphere $H^3 = SO(3)/A_5$, where A_5 is the even permutation group on five objects = the 60 orientation preserving symmetries of the dodecahedron. In fact $\pi_1(X) = \langle a,b \mid a^3 = b^5 = (ab)^2 \rangle \neq 0$. Now H^3 is also the boundary of the Milnor plumbing W^4 of 8 copies of the unit tangent disk bundle of S^2 according to the Dynkin diagram

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o-o-o-o-o-o-o
        |
        o
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(For this and other descriptions of H^3 see [9]). It is then an excellent exercise for an algebraic topology class to show that $\sigma(\overline{W}^4) = 8$, thus demonstrating that the homomorphism $\alpha\colon \theta_3^H \to Z_2$ is surjective.

We now have a short exact sequence

$$0 \to \ker(\alpha) \to \theta_3^H \overset{\alpha}{\to} Z_2 \to 0$$

Not one concrete fact is known about θ_3^H other than the existence of the surjection $\alpha\colon \theta_3^H \to Z_2$. For instance, it is not even known if θ_3^H is finitely generated.

The importance of the group $\ker(\alpha)$ is explained by

Theorem 1. (Galewski-Stern [6], [7], T. Matumoto [13]). *Let* M^m *be a topological manifold with* ∂M *triangulated as a simplicial complex. If* $m \geq 6$, *then there is an element* $\tau(M) \in H^5(M,\partial M;\ker(\alpha))$ *such that* $\tau(M) = 0$ *if and only if* *there is a simplicial triangulation* K *of* M *with* $K|\partial M$ *compatible with the given triangulation on* ∂M. *Moreover, there are* $|H^4(M,\partial M;\ker(\alpha))|$ *such triangulations on* M *up to concordance rel* ∂M.

Two triangulations K_0 and K_1 of M are *concordant rel* ∂M if there is a triangulation K of $M\times I$ such that $K|(\partial M\times I)$ is compatible with the given triangulation on $\partial M\times I$ and such that $K|M\times\{i\}$ is compatible with K_i for $i = 0,1$.

It should be noted that *a priori* there are two obstructions in Theorem 1 to triangulating M, but one of these obstructions vanish since

Theorem 2. (R. D. Edwards [3]). *Let* H^3 *be any* PL *homology* 3-*sphere. Then* $S^2 * H^3 = \Sigma^3 H^3$ *is homeomorphic to* S^6.

The importance of even a little understanding of θ_3^H is explained by

Corollary 3. (Galewski-Stern [6], [7], T. Matumoto [13]). *If there is an element* $x \in \theta_3^H$ *with* $\alpha(x) = 1$ *and* $2x = 0$, *then all topological* m-*manifolds,* $m \geq 6$ (≥ 7 if $\partial M \neq \emptyset$) *can be triantulated as simplicial complexes.*

This leads us to the following well-known conjectures.

Conjecture 1. *The group* θ_3^H *contains an element of order two.*

Conjecture 2. *The group* $\ker(\alpha)$ *is the zero group.*

In the remainder of this paper we will first consider the relationship between the simplicial triangulation obstruction of Theorem 1 and the combinatorial triangulation obstruction of Kirby-Siebenmann [10]. These results are implicit in [7]. We will then show how elements of θ_3^H yield exotic PL manifold structures on $I^k \times T^n$, $k + n \geq 5$, which are the standard PL structure on $\partial I^k \times T^n$, and how elements of $\ker(\alpha)$ yield *possibly* exotic simplicial manifold structures on $I^k \times T^n$, $k + n \geq 5$, which are the standard PL manifold structure on $\partial I^k \times T^n$. Furthermore, we will demonstrate that the exotic PL manifold structures on $I^k \times T^n$ are non-exotic when considered as simplicial manifold structures. Here, T^n is the cartesian product of S^1 n-times, and I^k is the cartesian product of the unit interval k-times.

Recall the short exact sequence

$$0 \to \ker(\alpha) \to \theta_3^H \overset{\alpha}{\to} Z_2 \to 0$$

and the resulting Bockstein exact coefficient sequence

$$\cdots \to H^4(M;\ \theta_3^H) \overset{\alpha}{\to}{}_* H^4(M;Z_2) \overset{\beta}{\to} H^5(M;\ker(\alpha)) \to \cdots$$

If M is a polyhedral homology m-manifold, there is an element $r(M) \in H^4(M;\theta_3^H)$ such that $r(M) = 0$ if and only if there is a PL m-manifold N and a PL acyclic map $f\colon N \to M$ (see [1], [2], [11]). If M is also a topological manifold, let $\Delta(M) \in H^4(M;Z_2)$ denote the Kirby-Siebenmann obstruction to putting a PL manifold structure on M [10]. Theorem 11.1 of [7] shows that

$$(4) \qquad \alpha_* r(M) = \Delta(M)$$

Note that this identifies $\Delta(M)$ as a simplicial cohomology class in the case that M has a simplicial triangulation K. For then $\Delta(M)$ is represented by the cocycle $\alpha'\colon C_4(M) \to Z_2$, where $C_4(M)$ is the free abelian group generated by the dual 4-cells e_σ^4 of the $(m-4)$-simplices σ of K and $\alpha'(e_\sigma^4) = \alpha(\text{link}(\sigma,K))$. In particular, a topological m-manifold M,

$m \geq 5$ (≥ 6 if $\partial M \neq \emptyset$) has a PL manifold structure if and only if there is a simplicial triangulation K of M such that the Kervaire-Milnor-Rohlin invariant of every 3-dimensional link of K is zero.

Since the obstruction to putting a PL manifold structure on a topological manifold M and the obstruction to putting a simplicial manifold structure on M are lifting obstructions, standard obstruction theory shows that

$$(5) \qquad \beta\Delta(M) = \tau(M)$$

Furthermore, we have by Corollary 12.5 of [7] that if $\beta_*: H^4(M;Z_2) \to H^5(M;Z)$ is the integral Bockstein homomorphism, then $\beta_*\Delta(M) = 0$ implies that $\tau(M) = 0$.

Now to the construction of exotic triangulations on $I^k \times T^n$, $k + n \geq 5$. Let $\mathcal{S}_{PL}(I^k \times T^n, \partial)$ denote the set of concordance (hence isotopy) classes of PL manifold structures on $I^k \times T^n$ extending the standard PL manifold structure on $\partial I^k \times T^n$. Similarly, let $\mathcal{S}_{TRI}(I^k \times T^n, \partial)$ denote the set of concordance classes of simplicial manifold structures on $I^k \times T^n$ extending the standard PL manifold structure on $\partial I^k \times T^n$. There is a natural map $\mathcal{S}_{PL}(I^k \times T^n, \partial) \to \mathcal{S}_{TRI}(I^k \times T^n, \partial)$. By the work of Kirby-Siebenmann [10], $|\mathcal{S}_{PL}(I^k \times T^n, \partial)| = |H^{3-k}(T^n;Z_2)|$ for $k + n \geq 5$; and by Theorem 1, $|\mathcal{S}_{TRI}(I^k \times T^n, \partial)| = |H^{4-k}(T^n;\ker(\alpha))|$ for $k + n \geq 6$. Our goal is to construct non-trivial elements of $\mathcal{S}_{PL}(I^k \times T^n, \partial)$ and $\mathcal{S}_{TRI}(I^k \times T^n, \partial)$ and show that the natural map $\mathcal{S}_{PL}(I^3 \times T^n, \partial) \to \mathcal{S}_{TRI}(I^3 \times T^n, \partial)$ is the zero map for $n \geq 2$.

Let H^3 be a PL homology 3-sphere and let $F^3 = H^3 - \text{int } I^3$. Then $F^3 \times I^n$ has as boundary a PL homology (n+2)-sphere H^{n+2}. Hence, by doing surgery on the interior of the parallelizable manifold $F^3 \times I^n$ we have that H^{n+2} bounds a contractible PL(n+3)-manifold P^{n+3} if $n \geq 2$. By identifying the I^n factor in $\partial P^{n+3} = H^{n+2} = \partial(F^3 \times I^n)$ so as to get T^n, we have a PL(n+3)-manifold M^{n+3} with boundary $S^2 \times T^n$. Note that M^{n+3} is homotopy equivalent rel ∂M^{n+3} to $I^3 \times T^n$. Since any manifold which is homotopy equivalent to $I^k \times T^n$ rel ∂, $k + n \geq 5$, is homeomorphic to $I^k \times T^n$ rel ∂ [16], we have that M^{n+3} is homeomorphic to $I^3 \times T^n$ rel ∂. The image of the PL manifold structure on M^{n+3} under this homemorphism yields a PL manifold structure Γ_{H^3} on $I^3 \times T^n$ extending the standard PL manifold structure on $\partial I^3 \times T^n$, hence determines an element $[\Gamma_{H^3}] \in \mathcal{S}_{PL}(I^3 \times T^n, \partial)$.

Theorem 6. $[\Gamma_{H^3}] = 0$ *if and only if* $\alpha(H^3) = 0$. *Furthermore, the natural map* $Z_2 \cong \mathcal{S}_{PL}(I^3 \times T^n, \partial) \to \mathcal{S}_{TRI}(I^3 \times T^n, \partial) \cong n \ker(\alpha)$ *is the zero map.*

Proof. Let M^{n+3} be as above and let $h\colon M^{n+3} \to I^3 \times T^n$ be a homeomorphism which is the identity over $\partial I^3 \times T^n$. Attach a copy of $I^3 \times T^n$ to M^{n+3} along ∂M^{n+3} to obtain a PL manifold M' homeomorphic to $S^3 \times T^n$. Note that M' bounds $Q^{n+4} = (cH^3 \times T^n) \cup \{\text{handles}\}$. By Theorem 2 Q^{n+4} is simplicially triangulated topological manifold which is an s-cobordism between M^{n+3} and $I^3 \times T^n$ rel ∂. By the observations following (4), $cH^3 \times T^n$ possesses a PL manifold structure extending the natural one on $H^3 \times T^n$ if and only if $\alpha(H^3) = 0$. This also follows from Theorem C of [17].

Let Q_*^{n+4} be the topological manifold obtained by adjoining the mapping cylinder of h to Q^{n+4} along M^{n+3}. Then

Q_*^{n+4} is homotopy equivalent to $I^4 \times T^n$ rel $\partial Q_*^{n+4} = S^3 \times T^n$, hence Q_*^{n+4} is homeomorphic to $I^4 \times T^n$ rel $\partial I^4 \times T^n$. This homeomorphism induces a simplicial manifold structure Σ_{H^3} on $I^4 \times T^n$ and a simplicial manifold concordance between Γ_{H^3} and the standard structure on $I^3 \times T^n$. It now follows that $[\Gamma_{H^3}] = 0$ if and only if $\alpha(H^3) = 0$ and that $\mathcal{S}_{PL}(I^3 \times T^n, \partial) \to \mathcal{S}_{TRI}(I^3 \times T^n, \partial)$ is the zero map for $n \geq 2$.

Perhaps Theorem 6 morally justifies Conjecture 2 above.

Note that in the proof of Theorem 6, the simplicial manifold structure Σ_{H^3} on $I^4 \times T^n$ can be assumed to be the standard PL manifold structure on $\partial I^4 \times T^n$ if $\alpha(H^3) = 0$, for we can assume that $h\colon M^{n+3} \to I^3 \times T^n$ is a PL homeomorphism. There results for each PL homology 3-sphere H^3 with $\alpha(H^3) = 0$ an element $[\Sigma_{H^3}] \in \mathcal{S}_{TRI}(I^4 \times T^n, \partial) \cong \ker(\alpha)$.

Theorem 7. *Let* H^3 *and* $\overline{H}^3$ *be* PL *homology* 3-*spheres with* $\alpha(H^3) = \alpha(\overline{H}^3) = 0$. *Then* $[\Sigma_{H^3}] = [\Sigma_{\overline{H}^3}]$ *if and only if there is a* PL *homology cobordism between* H^3 *and* $\overline{H}^3$.

Proof. Recall that Σ_{H^3} is the simplicial manifold structure on $I^4 \times T^n$ induced by a simplicial manifold structure on the topological manifold $Q^{n+4} = (cH^3 \times T^n) \cup \{\text{handles}\}$. Similarly, $\Sigma_{\overline{H}^3}$ is induced by a simplicial manifold structure on the topological manifold $\overline{Q}^{n+4} = (c\overline{H}^3 \times T^n) \cup \{\text{handles}\}$. Let X^4 be a PL homology cobordism between H^3 and $\overline{H}^3$. Now $H^4 = cH^3 \cup X^4 \cup c\overline{H}^4$ is a polyhedral homology manifold having the homology of S^4 and with $H^4 \times \mathbf{R}^2$ a topological manifold. By Theorem 1.4 of [7], $cH^4 \times \mathbf{R}^2$ is a topological manifold so that $P^{n+5} = (Q^{n+4} \times I) \cup (cH^4 \times T^n) \cup (\overline{Q}^{n+4} \times I)$ is a simplicially triangulated topological manifold. By attaching PL handles to $X^4 \times T^n \subset P^{n+5}$ we obtain a simplicially triangulated topological manifold $Y^{n+5} = (Q^{n+4} \times I) \cup (cH^4 \times T^n) \cup \{\text{handles}\} \cup (\overline{Q}^{n+4} \times I)$ homotopy equivalent to $I^5 \times T^n$. An application of the PL s-cobordism theorem to the boundary PL s-cobordism in Y^{n+5} between ∂Q^{n+4} and $\partial\overline{Q}^{n+4}$, each of which we identify with $\partial I^4 \times T^n$ via a PL homeomorphism, yields a simplicially triangulated topological manifold Y_*^{n+5} homotopy equivalent to $I^5 \times T^n$ and with $\partial Y_*^{n+5} = Q^{n+4} \cup S^3 \times T^n \times I \cup \overline{Q}^{n+4}$. There results a concordance between Σ_{H^3} and $\Sigma_{\overline{H}^3}$.

Conversely, suppose $[\Sigma_{H^3}] = [\Sigma_{\overline{H}^3}]$. Then there is a simplicially triangulated topological manifold W^{n+5} with $\partial W^{n+5} = Q^{n+4} \cup S^3 \times T^n \times I \cup Q^{n+4}$ and with a homeomorphism $f\colon W^{n+5} \to I^4 \times T^n \times I$ which is PL over $\partial I^4 \times T^n \times I$. Let $\pi\colon cH^3 \times T^n \to I^4 \times T^n$ and $\overline{\pi}\colon c\overline{H}^3 \times T^n \to I^4 \times T^n$ be homology equivalences which fiber over T^n. There is no obstruction to extending π and $\overline{\pi}$ to a homotopy equivalence $p\colon W^{n+5} \to I^4 \times T^n \times I$, with $p|f^{-1}(\partial I^4 \times T^n \times I)\colon f^{-1}(\partial I^4 \times T^n \times I) \to \partial I^4 \times T^n \times I$ a homotopy equivalence. Homotope p so that $p|f^{-1}(\partial I^4 \times T^n \times I)$ is PL transverse to $S^3 \times (\text{pt.}) \times I$. There results a PL cobordism X^4 between H^3 and $\overline{H}^3$. There is a transversality theory for maps of homology manifolds provided that the target is a PL manifold (Theorem 3.7 of [5]). So homotope p rel $f^{-1}(\partial I^4 \times T^n \times I)$ to be homology transverse to $I^4 \times (\text{pt.}) \times I$. There results a homology manifold cobordism X^5 between cH^3 and $c\overline{H}^3$ extending X^4. Now ∂X^5 bounds the homology manifold X^5, so that $r(\partial X^5) = 0$. But the only non PL 3-sphere links of ∂X^5 are H^3 and $\overline{H}^3$. Thus $H^3 \# -\overline{H}^3$

bounds a PL acyclic 4-manifold, hence H^3 and $\bar{H}^3$ are PL homology cobordant.

To construct non-trivial elements of $\mathcal{S}_{PL}(I^k \times T^n, \partial) \cong H^{3-k}(T^n;Z_2)$, $k + n \geq 5$, take the non-trivial element of $\mathcal{S}_{PL}(I^3 \times T^{n-k}, \partial)$ constructed above and identify the opposite ends of k- interval factors of I^3 to derive a non-trivial element of $\mathcal{S}_{PL}(I^k \times T^n, \partial)$. Note that these elements are trivial when considered as elements of $\mathcal{S}_{TRI}(I^k \times T^n, \partial)$. Similarly one constructs (possibly) non-trivial elements of $\mathcal{S}_{TRI}(I^k \times T^n, \partial) \cong H^{4-k}(T^n;\ker(\alpha))$, $k + n \geq 6$, from the elements of $\mathcal{S}_{TRI}(I^4 \times T^n, \partial)$ constructed above.

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