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ON ARC-SMOOTH CONTINUA

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This paper is a preliminary report on material which we intend to publish later in expanded form.

1. Introduction

A *smooth dendroid* [8] may be defined as a metric continuum X which is *hereditarily unicoherent at a point* p [14] and satisfies the following conditions.

- (1) *Arc condition.* For each x in $X - \{p\}$ there is an arc from p to x denoted by px .
- (2) *Smoothness condition.* If x_n is a sequence converging to x in $X - \{p\}$, then the sequence of arcs px_n converges to the arc px .

(*Note.* The fact that such a continuum is a dendroid is a consequence of Theorem 2.2 of [15].)

Smooth dendroids and their Hausdorff counterparts called *generalized trees* [45] arise naturally in the study of one-dimensional semigroups (see [20] and [25]) and partially ordered spaces (see [24] and [45]). These well-behaved one-dimensional continua have been the object of considerable attention in recent years (e.g., [6], [9], [11], [13], [17], [26], [30], [32], [33], [34], [39], [40] and [46]).

An obvious way to generalize the smooth dendroids is to impose weak forms of the "arc condition" and/or the "smoothness condition" on continua which are hereditarily unicoherent at some point. *Weakly smooth dendroids* [28], *smooth continua* [14] (see also [7], [15], [16], [17], [29], [33] and

[35]), *weakly smooth continua* [31], and *nearly smooth continua* [15] arise in this way. Mackowiak has considered a much more general notion of smoothness in [36], [37] and [38].

In this paper we generalize the smooth dendroids by retaining the full strength of the "arc condition" and the "smoothness condition" while dropping the requirement that the underlying continuum be hereditarily unicoherent at some point. The resulting continua, termed *arc-smooth*, may be considered as a higher dimensional analogue of the smooth dendroids. For one-dimensional continua the smooth dendroids and the arc-smooth continua coincide (unlike any of the other generalizations of smooth dendroids).

We shall show that, like the smooth dendroids, the arc-smooth continua arise naturally in the study of semigroups and partially ordered spaces. In fact we shall see that this class of continua has already been considered by several authors under various guises and for different reasons.

Our purposes here are to introduce some terminology involving arc-smooth continua, to establish some of their properties, and to illustrate how they arise in a variety of topological considerations.

2. The Definition

A *continuum* is a compact connected metric space. A *dendroid* is an arc-wise connected hereditarily unicoherent continuum. If X is a continuum, then $C(X)$ denotes the hyperspace of subcontinua of X with the Hausdorff metric (see [22]).

We define the continuum X to be *arc-smooth* at the point

p if there exists a function $A: X \rightarrow C(X)$ satisfying the following conditions.

- (1) *Arc condition.* For each x in $X - \{p\}$ the set $A(x)$ is an arc from p to x , denoted by px , and $A(p) = \{p\}$.
- (2) *Smoothness condition.* A is continuous; hence if x_n converges to x in $X - \{p\}$, then px_n converges to px .
- (3) *Compatibility condition.* If $x \in py$, then $px \subseteq py$.

(Note. (3) is automatically satisfied when X is a dendroid.)

A continuum is said to be *arc-smooth* provided it is arc-smooth at some point.

3. Some Examples

- (1) Clearly a dendroid is arc-smooth if and only if it is smooth.
- (2) Each starlike subcontinuum of E^n is arc-smooth.
- (3) Each convex subcontinuum of E^n is arc-smooth at each point.
- (4) The cone over any metric compactum is arc-smooth.
- (5) If P denotes the pseudo-arc, then the hyperspace $C(P)$ is arc-smooth.

4. The Partial Order

Let X be a continuum which is arc-smooth at p . We say that $x \leq y$ in case $px \subseteq py$.

Theorem 1. Let X be a continuum with $p \in X$. If X is arc-smooth, then \leq is a partial order on X satisfying (a) \leq is closed, (b) p is the unique minimal element, and (c) if $y \neq p$, then the lower set $L(y) = \{x \in X : x \leq y\}$ is an order

arc.

Conversely, if X admits a partial order \leq satisfying (a), (b) and (c), then X is arc-smooth at p .

This result is essentially Theorem 2.8 of [10]. It should be compared with the partial order characterizations of generalized trees given in [24] and [45].

Let X be an arc-smooth continuum. A metric d on X is called *radially convex* if $x \leq y < z$ implies that $d(x,y) < d(x,z)$.

Theorem 2. Every arc-smooth continuum admits a radially convex metric.

Proof. This follows from the previous theorem and the main result in [4].

5. Semigroup Actions

It is known that if X is a one-dimensional continuum which supports the structure of a semigroup with zero and unit, then X is a smooth dendroid [20]. In studying the converse, Koch and McAuley [25] introduced a class of spaces called *continua ruled by arcs*. Continua ruled by arcs are arc-smooth; in fact, the first four of the eight conditions in the definition of continua ruled by arcs are equivalent to our definition of arc-smooth continua. Eberhart studied arc-smooth continua under the name *ruled spaces* in [10]. Stadlander called them *K-spaces* in [42].

By a *thread* T we mean any topological semigroup (written multiplicatively) on the interval $[0,1]$ with 0 acting as a zero and 1 as a unit. We say that the thread T *acts*

naturally on the pointed continuum (X,p) if there is a mapping $m:T \times X \rightarrow X$ such that (a) $m(0,x) = p$, (b) $m(1,x) = x$, and (c) $m(s,m(t,x)) = m(st,x)$ for x in X and s,t in T .

The next characterization of arc-smooth continua follows from Theorem 1 of [42].

Theorem 3. Let (X,p) be a pointed continuum and T any thread. Then X is arc-smooth at p if and only if T acts naturally on X .

6. Contractibility

Smooth dendroids are contractible (see [9] and [39]), but contractible dendroids need not be smooth (e.g., [18]).

A special type of contractible spaces called *freely contractible* were utilized by Isbell [21] in the study of injective metric spaces. We shall show that the freely contractible continua coincide with the arc-smooth continua.

A *free contraction* of space Z to a point p is a homotopy $H:Z \times [0,1] \rightarrow Z$ such that (a) $H(z,0) = p$, (b) $H(z,1) = z$, and $H(H(z,s),t) = H(z,\min\{s,t\})$ for z in Z and s,t in $[0,1]$.

Theorem 4. The continuum X is arc-smooth at p if and only if X is freely contractible to p .

Proof. Suppose that H is a free contraction of X to p . Let T be the thread with $st = \min\{s,t\}$ for s,t in $[0,1]$. Define the map $m:T \times X \rightarrow X$ by $m(t,x) = H(x,t)$. Then T acts naturally on X and hence X is arc-smooth at p by Theorem 3. The converse is proved similarly.

Corollary 1. If X is arc-smooth at p , then X is locally contractible at p .

Proof. Let d be a radially convex metric on X and for each $t > 0$ let $N_t(p) = \{x \in X : d(p,x) \leq t\}$. For each t , $N_t(p)$ is a closed neighborhood of p which is arc-smooth at p . Thus $N_t(p)$ is contractible to p for each t .

Corollary 2. *If X is arc-smooth, then X is unicoherent.*

Proof. Every contractible continuum is unicoherent (see Theorem 7.3 in [48]).

Corollary 3. *If X is a finite dimensional continuum which is arc-smooth at each point, then X is an absolute retract.*

Proof. Since X is contractible and locally contractible, X is an absolute retract (see Corollary 10.5 of [3]).

Remark. Corollary 3 generalizes the fact that a dendroid which is smooth at each point is a dendrite (i.e., a one-dimensional absolute retract).

7. A Question

Let X be a continuum such that for each point p in X there exists a homotopy $H: X \times [0,1] \rightarrow X$ satisfying (a) $H(x,0) = p$, (b) $H(x,1) = x$, and (c) $H(H(x,s),t) = H(x,st)$ for x in X and s,t in $[0,1]$ (here st denotes ordinary multiplication). Bing [1] has asked whether such a continuum admits a *strongly convex metric*. Applying an argument analogous to that in the proof of Theorem 4 we see that Bing's question can be restated: Does a continuum which is arc-smooth at each point admit a strongly convex metric?

8. Selected Theorems

In this section we present a variety of theorems about arc-smooth continua and indicate their proofs (with the exception of Theorem 7).

Theorem 5. Let X be a one-dimensional continuum. Then X is a dendroid which is smooth at p if and only if X is arc-smooth at p .

Proof. It suffices to observe that every contractible one-dimensional continuum is a dendroid by Theorem 1 of [5].

Theorem 6. Every arc-smooth continuum is semi-aposyndetic (i.e., given distinct points, one of them is contained in the interior of a subcontinuum missing the other one).

Proof. This is a consequence of Theorem 4 of [8].

Theorem 7. For a planar continuum X the following are equivalent. (a) X is an absolute retract. (b) X is arc-smooth and semi-locally connected. (c) X is arc-smooth at each point.

Theorem 8. For a two-dimensional polyhedron X the following are equivalent. (a) X is collapsible. (b) X is arc-smooth. (c) X is arc-smooth at each point.

Proof. This follows from the equivalence of arc-smoothness and free contractibility and some results in [21]. That (a) and (b) are equivalent also follows from [44].

Remark. For one-dimensional continua and planar continua we have seen that arc-smoothness at each point characterizes the absolute retracts. Theorem 8 shows that the analogous statement is false in two-dimensions since there

are well-known examples of contractible 2-polyhedra which are not collapsible.

If X is an arc-smooth continuum we denote the set of maximal elements (relative to \leq) by $E(X)$ and call it the *end set* of X .

Theorem 9. Let X be a three manifold with boundary a two-sphere S^2 . Then X is a three cell if and only if X is arc-smooth at some point in such a way that $E(X) = S^2$.

Proof. This follows immediately from Theorem 3 of [43].

Smooth dendroids have the fixed point property (since dendroids do [2]). Arc-smooth continua need not have the fixed point property since there are cones which do not [23].

Theorem 10. Let X be an arc-smooth continuum. If X is uniquely arc-wise connected or embeddable in the plane, then X has the fixed point property.

Proof. The first part is a consequence of contractibility and Theorem 5 of [49]. The second part follows from Theorem 3 of [19].

Theorem 11. If the continuum X is arc-smooth at the point p , then each closed set containing p is the fixed point set of some selfmap on X .

Proof. The proof given in [47] for smooth dendroids also applies to arc-smooth continua.

A space is said to have the *complete invariance property* if each of its closed subsets is the fixed point set of some selfmap.

Corollary 4. If the continuum X is arc-smooth at each point, then X has the complete invariance property.

We remarked earlier that if P is the pseudo-arc, then the hyperspace $C(P)$ is an arc-smooth continuum. It was shown in [41] that $C(P)$ is a two-dimensional Cantor manifold. More generally, if Z is any continuum, then no closed zero-dimensional set separates $C(Z)$ [27]. A similar argument yields an analogous result for certain arc-smooth continua.

Theorem 12. If X is an arc-smooth continuum without separating points and $E(X)$ is a continuum, then no closed zero-dimensional set separates X .

The Hilbert cube is an arc-smooth continuum which is homogeneous [12].

Theorem 13. Every homogeneous non-degenerate arc-smooth continuum is infinite-dimensional.

Proof. The proof is similar to that given in [21] for compact injective spaces.

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