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PROBLEM SECTION

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PROBLEM SECTION

This year's problem section features problems posed by contributors to this issue of TOPOLOGY PROCEEDINGS, solutions to some problems in the first volume, an updating of M. E. Rudin's *Lectures on Set Theoretic Topology*, and a detailed treatment of four classic problems and many related problems of point-set topology.

CONTRIBUTED PROBLEMS

Before each problem there appears in parentheses the name of the contributor. In most cases, the articles in this volume will contain material closely related to the problem. The numbering of the problems in each category picks up where the first volume left off.

A. Cardinal Invariants

3. (*van Douwen*) Is every point-finite open family in a ccc space σ-centered (i.e. the union of countably many centered families)?

4. (*van Douwen*) For which $\kappa > \omega$ is there a compact homogeneous Hausdorff space X with $c(X) = \kappa$? (There is an example with $c(X) = 2^{\aleph_0}$. Here $c(X)$ denotes cellularity, i.e. the supremum of all possible cardinalities of collections of disjoint open sets.)

See also C3, K2, and Classic Problems V, VI, and VII.

B. Generalized Metric Spaces and Metrization

4. (*Burke and Lutzer*) Must a strict p-space with a G_δ-diagonal be developable (equivalently, θ-refinable)?

Remarks. In the proceedings of the Memphis State Topology Conference [Topology, Lecture Notes in Pure and Applied Mathematics, v. 24, Marcel Dekker] it was erroneously announced that J. Chaber had given an affirmative answer; however, Chaber did not claim to settle the question except in the cases where the space is locally compact or locally second countable [*On θ-refinability and strict* p*-spaces*, Fund. Math., to appear].

C. Compactness and Generalizations

3. (*van Douwen*) Is a compact Hausdorff space non-homogeneous if it can be mapped continuously onto βN? (Yes if $w(X) \leq c$).

4. (*Comfort*) Let $\beta(\kappa)$ denote the Stone-Čech compactification of the discrete space of cardinal κ. Let $\mathcal{U}_\lambda(\kappa) = \{p \leftarrow \beta(\kappa): |A| \geq \lambda \text{ for all } A \in p\}$, let $\mathcal{U}(\kappa) = \mathcal{U}_\kappa(\kappa)$ and $\kappa^* = \beta\kappa - \kappa$. Is it a theorem in ZFC that if $\lambda \neq \kappa$ then $\mathcal{U}(\lambda) \not\approx \mathcal{U}(\kappa)$? (This is true if $cf(\lambda) \neq cf(\kappa)$.) The symbol $\approx$ denotes homeomorphism.

5. (*Comfort*) With notation as in 4, is it a theorem in ZFC that $\omega_1^* \not\approx \omega_0^*$? (It is known that if $\kappa > \lambda \geq \omega_0$, and $\kappa^* \approx \lambda^*$ then $\lambda = \omega_0$ and $\kappa = \omega_1$.)

6. (*Comfort*) More generally, is it a theorem in ZFC that if $\kappa > \alpha \geq \omega_0$, $\lambda > \beta \geq \omega_0$, and $\mathcal{U}_\alpha(\kappa) \approx \mathcal{U}_\beta(\lambda)$, then $\lambda = \kappa$ and $\alpha = \beta$?

The following two problems were first posed in the article by Comfort in the AMS Bulletin 83 (1977), 417-455.

7. (*Comfort*) It is known [Ginsburg and Saks, *Some applications of ultrafilters to topology*, Pac. J. Math 57 (1975),

403-418, MR 52 #1633] that if $\{X_i: i \in I\}$ is a family of Tychonoff spaces such that $X_J = \Pi_{i \in J} X_i$ is countably compact for all $J \subset I$ with $J \leq 2^c$, then $X_I = \Pi_{i \in I} X_i$ is countably compact. Is 2^c the optimal test cardinal in this respect? Is there $\{X_i: i \in I\}$ with $|I| = 2^c$, X_J countably compact whenever $J \subset I$ and $J \neq I$, and X_I not countably compact? Is there X such that X^α is countably compact iff $\alpha < 2^c$?

8. (*Comfort*, communicated independently by N. Hindman and S. Glazer to him) For p, q $\in \beta N$, define p + q by $A \in p+q$ if $\{n | A - n \in p\} \in q$. Then $p+q \in \beta N$, and it is known that there exists $\overline{p} \in \beta N$ such that $\overline{p} + \overline{p} = \overline{p}$. Similarly (with $\cdot$ defined analogously) there is $\overline{q} \in \beta N$ such that $\overline{q} \cdot \overline{q} = \overline{q}$. Is there $p \in \beta N$ such that $\overline{p} + \overline{p} = \overline{p} \cdot \overline{p} = \overline{p}$?

9. (*Cameron*) Under what conditions is βX maximal countably compact?

10. (*Cameron*) Are all compact spaces strongly compact?

11. (*Cameron*) Are all countably compact spaces strongly countably compact?

12. (*Cameron*) Are all sequentially compact spaces strongly sequentially compact?

13. (*Cameron*) Are there maximal countably compact spaces which are not sequentially compact?

14. (*Cameron*) What are "intrinsic" necessary and sufficient conditions for a space to be maximal pseudocompact?

See also A4, L1, problem A20 of the subsection "Problems from other sources," and Classic Problems V and VI.

D. Paracompactness and Generalizations

8. (*van Douwen*) Is there a paracompact (metacompact

or subparacompact or hereditarily Lindelöf) space that is not a D-space? (X is a D-*space* if for every $V: X \to \tau X$ with $x \in V(x)$ for all x, there is a closed discrete $D \subset X$ such that $\cup\{V(x) | x \in D\} = X$).

9. (*Nyikos*) Is the finite product of metacompact σ-scattered spaces likewise metacompact? What if (weakly) θ-refinable, or screenable, or σ-metacompact, or meta-Lindelöf is substituted for metacompact?

10. (*Nyikos*) Is the product of a metacompact space and a metacompact scattered space likewise metacompact? (What about the other covering properties mentioned in 9.?)

11. (*Nyikos*) Is the finite product of hereditarily (weakly) $\delta\Theta$-refinable σ-scattered spaces likewise hereditarily (weakly) $\delta\Theta$-refinable? What about (weakly) $(\delta)\overline{\Theta}$-refinable spaces?

12. (*Williams*) Is $\square^{\omega}(\omega+1)$ always paracompact or normal?

13. (*Williams*) Is $\square^{\omega_1}(\omega+1)$ normal in any model of ZFC?

14. (*Williams*) Can there be a normal non-paracompact box product of compact spaces?

15. (*Williams*) Is the box product of countably many compact linearly ordered topological spaces paracompact?

16. (*Williams*) For directed sets D and E, define $D \leq E$ if there exists a function $T: D \to E$ preserving bounded sets; allow $D \equiv E$ if $D \leq E$ and $E \leq D$. For which directed sets D does $D \equiv {}^{\omega}\omega$ imply $\square^{\omega}(\omega+1)$ is paracompact? Does $\omega_1 \times \omega_2 \equiv {}^{\omega}\omega$ imply $\square^{\omega}(\omega+1)$ is paracompact? [*Note*. If $\kappa \leq c$ is an ordinal of uncountable cofinality, then each of $\kappa \equiv {}^{\omega}\omega$ and $\kappa \times c \equiv {}^{\omega}\omega$ imply $\square^{\omega}(\omega+1)$ is paracompact.]

See also B4 and Classic Problem VII.

E. Separation and Disconnectedness

4. (*Thomas*) If X is a k_w-space, is $\beta X-X$ necessarily an F-space?

See also K1.

G. Mappings of Continua and Euclidean Spaces

8. (*Grace*) Is there a monotonely refinable map (i.e. a map that can be ε-approximated by a monotone ε-map, for each positive ε) from a regular curve of finite order onto a topologically different regular curve of finite order?

H. Mappings of Other Spaces

2. (*Janos*) Let (X,d) be a compact metric space of finite dimension and f: $X \to X$ an isometry of X onto itself. Does there exist a topological embedding i: $X \to E^m$ of X into some Euclidean space E^m such that f is transformed into Euclidean motion? This would mean that there exists a linear mapping L: $E^m \to E^m$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad i \quad} & E^m \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle L} \\ X & \xrightarrow{\quad i \quad} & E^m \end{array}$$

commutes.

I. Infinite Dimensional Topology and Shape Theory

6. (*Hastings*) Is every (weak) shape equivalence of compact metric spaces a strong shape equivalence?

J. Group Actions

1. (*Wong*) Every finite group G can act on the Hilbert cube, Q, semi-freely with unique fixed point, which we term

based-free. Let G act on itself by left translation and extend this in the natural way to the cone C(G). Let Q_G (which is homeomorphic to Q) be the product of countably infinitely many copies of C(G). The diagonal action σ is based-free G-action on Q, and any other based-free G-action on Q_G is called *standard* if it is topologically conjugate to σ. Does there exist a non-standard based-free G-action on Q_G?

See also I1 through I5 (Vol. 1).

K. Connectedness

1. (*Guthrie, Stone, and Wage*) What is the greatest separation which may be enjoyed by a maximally connected space? In particular, is there a regular or semi-regular Hausdorff maximally connected space?

2. (*Guthrie, Stone, and Wage*) For which κ does there exist a maximally connected Hausdorff space of cardinal κ? In particular, is there a countable one?

3. (*Nyikos*) Does there exist a weakly σ-discrete, connected, normal space? (A space is *weakly σ-discrete* if it is the countable union of discrete subspaces.)

L. Topological Algebra

1. (*Henriksen*) Find a necessary and sufficient condition on a realcompact, rim compact space X in order that $C^{\#}(X)$ will determine a compactification (and hence the Freudenthal compactification) of X. To do so, it will probably be necessary to characterize the zero-sets of elements of $C^{\#}(X)$.

See also O2, O3, and O4.

M. Manifolds

1. (*W. Kuperberg*) Is it true that the orientable closed surfaces of positive genus are the only closed surfaces embeddable in the products of two one-dimensional spaces?

2. (*W. Kuperberg*) Suppose that T is a torus surface contained in the product of $X \times Y$ of two one-dimensional spaces X and Y. Do there exist two simple closed curves $A \subset X$ and $B \subset Y$ such that $T = A \times B$? (In other words, if π_1 and π_2 are the projections, is $T = \pi_1(T) \times \pi_2(T)$? Here = always denotes set-theoretic equality.)

See also D6 (following subsection).

N. Measure and Topology

1. (*Pfeffer*) Let $\alpha \leq \gamma$ and let μ be a diffused, γ-regular α-measure on a T_1-space X. Is μ moderated?

2. (*Pfeffer*) Let $\alpha > \beta$ and let μ be a β-finite, Borel α-measure on a metacompact space X containing no *closed* discrete subspace of measurable cardinality. Is μ β-moderated?

O. Theory of Retracts; Extension of Continuous Functions

1. (*Wong*) An absolute retract (AR) M is said to be *pointed* at a point $p \in M$ if there is a strong deformation retract $\{\lambda_t\}$ of M onto $\{p\}$ such that $\lambda_t^{-1}(p) = p$ for all $t < 1$. It is known that a point p in a compact AR is pointed if $M - \{p\}$ has the homotopy type of an Eilenberg-MacLane space of type $(\mathbb{Z}_n, 1)$ where $\mathbb{Z}_n = \mathbb{Z}/(n)$. Can we relax the condition on M {p}? In particular, is every point "pointed" in a compact AR?

2. (*Sennott*) Is the converse of Theorem 1 (these PROCEEDINGS) true? If not, is there a counterexample where S is P-embedded?

3. (*Sennott*) From Theorem 2 it is clear that if S is D-embedded in X, then there exists a s.ℓ.e. from $C_\mu(S)$ to $C_\mu(X)$ and from $C_p(S)$ to $C_p(X)$. Must there exist a s.ℓ.e. from $C_c(S)$ to $C_c(X)$?

4. (*Sennott*) Give characterizations (similar to those known for P- and M-embedding) for the other embeddings introduced in Section 2.

INFORMATION ON EARLIER PROBLEMS

B1. (*Przymusinski*) Can each normal (or metacompact) Moore space of weight $\leq c$ be embedded in a separable Moore space? *Consistency result* (*van Douwen and Przymusinski*). Under CH, the answer is yes even if "normal" and "metacompact" are completely dropped. To appear in Fund. Math.

B3. (*Alster and Zenor*) Is every locally connected and locally peripherally compact normal Moore space metrizable? *Solution* (*Zenor*). Yes, these PROCEEDINGS.

C1. (*Przymusinski*) Can each first countable compact space be embedded into a separable first countable space? A separable first countable compact space? *Remarks.* The answer to both questions is Yes if CH is assumed. The first answer can be found in the research announcement by Przymusinski in Vol. 1 of these PROCEEDINGS. The second answer can be found on p. 143 of R. Walker's book, *The Stone-Čech Compactification* (Springer-Verlag, 1974). However, the proof of Parovičenko's result on which this relies (p. 82) has a gap in it; but this gap can be filled.

C2. (*Woods*) Is it consistent that there exists a normal countably compact Hausdorff F-space X such that $|C^*(X)| = 2^{\aleph_0}$ and X is not compact? *Solution* (*van Douwen*). Yes, in fact the assertion is equivalent to $\neg$CH. There is an absolute example of a countably compact normal basically disconnected space which is not compact and satisfies $|C^*(X)| = \aleph_2 \cdot 2^{\aleph_0}$. E. K. van Douwen, *A basically disconnected normal space* Φ *with* $|\beta\Phi - \Phi| = 1$, to appear. This example may shed some light on D1.

D6. (*Alster and Zenor*) Is every perfectly normal, locally Euclidean space collectionwise normal? *Remarks*. If PMEA is consistent, so is a Yes answer to this question, even if "perfectly" is dropped. Under PMEA, every first countable normal space is collectionwise normal. [P. Nyikos, *A provisional solution to the normal Moore space problem*, to appear.]

E3. (*Bankston, attributed to R. Button*) Are ultraproducts of scattered Hausdorff spaces scattered? [Non-Hausdorff counterexamples are known.] *Remarks by E. K. van Douwen*. The trivial answer to E3 is: almost never. Bankston has misstated Button's question: Button's question deals with enlargements, not with ultrapowers, and is not such a shameless triviality as is Bankston's question. Let ^{1} be the derived set operator, $^{(n)}$ its n-fold iteration. Suppose $X^{(n)} \neq \emptyset$ for all natural numbers n. Then the following subspace of $\Pi_D X$ (countably many copies of X, D a free ultrafilter on ω) is dense in itself: $\{x_D\colon \exists n$ such that $\forall k > n$, level $(x_k) <$ level $(x_{k+1})\}$.

See also the subsection on Classic Problems.

PROBLEMS FROM OTHER SOURCES

In the second printing of M. E. Rudin's *Lectures on set theoretic topology* [Regional Conference Series in Mathematics No. 23, published by the AMS], many problems that appeared at the end of the first printing have been replaced by new ones. Since the logistics of the printing process did not allow for the solutions of these omitted problems to appear there, we felt it appropriate to list these problems here, along with the information we have on their current status. *The following problems have been solved in full:*

A18. Is every scattered completely regular space zero-dimensional? *Solution.* No. Solomon, Bull. London Math. Soc. 8 (1976), 239-240.

A21. Is every normal space X countably compactifiable? *Solution.* No. Burke and van Douwen, pp. 81-89 in: *Set-theoretic topology*, Academic Press, 1977.

B1. If X is ccc and Y is ccc, but X×Y is not ccc, then is there a Souslin line? *Solution.* No. Laver and Galvin: such an X and Y exist under CH. Jensen: CH does not imply the existence of a Souslin line.

B2. Is there a Souslin line if there is a "small Dowker space"? *Solution.* No, CH also works. See Classic Problem VII for further details.

B5. Is every compact space supercompact? *Solution.* No. Murray Bell, Gen. Top. Appl., to appear.

B6. Is there a Baire space whose square is not Baire? *Solution.* Yes, even metrizable examples. Fleissner, *Barely*

Baire spaces, to appear.

B7. Is the set of remote points in βR dense in $\beta R-R$? *Solution*. Yes. van Douwen.

B12. Is there an infinite homogeneous extremally disconnected compact space? *Solution*. No. In fact (Kunen) there is no infinite homogeneous compact F-spcae. c.f. W. W. Comfort, AMS Bulletin 83 (1977), 417-455.

B14. Can a compact space be decomposed into more than c closed G_δ's? *Solution*. No, Arhangel'skij. The proof by R. Pol of Arhangel'skij's famous theorem that every first countable compact (Hausdorff) space is of cardinal $\leq c$ can be adapted here, replacing points by closed G_δ subsets.

C1. Is there a first countable, ccc, density $\leq c$ space with uncountable spread? *Solution*. Yes. In fact, the Sorgenfrey plane is a separable example, and the square of Alexandroff's "two arrows" is compact as well. Examples are so easy to come by, this problem was almost certainly misstated.

Yes, if loc. comp. + MA + 7CH

D8. Is every perfectly normal collectionwise normal space paracompact? *Solution*. No. R. Pol, Fund. Math. 97 (1977), 37-42. What about realcompact

Any negative solution to D8 is also a negative solution to:

D9. Is every perfect space Θ-refinable?

D13. Is every countably compact space with a G_δ-diagonal metrizable? *Solution*. Yes. J. Chaber, Bull. Acad. Pol. Sci. Ser. Math. 24 (1976), 993-998.

D15. Is every linearly ordered space with a point-countable base quasi-metrizable? No. G. Gruenhage, Canad. J. Math. 29 (1977), 360-366.

D21. This is a long problem on continuous extenders, answered for the most part in the paper by van Douwen, Lutzer, and Przymusinski, *Some extensions of the Tietze-Ursohn theorem*, Amer. Math. Monthly 84 (1977), 435-441.

E1. Is there a collectionwise Hausdorff non-normal Moore space? *Solution.* Yes. Wage, Canad. J. Math., to appear.

Remarks. The new problem E7 is closely related: is there a strongly collectionwise Hausdorff Moore space which is not normal? Yes, if $MA + \neg CH$, for the $\exists$ paralind. non-normal Moore

E4. Does every Moore space of cardinal $\leq c$ have a point-countable separating open cover? *Solution.* No. Burke, p. 17 in *Topology*, Lecture Notes in Pure and Applied Mathematics, v. 24, Marcel Dekker, 1976. Recently M. Wage has constructed a simpler example.

E6. Should have been stated as follows: If $G_1, G_2, \cdots,$ is a development for a Moore space X and $G^*_{n+1}(p) \subset G^*_n(p)$ for all n, does every conditionally compact subset of X have compact closure? *Soluticn.* No, Laurie Gibson.

E7. Can every first countable space X with $|X| \leq c$ be embedded in a separable, first countable space? *Solution.* Independent of ZFC: Yes if CH, no if $\neg T(c)$. van Douwen and Przymusinski, *Separable extensions of first countable spaces*, Fund. Math., to appear.

E8. Can every Moore space X with $|X| \leq c$ be embedded in a separable, first countable space? *Solution.* Same as for E7.

Remark. The new problem E4 is closely related: Can a Moore space of weight $\leq c$ [equivalently, cardinality $\leq c$] be embedded in a separable Moore space if it is locally compact? Or has a point countable base? or is metacompact [equivalently, has a σ-point-finite base]?

E9. Can every metric space X with $|X| \leq c$ be embedded in a pseudocompact Moore space? *Solution.* Yes. Reed and van Douwen.

F1. Is there a (first countable separable) paracompact space X such that X^2 is normal but not paracompact? *Solution.* Yes. Przymusinski, Fund. Math., to appear. See also his paper in these PROCEEDINGS.

F4. Is there a locally compact normal space X and a metric space Y such that $X \times Y$ is not normal? *Solution.* Yes. van Douwen, *A technique for constructing honest locally compact submetrizable examples*, to appear.

H3. If X is the metrizable image of a complete metric space under a k-covering map, does X have a complete metric? *Solution.* No. Appearing in a recent issue of Michigan Math J.

Two problems were omitted because they were equivalent to others:

They were A5 (equivalent to C8) and D7 (equivalent to D6). Incidentally, the "answer" to A5 was mis-stated; it ought to read "No if CH."

Problem A20 *was omitted, but is still completely unsolved* (even consistency results are lacking). It reads: Is the property *initially* $\underline{m}$*-compact* productive for regular uncountable $\underline{m}$? (Every open cover of cardinality $\leq \underline{m}$ on X has a finite subcover if X is initially $\underline{m}$-compact).

Partial answers have been gotten to the following problems:

A6. If X is a regular space of countable spread, does $X = Y \cup Z$ where Y is hereditarily Lindelöf? *Reply.* No if CH or there exists a Souslin line. Roitman, Gen. Top. Appl. 8 (1978), 85-91.

C4. Is there a (regular) hereditarily separable space X with $|X| > 2^{\omega_1}$? *Reply.* Clearly, no if CH if X is regular, because $w(X) \leq c$ and so $|X| \leq 2^c$.

C7. Is the density $\leq$ the smallest cardinal greater than the spread for compact spaces? regular spaces? regular hereditarily Lindelöf spaces? *Reply.* Shapirovskij has shown that every compact spaces of countable spread has density $\leq \aleph_1$. See reference [126] of Rudin's lecture notes.

D18. In screenable spaces do normal and collectionwise normal imply countably paracompact? *Reply.* See the subsection on Classic Problems.

D20. Does a regular ** p-space (or $\omega\Delta$-space) have a countable base? For ** use ω_1-compact with a point-countable separating open cover, or hereditarily ccc, or hereditarily ccc with a G_δ-diagonal. *Reply.* No for the first interpretation of **: van Douwen, *A technique for constructing honest locally compact submetrizable examples*, to appear. No for the second: Alexandroff's "double arrows" is a compact, hereditarily separable and hereditarily Lindelöf counterexample. No is consistent for the third, under CH: Juhász, Kunen, and Rudin, Canad. J. Math., 28 (1976), 998-1005.

The following problems were repeated, but they have been answered by now.

A17. Is every image of a scattered space under a closed map scattered? *Solution.* No. Kannan, *Scattered spaces II*, Ill. J. Math., to appear.

B8. Is there a P-point in $\beta N-N$? *Solution.* Independent! S. Shelah has invented a new forcing technique which allows the construction of a model where there are no P-points in $\beta N-N$. W. Rudin, back in 1956, showed that CH implies there are P-points in $\beta N-N$.

B9. Does every hereditarily separable compact space have a point of countable character? a nontrivial converging sequence? a butterfly point? *Solution.* No if Φ (Fedorchuk), yes if MA + $\neg$CH (Szentmiklossy). See Classic Problem VI for further details.

B13. If X is Lindelöf and Y is realcompact, does X closed in $X\cup Y$ imply that $X\cup Y$ is realcompact? *Solution.* There is a trivial counterexample, as pointed out by Kato. One uses the Dieudonné plank or Thomas's plank (the latter is constructed by removing the "doubly nonisolated" point from the product of $\omega+1$ and the one-point compactification of an uncountable space), where X is countable and Y is the product of $\omega+1$ with an uncountable discrete space.

C8. Could a compact hereditarily separable space have cardinality greater than c? *Solution.* Yes if Φ (Fedorchuk), no if MA + $\neg$CH (Szentmiklossy). See Classic Problem VI for further details.

D17. Are compact (or paracompact Σ) spaces with a $\delta\Theta$ base metrizable? *Solution.* Yes. Chaber, Fund. Math. 94 (1977), 209-219. One uses the fact that every Σ-space is a β-space, and that every Θ-refinable space with a base of countable order is a Moore space, together with Chaber's theorem that every monotonic β-space with a $\delta\Theta$-base has a base of countable order. One can even replace "paracompact Σ" with "collectionwise normal Σ" because they are equivalent in the presence of a $\delta\Theta$-base: the sets C(x) used in defining "Σ-space" are then compact, and E. Michael has shown that such Σ-spaces (called "strong Σ-spaces") are subparacompact, hence Θ-refinable.

F12. Does $\dim(X \times Y) \leq \dim X + \dim Y$ hold for completely regular spaces? *Solution.* No, Wage and Przymusinski have come up with many different counterexamples, including cases where one factor is locally compact, or Lindelöf, or perfectly normal.

F14. Is a Σ-product of metric spaces always normal? *Solution.* Yes. Mary Ellen Rudin and S. Gul'ko, the latter's paper appearing in Dok. Akad. Nauk SSSR.

G5. Let S be the pseudo-arc and suppose $f: S \to S$ is a map which is fixed on some nonempty open set. Is f the identity? *Solution.* No. W. Lewis, example to appear in Canad. J. Math. *Remarks.* Lewis's example is a homeomorphism, so G2 ("is f a homeomorphism?") remains open.

Problem E5 *was mis-stated in both printings, and ought to read:* Does every noncompact Moore space which is closed in every Moore space in which it is embedded have a dense subset which is conditionally compact? (*Conditionally compact* means that every infinite subset has a limit point somewhere in the space, though not necessarily in the subspace.)

Finally, Nyikos's result that the Product Measure Extension Axiom (PMEA) *implies every first countable normal space* *is collectionwise normal* provisionally solves problems D1, D2, D3, and E10.

CLASSIC PROBLEMS

This past year has been a great one for the solution of long-outstanding problems of general topology. The two most sensational discoveries were the solution by S. Shelah of the problem of whether there are P-points in $\beta N-N$ and the solution (assuming the consistency of there being a strongly compact cardinal) by P. Nyikos of the normal Moore space problem. But there were others: the proof by H. Junnila that the closed image of a Θ-refinable space is Θ-refinable, along with numerous other results greatly advancing the theory of metacompactness, and Θ-refinability; E. van Douwen's proof that the subset of remote points of $\beta R-R$ is dense, as well as many related observations which advance the theory of remote points; the construction by G. Gruenhage of a zero-dimensional symmetrizable space with a σ-locally countable and σ-disjoint base which is not even countably metacompact, answering several old and oft-repeated questions simultaneously; the proof by P. Nyikos that a certain compact nonmetrizable space has a hereditarily normal square under MA + $\neg$CH, "consistently" answering a question posed by Katetov back in 1948; the construction by G. Kozlowski and P. Zenor, under CH, of a perfectly normal, hereditarily separable, nonmetrizable smooth (hence also analytic) manifold; the proof by M. E. Rudin and S. Gul'ko, independently of each other, that a Σ-product of metric spaces is always normal, solving a problem posed in 1959 by Corson; the constructions by M. E. Rudin (assuming

CH and SP) and T. Przymusinski (assuming CH alone)* of a normal space with a σ-disjoint base which is not paracompact, "consistently" answering a 1955 question of Nagami; and the proof by Z. Szentmiklossy that MA + $\neg$CH implies there are no compact S-spaces. General topology will never be quite the same again.

But general topology is so inexhaustible that even now there is no end of difficult, challenging, and basic problems, even of old ones. For example, although Classic Problem III has been partially solved, and minor dents have been made in Classic Problems I and II, they still stand, and even the ancient Moore space problem is very much alive. And the four additional classic problems we will expound on below are only a small sample of the ones that deserve to be included.

But first, here is a rundown on the status of the first four Classic Problems.

One related problem listed under Classic Problem I was already solved last year and was included by mistake: "Does every compact hereditarily normal space contain a point with a countable π-base?" The answer is Yes: Shapirovskij (reference #4). Another related problem has been partly solved by Fedorchuk: Does every totally disconnected compact space contain either a copy of $\omega+1$ or a copy of βN? (Equivalently: Does an infinite Boolean algebra have either a countable infinite or a complete infinite homomorphic image?) The answer is No under PH, as shown by Fedorchuk [Proc. Cambridge Phil. Soc., 81 (1977), 177-181], who showed that in a well-known Cohen model of set theory one can construct a zero-dimensional compact space of cardinal $2^{\aleph_0}$ with no nontrivial convergent sequences.

*See footnote next page.

Classic Problem II remains unchanged unless one is willing (and many set theorists are!) to assume that it is consistent that there be a strongly compact cardinal. In that case, the Product Measure Extension Axiom (PMEA) is also consistent, and implies that every normal (never mind "perfectly"!) space with a point countable base is collectionwise normal, and so the first related problem would then be completely answered: every perfectly normal space with a point countable base is collectionwise normal under PMEA, while under MA + $\neg$CH there are normal Moore counterexamples.

The most exciting break-through was on Classic Problem III: is every normal screenable space paracompact? M. E. Rudin and T. Przymusinski* have independently constructed "consistent" examples of normal spaces with σ-disjoint bases that are not paracompact. Not knowing whether either space is collectionwise normal, we do not know whether "screenable and normal implies collectionwise normal" or "screenable and collectionwise normal imply paracompact" remains untouched.

Whether one says "paracompact" or "countably paracompact" is immaterial: they are equivalent for normal screenable spaces, as Nagami showed in 1955. Recently, G. Gruenhage has shown that one may as well ask whether normal screenable spaces are countably orthocompact, observing that Construction 1 in the paper by Heath and Lindgren in *Set-Theoretic Topology* [Academic Press, New York, 1977] preserves screenability (as well as [collectionwise] normality) but destroys countable orthocompactness if one starts with a non-countably metacompact space.

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*Added in proof: T. Przymusinski has withdrawn his claim.

One other simple observation on this problem was made by H. Junnila: every hereditarily collectionwise normal, weakly Θ-refinable space is screenable. One uses the characterization of weakly Θ-refinable spaces given in the Bennett-Lutzer paper [Gen. Top. Appl. 2 (1972), 49-54] where for each point p there is an n such that $\mathrm{ord}(p, U_n) = 1$.

The status of Classic Problem IV (Is every stratifiable space M_1?) has not changed at all.

Classic Problem V. (*Arhangel'skij*) Does every compact hereditarily normal [abbreviated T_5] space of countable tightness contain a nontrivial converging sequence? In this classic problem and the next, "space" means "infinite Hausdorff space." A space X is of *countable tightness* if $\bar{A} = \cup\{\bar{B} | B \subset A, B$ countable$\}$ for all $A \subset X$.

Related problems. Is every separable compact T_5 space of countable tightness? of cardinal $\leq c$? sequentially compact? sequential?

Equivalent problems. Let P be a closed-hereditary property: that is, one that is true for every closed subset of a space with the property. The problem of whether every compact space satisfying P contains a nontrivial convergent sequence is equivalent to that of whether every compactification of $\mathbf{N}$ (the countable discrete space) satisfying P contains a nontrivial convergent sequence. The problem of whether every compact space satisfying P is sequentially compact is equivalent to that of whether every compactification of $\mathbf{N}$ satisfying P has a point x and a sequence of distinct points of $\mathbf{N}$ converging to x. Hence "separable" is redundant in the third part of the last related problem.

Along the same lines, here is an implication which goes only one way: if every separable compact space satisfying a closed-hereditary property P is sequential, then every compact space of countable tightness satisfying P is sequential.

Consistency results. Under Axiom Φ (which follows from V = L and resembles $\diamondsuit$) Fedorchuk has constructed a hereditarily separable (hence of countable tightness) compact T_5 space of cardinality 2^c which has no nontrivial convergent sequence.

If $2^{\aleph_0} < 2^{\aleph_1}$, then (F. B. Jones) every separable T_5 space is of countable spread. Now Shapirovskij and Archangel'skij have shown independently that every compact space of countable spread is of countable tightness. Thus, under $2^{\aleph_0} < 2^{\aleph_1}$, every separable compact T_5 space X is of countable tightness.

It can also be shown, assuming $2^{\aleph_0} < 2^{\aleph_1}$, that any compact T_5 space which does not contain an S-space is sequentially compact, and if it has countable tightness it is then Fréchet-Urysohn, hence sequential.

Under MA + $\neg$CH, every compact space of cardinal $<2^c$ is sequentially compact (Malyhin and Shapirovskij), so that "Yes" to the second part of the last related problem implies "yes" to the third. On the other hand, it is not known whether every separable T_5 compact space is of cardinal $<2^c$ under MA + $\neg$CH. In fact, it is a mystery what happens to any of these problems under MA + $\neg$CH. It is not even known whether the Franklin-Rajagopalan space $\gamma\mathbf{N}$ (a compactification of $\mathbf{N}$ with growth ω_1+1, hence not of countable tightness) can be T_5 under MA + $\neg$CH.

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See also Classic Problem I (vol. 1) and Classic Problem VI.

Classic Problem VI. *(Moore and Mrówka)* Is every compact space of countable tightness sequential? [A space X is *of countable tightness* if for every $A \subset X$, $\overline{A} = \cup\{\overline{B}: B \subset A, B$ is countable$\}$. A subset A of X is *sequentially closed* if no point of X outside A has a sequence in A converging to it; X is *sequential* if every sequentially closed subset is closed.]

Related problems. A. Is there a hereditarily separable, countably compact, noncompact space?

B. *(Efimov)* Does a compact space of countable tightness have a dense set of points of first countability?

C. (*Hajnal and Juhász*) Is there a hereditarily separable compact space of cardinal $>c$? [Note: this problem has recently been solved, but is included here to make the discussion easier.]

D. Is there a compact space of countable tightness that is not sequentially compact?

E. Is every separable, countably compact space of countable tightness compact? What if it is locally compact?

F. (*Franklin and Rajagopalan*) Is every separable, first countable, countably compact [hence sequentially compact] space compact? What if it is locally compact?

Consistency results. Under Axiom Φ, Fedorchuk has constructed a hereditarily separable compact space of cardinal 2^c [see Classic Problem V for the reference].

Assuming $\diamond$, Ostaszewski has constructed a perfectly normal, hereditarily separable, locally compact, locally countable (hence first countable) countably compact space which is not compact.

Assuming CH, Rajagopalan has constructed a compact space of countable tightness which is not sequential.

Assuming CH, Hajnal and Juhász have constructed a hereditarily separable, hereditarily normal, countably compact, noncompact topological group of cardinality $\aleph_1$.

Assuming P(c) [referred to as "Booth's Lemma" in [8] and ⊛ by Rajagopalan] Rajagopalan has constructed a locally compact, locally countable (hence first countable), separable, countably compact, noncompact space. E. van Douwen has shown that BF(c), which is strictly weaker than P(c), is enough for constructing such a space, by showing that BF(c) is equivalent to the axiom that every first countable regular space of cardinality $<c$ has property wD. A space has *property* wD if every infinite closed discrete subset A contains an infinite subset A' such that the points of A' can be put into a discrete family of open sets, each of which meets A' in a single point. [Note that every subspace of a first countable, countably compact space has property wD.]

Z. Szentmiklossy has recently shown that MA + $\neg$CH implies that every compact, hereditarily separable space is hereditarily Lindelöf--in other words, there are no compact S-spaces. This implies the same result for locally compact spaces, since the one-point compactification of a locally compact, hereditarily separable space is also hereditarily separable. Since every hereditarily Lindelöf, compact space is first countable, it follows by Arhangel'skij's theorem that under MA + $\neg$CH the answer to C is negative, while under Φ it is affirmative.

Remarks. There are numerous connections between the various questions and also with Classic Problem V and its related questions. Some of them are immediately apparent; others require a little digging.

In those models of set theory where there exists a hereditarily separable, compact space of cardinal $>c$, one can take a countable dense subspace and attach limit points to each of its sequences, then limit points to each countably infinite subset of the resulting subspace, and so by induction (which ends at ω_1, if not sooner) construct a countably compact subspace of cardinal $\leq c$ which is not compact (because it is dense in the whole space) and is hereditarily separable.

From Szentmiklossy's result, we know there is no *locally compact*, hereditarily separable, countably compact, noncompact space under MA + $\neg$CH; however, in those models of set theory where such a space exists, its one-point compactification gives a negative solution to Classic Problem VI. This is because (cf. the review of [3]) a compact space is sequential if, and only if, it is sequentially compact and every countably compact subspace is closed. This fact can be exploited to give further connections between the various problems.

For example, if Y is a compact, sequentially compact space of countable tightness that is not sequential, then it contains a separable, sequentially compact, noncompact subspace (which will be of countable tightness--countable tightness is a hereditary property). Indeed, let X be a sequentially closed (hence sequentially compact) subspace of Y which is not closed, and let $y \in c\ell X - X$. Pick a countable subspace Q of X having y in its closure, then the closure of Q in X is the desired subspace. Thus if there is a counterexample to Classic Problem VI, it is either an example of D or contains a counterexample to E.

In the above construction, the closure of Q in X need not be locally compact. The following lemma gives one possible way of achieving that.

Lemma. Let Y *be a compact space which contains an non-isolated point* y *which is not the limit of a nontrivial convergent sequence in* Y. *Then* $Y - \{y\}$ *is sequentially closed in* Y, *and locally compact and countably compact.*

Now if Y can be made hereditarily separable in this lemma, we have a counterexample to A and, of course, to the Moore-Mrówka problem. If Y can be made of countable tightness, we have either an example of D or a locally compact, sequentially compact counterexample to E: just pick a countable subset Q of Y such that $y \in \overline{Q} - Q$, then $\overline{Q} - \{y\}$ is the desired counterexample.

Another way of getting local compactness is to start with a scattered space:

Lemma. If Y *is a compact scattered space of countable tightness which is not sequential, then* Y *contains a sequentially compact, locally compact, separable subspace which is not compact.*

The proof uses much the same ideas as above; this time we let X be as before the previous lemma, and let y be isolated in the relative topology of $c\ell X - X$. Use the fact that a scattered compact space is zero-dimensional to pick a clopen neighborhood V of y in Y which misses the rest of $clX - X$, and do everything else inside V. (Sequential compactness comes from the fact that every countably compact scattered space is sequentially compact.)

There are also connections with Classic Problem V which may not be obvious at first. It can be shown that:

Theorem. $[2^{\aleph_0} < 2^{\aleph_1}]$ *Let* X *be a compact* T_5 *space. At least one of the following is true.*

(1) X *is sequentially compact.*

(2) X *contains an* L*-space and an* S*-space.*

(3) X contains a compact S-space of cardinality $>c$.

Thus (see the discussion of Classic Problem V) any compact T_5 space which is not sequentially compact also gives us an example of D and, under $2^{\aleph_0} < 2^{\aleph_1}$, either gives us both S and L spaces or a T_5 example of C and hence of A.

Arhangel'skij's theorem [1] that every sequential ccc space of point countable type (in particular, every compact ccc sequential space) is of cardinality $\leq c$ is also relevant here (showing an example of C is also a counterexample to the main problem) and to Classic Problem V.

Finally, although there are many examples of Hausdorff, nonregular, hereditarily separable spaces that are not Lindelöf [8], we are unaware of any countably compact examples.

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Most of the references to Classic Problem I (vol. 1) and Classic V are also relevant here.

Classic Problem VII. Does there exist a "small" Dowker space? More precisely, does there exist a normal space which is not countably paracompact and is one or more of the following:

A. first countable?

B. [hereditarily] separable?

C. of cardinality $\aleph_1$?

D. submetrizable?

E. locally compact?

Related problems. Is there a pseudonormal space (a space such that two disjoint closed subsets, one of which is countable, are contained in disjoint open sets) which is not countably metacompact, and is one or more of the above? Is there a realcompact Dowker space? Is there a monotonically normal Dowker space? See also Classic Problem III, vol. 1.

Consistency results. Assuming $\diamondsuit$, Juhász, Kunen, and Rudin have constructed a locally compact, locally countable (hence first countable), hereditarily separable, σ-countably compact Dowker space of cardinality $\aleph_1$. Assuming CH, they have constructed an example with all of these properties except local compactness and σ-countable compactness. These examples are submetrizable, hence realcompact.

Assuming the existence of a Souslin line, Mary Ellen Rudin has constructed a hereditarily separable Dowker space and also one that is first countable and of cardinality $\aleph_1$, as well as realcompact.

Assuming ♣, P. de Caux has constructed a Dowker space of cardinality $\aleph_1$ which is separable, locally countable, and weakly first countable. It is neither first countable nor locally compact nor realcompact, but it is weakly Θ-refinable, collectionwise normal, and $\aleph_1$-compact.

It is possible to construct a pseudonormal example with all these properties except normality (and perhaps non-realcompactness), which is not countably metacompact, and is collectionwise Hausdorff, by the following axiom, obviously implied by ♣:

To each countable limit ordinal λ it is possible to assign a subset $T(\lambda)$ of $[0,\lambda]$ converging to λ, such that if A is an uncountable subset of ω_1, there exists λ such that $A \cap T(\lambda)$ is infinite.

One simply uses the construction in [1], substituting this assignment $T(\lambda)$ for the one given by de Caux.

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Classic Problem VIII. Is every γ-space quasi-metrizable? Let X be a space and let τ be the collection of open subsets of X. Let $g: \omega \times X \to \tau$ be a function such that for each x and n, $x \in g(n,x)$. A space X is a *γ-space* if it admits a g such that for each x and each n, there exists $m \in \omega$ such that if $y \in g(m,x)$, then $g(m,y) \subset g(n,x)$ and such that $\{g(n,x)\}_{n=1}^{\infty}$ is a local base at x.

Theorem. A space X *is quasi-metrizable if, and only if, it is a γ-space with a function* g *as above such that* $m = n + 1$ *for all* x *and all* n.

Equivalent problem. Does every space with a compatible local quasi-uniformity with countable base have a compatible quasi-uniformity with countable base?

Related problems. Is every paracompact (or Lindelöf) γ-space quasi-metrizable? Is every γ-space with an orthobase quasi-metrizable? Is every linearly orderable γ-space quasi-metrizable?

Consistency results. None.

Remarks. These problems are probably not as well known as most of the others in this sub-section, but there are a number of reasons why the main one deserves to be called a classic. It is old enough, going back to Ribeiro's paper of 1943 where a theorem which says in effect that every γ-space is quasi-metrizable is given, but the proof is at best incomplete. The concept of a γ-space has been "discovered" independently by quite a few researchers over the years, and [1] lists 5 aliases and 13 conditions equivalent to being a γ-space, some of them bearing little resemblence to that given here. Moreover, consider the equivalent problem stated above: if one drops "quasi" in both places, one gets the classic metrization theorem of A. H. Frink, and there may be a neat general theory to be had if this "quasi" analogue turned out to be right also. Not to mention the convenience of having one less kind of "generalized metric space" to deal with. On the other hand, a γ-space that is not quasi-metrizable would probably break some exciting new topological ground, as did Kofner's example several years ago of a quasi-metrizable space which does not admit a non-Archimedean quasi-metric.

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