TOPOLOGY PROCEEDINGS
Volume 3, 1978
Pages 1–36
http://topology.auburn.edu/tp/

A SURVEY PAPER ON SOME BASE AXIOMS

by

C. E. Aull

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

A SURVEY PAPER ON SOME BASE AXIOMS

C. E. Aull

Dedicated to the memory of E. W. Chittenden. (see postscript)

1. Introduction

Base axioms along with the related fields of metrization and generalized metric spaces are among the most active areas of research in general topology. For excellent recent surveys in these fields see [B13], [H10] and [N4]. In this survey we discuss costratifiable base axioms and study them in regard to metrization, mappings, the conversion of covering and local properties to base properties, countable product theorems and the impact of set theory on these axioms. The surveyed axioms range from the early ones like the development and the point countable base to more recent ones such as the weak-uniform base. This is an attempt to bring some organization into the vast amount of work done in this field. There will be a few minor new results and a list of unsolved problems at the end.

2. Costratifiable Base Axioms

A base axiom (or the base itself) is called *costratifiable* if a T_1, stratifiable (M_3) space is metrizable iff it satisfies the base axiom. Most base axioms in the literature are costratifiable, but there are others, such as the M_i axioms of Ceder [C1], σ-minimal bases and the quasi-uniform bases of Lutzer [L2] that are not. Important results on M_i spaces have been obtained recently, including the equivalence of M_2 and M_3, proved independently by Gruenhage [G4] and

Junnila [J2], and the result of Burke, Engelking and Lutzer [B12] that a hereditarily T_3 space with a σ-HCP base (HCP = Hereditarily Closure Preserving) is metrizable. For earlier results, see particularly works of Borges [B8], [B9], and [B10].

In response to a question of the author [A5], Bennett and Berney [B1], and Lutzer [B5] have explored the relationship between σ-minimal bases and quasidevelopments and have also obtained other results. It is shown in [B1] that σ-minimal bases are not costratifiable.

The costratifiable bases will be divided into three overlapping configuration types according to whether they compare with developments, or with point-countable bases, or with bases of quasimetrizable spaces. Two bases, the weak uniform base and the orthobase, do not fit any of these configurations. However, if a WUB space has no isolated points, it has a point-countable base, so it will be treated with the point-countable configuration.

3. The Developmental Configuration

Definition 1. For a topological space $(X,\mathcal{J})$, a subcollection $\mathcal{V}$ of $\mathcal{J}$ is called a

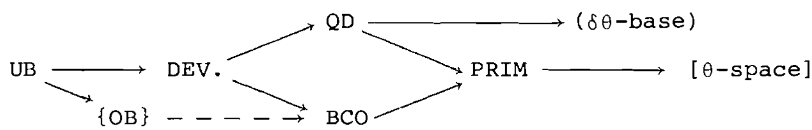
(a) *uniform base (orthobase,* (OB)*)*, if every infinite subfamily $\mathcal{V}'$ of $\mathcal{V}$ containing a given point is a base at some point (or $\cap\{V: V \in \mathcal{V}'\}$ is open),

(b) *quasidevelopment* (QD) if $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{G}_i$ and if $x \in T \in \mathcal{J}$, then there exists an n and a G such that $x \in G \in \mathcal{G}_n$, and if $x \in G' \in \mathcal{G}_n$ then $G' \subset T$. We will refer to each $\mathcal{G}_k$ as a *collection* of the quasi-development. If each member of the sequence covers X then the sequence is a *development*. A T_3

space with a development is a *Moore space*,

(c) *base of countable order* (BCO) if any sequence $B_1 \supset B_2 \supset \cdots$ of distinct members of $\mathcal{V}$, each of which contains the point p, forms a neighborhood base at p,

(d) *primitive base* (PRIM) if $\mathcal{V} = \bigcup_{i=1}^{\infty} W_n$ and each W_n is a well-ordered open cover of X, and whenever $x \in U$, for U open in X, there are positive integers n and k such that x belongs to n elements of W_k and the n^{th} such element is a subset of U.



The developmental configuration. Parentheses [brackets] indicate that the axiom is treated under the point-countable [quasimetric] configuration. Dotted lines indicate an implication if the space is connected. The axiom {OB} is not strictly part of the developmental configuration.

The oldest of the costratifiable bases is the development. To the author's knowledge this axiom first appeared in a paper by Chittenden and Pitcher [C7] in 1919. In this paper it was proved that a compact T_3 developable space is countably compact and metrizable. Some related ideas appeared in a paper by Hedrick in 1911 [H8]. The term development was used in a book by E. H. Moore [M3] for something far more general than current usage. A method for putting a distance function on a development was introduced in that book. The evolution of the usage of the term developable can be traced through

[C6], [C5] and [M4]. An important landmark in developable spaces was the Alexandroff Urysohn Metrization Theorem [A5]. However, developable spaces were not studied extensively and in depth until R. L. Moore's classic exposition "Foundations of Point Theory" appeared. Some work on these spaces is also in Tukey [T4]. Two other landmarks are Jones' paper [J1] introducing the normal Moore space problem to the literature and Bing's classic metrization paper [B7] which includes the result that collectionwise normal T_1 developable spaces are metrizable.

Jones [J1] showed that, under the continuum hypothesis, separable T_4 spaces have the property that every uncountable subset has a limit point, so that normal separable Moore spaces are metrizable under this hypothesis.

It has been shown by Przymusinski and Tall [P4] that under (MA + ~CH) there exists a metacompact c.c.c., nonmetrizable, normal, nonseparable, Moore space. Worrell [W9] and Wage [W1] have constructed examples, without set theoretic assumptions, of nonnormal collectionwise T_2 Moore spaces. Results on the general case and many special cases of the normal Moore space conjecture can be found in Rudin's recent book [R5] on set theory and topology and in G. M. Reed's review of a paper of Alster and Pol [A6]. Some of these results are included in the following theorem:

Theorem 1. *Let* $(X,\mathcal{J})$ *be a normal Moore space. Then*

(a) X *is metrizable iff* X *is collectionwise normal.* (Bing [B7])

(b) Under CH, X *is metrizable if* X *is separable.* (Jones [J1])

(c) Under MA + ~CH, X *may be separable and locally compact or locally connected and not metrizable.* (Rudin [R5])

(d) X *is metrizable if* X *is metalindelöf and locally separable or locally compact.*

(e) X *is metrizable if* X *is locally connected and locally compact.* (Reed and Zenor [R4])

(f) Under V = L, X *is metrizable if* X *is locally compact.* (Fleissner [F4])

(g) Under the axiom of constructability, X *is collectionwise Hausdorff.* (Fleissner [F4])

(h) X *is metrizable if* X *is collectionwise Hausdorff and locally separable.* (Worrell [W9] and Alster and Pol [A6] independently)

(i) It is consistent with ZFC *that* X *be collectionwise Hausdorff and not metrizable.* (Fleissner [F3]) M. E. Rudin [R5] has conjectured the existence of such a space without any set theoretic assumptions beyond the axiom of Choice.

The results of Worrell are results from his 1961 thesis that have been published recently [W9].

Theorem 2. (Arhangel'skiǐ [A9] and Heath [H4]) *A* T_1*-space is the open* π*-image of a metrizable space iff it is a developable space.* (A map f: X → Y is called a π-map if, for each $y \in Y$ and each open set U containing y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$.)

See Gitting's survey paper [G1] on mappings for background and for Ponomarev's contributions.

Results on developable spaces have been so numerous that it is beyond the scope of this paper to mention them all. However, one might look at the methods of Reed for constructing examples, starting with [R2], and Green's work on completeness and the embedding of Moore spaces, starting with [G3].

A special case of the developable space is the space with a uniform or point-regular base of Alexandroff [A2] which is a metacompact developable space. These spaces have a very interesting mapping characterization:

Theorem 3. (Arhangel'skiǐ [A7] and Hanai [H1]) *A* T_1*-space has a uniform base iff it is the compact open image of a metric space.*

However, the composition of two compact open maps is not necessarily a compact open map. The class of all successive compact images of a space is referred to as the Arhangel'skiǐ class MOBI, which will be discussed later.

Orthobases were introduced by Nyikos [N7]; Lindgren noted that they are direct generalizations of uniform bases, and Phillips [P1] showed that a connected space with an orthobase has a BCO. Nyikos [N7] had previously shown that a paracompact, OB, T_2-space that is connected and locally connected or locally compact is metrizable. In an OB-space, k, q, sequential, E_0 and first countable are equivalent [N7]. See [G1], [L1], [N7], [N8] and [C1] for properties of OB-spaces in connection with non-archimedean spaces, protometrizable spaces, dimension theory and other base axioms. Also see the sections on the other base axioms which follow.

Quasidevelopable spaces were introduced by Grace and studied by Bennett [B3]. A little earlier, Wicke and Worrell [W5] had introduced the θ-base. Bennett, Lutzer and Burke [B5] showed the equivalence of the above two concepts.

Theorem 4. *The following are equivalent for a topological space* $(X,\mathcal{J})$*: (In* [A13], *except* (a) $\leftrightarrow$ (b) *as noted above)*

(a) $(X,\mathcal{J})$ *has a* Q-D.

(b) $(X,\mathcal{J})$ *has a* θ*-base.* (Definition 2c).

(c) $(X,\mathcal{J})$ *has a* θ*-base such that* $\mathcal{B}(n_x)$ *has order* 1 (line 4, Definition 2c)

(d) If $(X,\mathcal{J})$ *is essentially* T_1, *there is a base* $\mathcal{B} = \cup\mathcal{B}_n$ *for* $(X,\mathcal{J})$ *such that every* σ*-refinement is a base.* (A family $\mathcal{A} = \cup\mathcal{A}_n$ is a *σ-refinement* of a family $\mathcal{B} = \cup\mathcal{B}_n$ if, for each n, $\cup\{A: A \in \mathcal{A}_n\} = \cup\{B: B \in \mathcal{B}_n\}$ and the family $\mathcal{A}_n$ is a refinement of $\mathcal{B}_n$.)

(e) There is a base $\mathcal{B} = \cup\mathcal{B}_n$ *for* $(X,\mathcal{J})$ *such that every* σ*-refinement is a* QD.

(f) $(X,\mathcal{J})$ *is a weak* σ*-space and has a* $\delta\theta$*-base.*

Property (e) is readily seen to be satisfied by developable spaces, with the stipulation that each $\mathcal{B}_n$ covers X, and (d) can be extended in a similar manner. From (e) one would expect that if a QD-space has a certain base property then there is a QD for the space with that base property. From (c), (d), and (f) various results follow on converting hereditary covering properties or hereditary collectionwise normal properties into corresponding base properties. For instance, a hereditarily metalindelöf QD-space has a point-countable base and is hence metrizable if it is locally separable. We note that from F. B. Jones' [J4] result (that under CH, a completely T_4 separable space is hereditarily $\aleph_1$-compact) it follows that, under CH, completely T_4 locally separable QD-spaces are metrizable. We note also that perfect (T_3, metacompact, complete) QD-spaces are (completely) developable [B3].

To the author's knowledge, relatively little is known about the mapping properties of QD-spaces. Burke [B11] has some interesting results on maps involving some special cases involving perfect maps, but as of yet it is unknown as to whether QD-spaces are preserved under perfect maps. It would be interesting to know whether class MOBI spaces are quasi-developable, a question posed by Bennett.

Another interesting generalization of developable bases is the BCO. Of the costratifiable bases studied here, it is one of the two known to be preserved under both perfect and compact open maps (regularity is needed for the latter). No other of the base axioms discussed can be said to have a better mapping theory; for, in addition, we have:

Theorem 5. [W8]

(a) A T_1*-space* S *has a* BCO *iff there is a metric space* (M,d) *and an open continuous mapping* f *of* M *onto* S *such that, for each* $x \in S$, $f^{-1}(x)$ *is complete with respect to the metric* d.

(b) A T_1*-space* S *has a monotonically complete* BCO *iff* S *is the open continuous image of a complete metric space.*

The monotonically complete BCO spaces in (b) are often called complete Aronszasjn spaces [A11]. This special class of BCO spaces originated in 1930, but the BCO was introduced by Arhangel'skiǐ [A7] in 1962.

Every θ-refinable (hereditary weakly θ-refinable) [paracompact] BCO space is developable (QD) [metrizable] ([W4], [B4], [W4]). Bennett and Berney [B4] have shown that a QD β-space has a BCO, and this has been generalized by Fletcher and Lindgren [F8] to show that a quasi-β-space that is a θ-space has a BCO if it is regular. Reed [R1] has shown that there is an additional tie between BCO's and QD's in that every BCO space has a dense QD-subspace.

The primitive base is a generalization of both the QD and the BCO. Wicke and Worrell [W4] have shown that a space is essentially T_1 and has a base of countable order iff it has a primitive base and closed sets are sets of interior condensation locally.

Wicke [W2] has shown that primitive bases are preserved by compact open mapping and, in fact, by inductively open mappings.

4. The Point-Countable Configuration

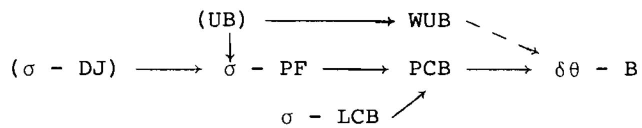
Definition 2. For a topological space $(X, \mathcal{J})$, a base $\mathcal{B}$ is

(a) a *point-countable* base if, for each $x \in X$, $\mathcal{B}$ is of countable order; (abbreviated PCB).

(b) a *σ-point finite* (*σ-disjoint*) [*σ-locally countable*] base if $\mathcal{B}$ is the countable union of point-finite families (pairwise disjoint families) [locally countable families]; (abbreviated σ-PF, (σ-DJ) and [σ-LC]).

(c) a θ-*base* (δθ-*base*) if $\mathcal{B} = \cup_n \{\mathcal{B}(n): n \geq 1\}$ and, given an open set $T \subset X$ and a point $x \in T$, there is an n_x such that $\mathcal{B}(n_x)$ has finite (countable) order at x and there exists $B \in \mathcal{B}(n_x)$ such that $x \in B \subset T$. A family $\mathcal{M}$ is of finite (countable) order at x if x belongs to a finite (countable) number of members of $\mathcal{M}$.

(d) a *weak uniform* base if no two point subset of X is contained in infinitely many members of $\mathcal{B}$.



The point countable configuration. The axiom (UB) is in the developmental configuration. The dotted line indicates an implication subject to restrictions on the isolated points.

Point-countable bases were introduced by Alexandroff and Urysohn in 1929, [A4], where it was proved that locally separable spaces with point-countable bases are metrizable. Spaces with point-countable bases have mapping theorems comparable to those of BCO spaces. Point-countable bases are also preserved under both perfect [F2] and compact open mappings. Also we have the following beautiful result of Ponomarev [P2]:

Theorem 6. *A space has a point-countable base (first countable base) iff it is the open* S-*image (open image) of a metric space.* (The term S-*image* means the image under a map whose point inverses satisfy the second axiom of countability.)

Unlike the compact open mappings, the open S-mappings are closed under composition so that the property of having a point-countable base is preserved under these mappings.

Using his result [M2] that a strong Σ-space with a point-countable separating open cover is a σ-space, combined with Heath's results [H5] that a semimetrizable space with a PCB is developable and that a σ-space is semistratifiable, Michael established that a paracompact Σ-space with a PCB is metrizable. These and other results led the author [A13] to introduce the $\delta\theta$-base, a generalization of both the PCB and the QD. In particular, the author was able to replace PCB with $\delta\theta$-B and use Heath's method to prove that a semistratifiable space is developable iff it has a $\delta\theta$-base. Chaber [C3] used some interesting new methods to generalize Michael's metrization result by replacing PCB with $\delta\theta$-B and replacing Σ with a monotonic β-space (Definition 3). Most metrization theorems involving a PCB have been generalized to ones involving a $\delta\theta$-B, a notable exception being the local separability theorem of Alexander and Urysohn [A4] mentioned earlier. Questions 2, 3, and 4 of [A16] have been answered in the affirmative.

The spaces with σ-point finite bases (σ-disjoint bases) are precisely the hereditarily σ-metacompact (hereditarily screenable) QD-spaces [A13]. If they are perfect (perfectly T_4) then the spaces have a uniform base (are metrizable, [A15]). Arhangel'skiǐ [A9] proved that a T_1-space with a σ-point finite base is metrizable iff it is perfectly normal and collectionwise normal. Wicke and Worrell [W3] were able to replace σ-PF by QD in this theorem. R. W. Heath and

G. M. Reed [B11] have shown that a σ-DJ base is not necessarily preserved under a perfect mapping in contrast to Filippov's result that a σ-PF base is preserved under a perfect mapping. In regard to σ-DJ bases, Hunsaker and Lindgren [H13] have shown that a space has a σ-DJ base iff it has a QD$\{G_i\}$ with the property that, for each $x_o \in X$, $\{St^2(x,G_n): x_o \in St^2(x,G_n)\}$ is a base for the neighborhoods of x_o.

Another interesting special case of the PCB is the σ-locally countable base. Fedorcŭk [F1] has shown that T_3 spaces with σ-LCB's are metrizable iff they are paracompact. The author [A12] has shown that there is a pattern of conversion from covering properties similar to that of developable spaces (see Theorem 6); some of the same patterns were found independently by Shiraki [S2]. More recently, extending this pattern, Fleissner and Reed [F5] have shown that a subparacompact space with a σ-LCB is developable. Significantly they have shown that in general a T_3-space with a σ-LCB may not be even normal or paralindelöf, and even that a developable space may not be normal or countably paracompact. Gruenhage has constructed a σ-LCB space that is not developable, perfect or θ-refinable. It would be interesting to know if collectionwise normal T_1-spaces with a σ-LCB are metrizable.

Another interesting base which fits in here, if one ignores isolated points, is the weak uniform base. In general, a space with such a base is not even first countable. To the author's knowledge the only publications on this base are [H7] and [D1]. In [H7] it was established that WUB-spaces have a G_δ diagonal and that countably compact T_3 WUB-spaces are metrizable and, with various conditions on the isolated points, the WUB-spaces will have a first countable base, a $\delta\theta$-B or a PCB, according to the restrictions. Furthermore, based on a result of Slaughter [S4], this base is not preserved under perfect mappings; in fact, it is not preserved under finite-to-one closed mappings. In [D1] further studies on the restrictions on the isolated points were made and it was shown that a WUB-space with at most c isolated points has a PCB and that there exists a Moore space with a WUB that is not metacompact. It was further shown that, under Martin's axiom and $\omega_2 < 2^{\omega_0}$, there is a normal Moore space with a WUB that has no PCB. Of interest is an unpublished result of Lindgren that a space with a WUB and a σ-Q base without isolated points has a σ-PF base. Since a WUB-space has a G_δ diagonal, a WUB-space with an orthobase is QD and, if the isolated points form an F_δ-set, it is developable and σ-Q, [L1] (if there are no isolated points it is UB). See [L1] for results on G_δ-diagonals and quasi-G_δ diagonals, in connection with OB-spaces.

5. The Quasimetric Axis

Definition 3. A topological space $(X,\mathcal{J})$ is

(a) a σ-Q *space* [F9] if there is a base for the space which is the countable union of Q families. (A Q *family* is a family such that the intersection of the members of an arbitrary subfamily is open.)

(b) a θ-*space* [H11] if there is a function g from $N \times X$ into $\mathcal{J}$ such that (i) $x \in \bigcap_{n=1}^{\infty} g(n,x)$ for each $x \in X$, and (ii) if $y_n \in g(n,p)$ and $\{p,x_n\} \subset g(n,y_n)$ for $n = 1,2,3,\cdots$ then p is a cluster point of $\{x_n\}$. If we do not require $p \in g(n,y_n)$ then X is called a γ-*space*. (Note: a space is called a β-*space* if it satisfies (i) and if $p \in g(n,x_n)$ for $n = 1,2,\cdots$, then the sequence $\{x_n\}$ has a cluster point. If the cluster point is specified to be p then the space is semi-stratifiable. Some generalizations are *quasi*-β, defined in [F9], and *monotonic* β and *monotonic semistratifiable* spaces as defined in [C3]. If we replace (ii) by: if $\{p,x_n\} \subseteq g(n,y_n)$ for $n = 1,2,\cdots$ then p is a cluster point of the sequence $\{x_n\}$, then we have a characterization of developable spaces.)

(c) an O-*metric* space if there is a function ρ mapping $X \times X$ into the set of nonnegative reals such that $\rho(x,y) = 0$ iff $y = x$ and a set $F \subset X$ is closed iff $\rho(x,F) > 0$ whenever $x \notin F$. X is *quasimetric* (*nonarchimedean quasimetric*) if, in addition, for $x,y,z \in X$ the triangle inequality $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ holds, ($\rho(x,z) \leq \max\{\rho(x,y),\rho(y,z)\}$ holds).

$$(\sigma - DJ) \longrightarrow (\sigma - PF) \longrightarrow \sigma - Q \longrightarrow QM \longrightarrow \gamma \longrightarrow \theta$$

$$\{UB\} \nearrow (\sigma - PF)$$

The QM-axis. Parentheses {brackets} indicate parts of the point-countable configuration {developable configuration}.

Heath first characterized developable spaces and Nagata spaces by the method in Definition III (b) and Hodel [H11] summarized the study of these spaces and introduced new characterizations and new axioms, including β, γ and θ. Chaber [C3] studied monotonic generalizations of some of these. The axiom β is not a base axiom and will be studied in connection with the metrization of costratifiable base axioms. The axioms σ-Q, γ and θ are important in regard to quasimetrization and quasiuniform structures. Sion and Zelman [S3] and Norman [N6] proved that having a σ-point finite base was a sufficient but not necessary condition for the quasimetrizability of a T_3 space.

The σ-Q base was probably first studied by Cedar [C1] in a paper in which he showed that an M_1 space with a σ-Q base is metrizable. Fletcher and Lindgren [F10] showed that, for a T_1 space, the existence of a σ-Q base is equivalent to the existence of a non-archimedean quasi-metric, which in turn is equivalent to having a compatible transitive quasi-uniformity with a countable base. Kofner [K1] proved similar results and the result that having a σ-Q base is equivalent to satisfying Hodel's strong first axiom of countability.

Fletcher and Lindgren [F7] also showed that a topological space is a γ-space iff it either:

(a) is a co-Nagata space; or

(b) admits a local quasi-uniformity with a countable base; or

(c) is a first countable Nagata space; or

(d) has an O-metric such that (1) for $n \in N$, $g(n,x) = \{y \mid \rho(x,y) < 1/2^n\}$, and (2) if F is a closed subset of X, K is a compact subspace of X, and $F \cap K = \emptyset$, then $\rho(F,K) > 0$; or

(e) satisfies (d), (1) and (2), and the property that if $\lim_{n\to\infty} \rho(x,x_n) = 0$ and $\lim_{n\to\infty} \rho(x_n,y_n) = 0$, then $\lim_{n\to\infty} \rho(x,y_n) = 0$.

Using (e), Nedev and Ĉhoban [N5] proved that γ-spaces are preserved under perfect maps.

The metrization of γ-spaces has been studied by Martin [M1] using generalizations of Nagata spaces. Previously, Hodel [H10] had shown that a T_3 γ-space is metrizable iff it is a Nagata space or an M_3 space, and that a γ-space is developable iff it is a β-space.

In [H11] Hodel introduced θ-spaces and showed that semi-stratifiable θ-spaces are developable. Fletcher and Lindgren have made an extensive study of θ-spaces and their equivalences, and their relationships with other base axioms and linearly ordered spaces. For instance, a space with a primitive base is a θ-space, and quasi-β-spaces that are θ-spaces are BCO spaces. All but four of the other costratifiable base axioms are stronger than this axiom. We might note that, unfortunately, the term θ-base is used in connection with QD-spaces which are considerably stronger than θ-spaces. Nyikos [N8] has shown that σ-Q is equivalent to θ if X is a paracompact OB-space.

6. Conversion Patterns

Definition 4. A family of sets is

(a) σ-0 if it is σ-disjoint,

(b) σ-1 if it is σ-relatively locally finite, i.e., if it is the union of a denumerable number of families, each locally finite at each point of its union,

(c) σ-2 if it is σ-point finite, and

(d) σ-3 if it is point countable.

A topological space is σ-k *refinable* if every open cover has a σ-k refinement. A topological space is CN-k if, for every discrete family $\{D_a\}$, there is a σ-k family $\{G_a\}$ such that $D_a \subset G_a$ for each a, and $D_b \cap G_a = \emptyset$ for $b \neq a$. HCN-k will indicate that every subspace is CN-k.

Theorem 7. [In [A12] or [A13] except as noted and except for results on Q.]

Consider the following properties:

(a) X *has a* σ-k *base,*

(b) X *is* σ-k *refinable,*

(c) X *is hereditarily* σ-k *refinable,*

(d) X *is* CN-k, *and*

(e) X *is hereditarily* CN-k *refinable.*

Then, (1) for any space X, *(a)* $\rightarrow$ *(c)* $\rightarrow$ *(b) and (e)* $\rightarrow$ *(d);*

(2) if k = 1, *and* X *is normal and hereditarily countably paracompact, then (c)* $\rightarrow$ *(e);*

(3) if k = 2 *and* X *is hereditarily metacompact then (c)* $\rightarrow$ *(e);* (based on [N2] and [S5])*;*

(4) if k = 3 *then (c)* $\rightarrow$ *(e);*

(5) for BCO *spaces, (c)* $\rightarrow$ *(a) for* k = 0, 1, *and* 2*;*

(6) for developable spaces, [A13]*, (d)* $\rightarrow$ *(a) for* k = 0,1,2,3 *and* Q ([F9]) *and (c)* $\rightarrow$ *(a) for* k = 3 *and* Q*;*

(7) for QD-*spaces, (c)* $\rightarrow$ *(a) and (e)* $\rightarrow$ *(a);*

(8) if X *is a* σ-LCB *space then (c)* $\rightarrow$ *(a) for* k = 0,1,2,3*;*

(9) if X *is* σ-LCB *and is weakly* θ*-refinable* (subparacompact) *then* X *is* QD, [A12] *or* [S2], (developable, [F5]).

We note that it is proved in [L1] that orthocompact, hereditarily orthocompact and OB are equivalent in developable spaces. As a consequence, σ-QB and OB are equivalent in a developable space, [L1].

Some of the bases are known to have the property that if $\mathcal{U}$ is an open cover of a space such that each member has the base property in the relative topology, then the space has the base property. This is true of BCO spaces, primitive bases and θ-spaces. This is not true for developable spaces or spaces with point-countable bases.

Theorem 8. *Locally* P → *globally* P, *if* P *is* BCO, *primitive base or* θ. *If a space is* θ*-refinable* (*metacompact*) *then locally* P → *globally* P, *if* P *is developable* (if P is UB) [W7]. *If* X *is hereditarily weakly* θ*-refinable* (*hereditarily* σ*-metacompact*) *and locally* β, *then locally* QD → QD (*locally* σ*-PF* → σ*-PF*).

The last results follows from the fact that QD → θ. Also, since θ is hereditary and β is preserved with respect to closed sets, then X is locally BCO by a result of Fletcher and Lindgren, so that X is BCO and, by a result of Bennett and Berney, is QD. The result for σ-PF bases follows then from Theorem 7.

7. Metrization

Theorem 9. *The following are satisfied for costratifiable base axioms.* (For OB, first countability is added.)

(1) T_3 + β + *paracompact* + *base axiom* ⇔ *metrizable;*

(2) β + θ*-refinable* + *base axiom* → *developable;*

(3) T_3 + *semistratifiable* + *base axiom* + *collectionwise normal* ⇔ *metrizable;* (semistratifiable spaces were introduced in [C9]); *and,*

(4) T_3 *+ semistratifiable + base axiom → developable.*

Proof. It will be sufficient to show that these are satisfied for $\delta\theta$-bases, WUB and θ-spaces. (1), (3), and (4) have been discussed for a $\delta\theta$-base. (2) follows from Chaber's result [C3] that monotonic B + $\delta\theta$-base ⇒ BCO. Heath and Lindgren showed that (3) and (4) are satisfied for a WUB, since a WUB is a $\delta\theta$-base if the isolated points form an F_σ-set. Hodel [H9] has shown that a θ-refinable β-space with a weak G_δ-diagonal is semistratifiable. It follows that (1) and (2) are also satisfied for a WUB base since WUB spaces have a G_δ-diagonal. Hodel [H11] has shown that (3) and (4) are satisfied for θ-spaces. (1) and (2) follow from the result of Fletcher and Lindgren [F9] that a T_3, quasi-β, θ-space has a BCO. Lindgren and Nyikos [L1] have shown that a semi-stratifiable space with an orthobase is developable and Phillips [P1] has shown that a first countable monotone β-space with a monotonic orthobase has a BCO; so (1), (2), (3), and (4) are satisfied for first countable orthobases. We are using, in most of these results, Bing's result that collectionwise T_4 developable spaces are metrizable and Wicke and Worrell's result [W4] that θ-refinable BCO spaces are developable.

Many metrization theorems involving M-spaces [M6], σ-spaces [O1], Σ-spaces [N3], and M_i spaces [B9], follow from (1) and (3) since all of these spaces are β-spaces [H12], and some are semistratifiable. It is clear from (1) that a compact T_3 space satisfying any of the costratifiable base axioms is metrizable. In case of an orthobase it follows from Σ + OB + T_3 + paracompact $\Rightarrow$ metrizability, [N7]. In [H11] Hodel showed that a space that satisfies both β and γ is developable. Hence a countably compact T_3 γ-space is metrizable. As a consequence of a theorem of Worrell and Wicke announced in 1966 that all countably compact weak $\delta\theta$-refinable spaces are compact, it follows that countably compact T_3-spaces with a $\delta\theta$-base are metrizable. Proofs of this theorem have been obtained by Wicke and Worrell [W5], Burke and Lutzer [B13], and Chaber [C3]. Heath and Lindgren proved that a countably compact T_3-space with a WUB is metrizable. The space of countable ordinals is an example of a countably compact T_3 BCO-space that is not metrizable. The question is unsettled for OB spaces. Thus, for all bases except BCO, primitive bases, θ-spaces, and OB-spaces:

(5) *countably compact* + T_3 + *base axiom* $\Rightarrow$ *metrizable.*

Finally we note that with Smirnov's result that a T_3 locally metrizable space is metrizable iff it is paracompact, we can improve (1) as follows:

(6) T_3 + *locally* β + *locally base axiom* + *paracompact* $\Leftrightarrow$ *metrizability.* (*For* OB, *first countability added.*)

We note in connection with this result that β is closed hereditary, and that all the costratifiable base axioms are hereditary. For most base axioms this conclusion is immediate but for some, like BCO [W3], the proof is involved.

8. The Impact of Certain Countability Conditions

Countability conditions alone have little or no metrization effect on the axioms of the quasimetric axis. Hodel [H11] showed that the Sorgenfrey line has a σ-Q base, but it is well known to be a hereditarily Lindelöf, hereditarily separable, T_4 space. However, for any base axiom in the developmental configuration (OB not included):

(7) T_3 + *hereditary Lindelöf* + *base axiom* $\Rightarrow$ *metrizable.*

And, for any base axiom in the point-countable configuration we have:

(8) T_3 + *hereditarily separable* + *base axiom* $\Rightarrow$ *metrizable.*

There is an example consistent with V = L (Ostaszewski, [O2]), that is perfectly T_4, hereditarily separable, countably compact, and not metrizable and which, Burke and Lutzer [B13] point out, satisfies BCO. On the other hand, Tall [T1] has shown that, if $2^{\aleph_0} < 2^{\aleph_1}$, then there is a T_3 hereditarily Lindelöf space with a point-countable base that is not metrizable. Thus, even with the additional requirement of a co-stratifiable base, we are involved with the problem of the existence of T_3 S-spaces and T_3 L-spaces. In a T_3 QD space each of hereditary separability, hereditary Lindelöf and hereditary $\aleph_1$-compact is equivalent to separability and metrizability [B3]. We have already mentioned the impact of separability (local separability) in connection with developable and QD-spaces (point-countable bases).

9. Countable Products

Except for WUB and OB, all of the costratifiable bases are countably productive. One might note that WUB and OB are the two bases not satisfying the first axiom of countability. Heath proved that a WUB base is not finitely productive by showing that the product of any non first countable WUB with the real line does not have a WUB; if the product had a WUB, then being without isolated points it would satisfy the first axiom of countability. The cartesian product of an example of Corson and Michael (which has a σ-disjoint base that is an OB and has a G_δ-diagonal, but which is not developable) with the real line does not have an OB, since an OB space with a G_δ-diagonal and no isolated points is developable. Lindgren first noted this. The countable productivity of primitive bases [W3] and BCO's [W4] are due to Wicke and Worrell, QD's to Bennett, θ-spaces to Fletcher and Lindgren [F9], and γ-spaces to Abernathy [A1]. The proofs for $\delta\theta$-base, PCB (folklore), σ-PF, σ-DJ, σ-LC and σ-Q are straightforward. The results for developable spaces (folklore) and uniform bases can be obtained from the countable productivity of QD's and σ-PF bases respectively, using the countable productivity of σ-spaces [O1].

10. Perfect Maps, Compact Open Maps and the Class MOBI

Stone [S6] and Morita and Hanai [M7] proved that metrizability is preserved under a perfect map. Worrell [W10] proved the analogous theorem for developable spaces and Fillipov [F2] proved it for PCB's and σ-PF bases. We note that a proof of Fillipov's last result can be found in [B11]. If we combine Worrell's result with these results, we have that uniform bases are preserved under perfect maps, a theorem of Arhangel'skiǐ [A7]. Heath and Reed [B11] have an example of a space with a σ-DJ base that is not preserved under perfect maps; and Heath and Lindgren [H7], using an example of Slaughter [S4], have shown that WUB bases are not preserved under perfect maps. Nyikos has shown that OB's are not preserved under these maps. We have already mentioned that Wicke and Worrell showed that perfect maps preserve BCO's. Burke has announced that σ-LCB's are also preserved.

Nedev and Ĉhoban [N5] have proved that perfect mappings preserve γ-spaces, using formulations of Fletcher and Lindgren. Below we show that σ-Q bases are preserved under these maps. (Compare with [S1], Lemma 2.4.) At present the question is unsettled for θ-spaces, primitive bases, δθ-bases, σ-LCB bases, and QD's; Burke [B11] has proved some special cases involving the latter.

Theorem 10. *The following types of bases are preserved under perfect maps* (i.e. under a perfect map the image of a space with a certain type of base also has this same type base):

development, UB, PCB, σ-PF, BCO, γ, σ-Q, *and* σ-LCB. *The following are not:* σ-DJ, WUB, *and* OB.

Proof that σ-Q *spaces are preserved under a perfect map:* Let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ be a σ-Q base, where each $\mathcal{B}_n$ is a Q-family, for the space $(X,\mathcal{J})$. If $\mathcal{V}_n = \{\cup\{B: B \in S\}: S \subset \bigcup_{k=1}^{n} \mathcal{B}_k\}$, then $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ is a σ-Q base for $(X,\mathcal{J})$. Also for K compact, $T \in \mathcal{J}$, and $K \subset T$, there is $V \in \mathcal{V}$ such that $K \subset V \subset T$. We note that each $\mathcal{V}_n$ is a Q family. Let f be the perfect map from $X \to Y$. Then $\mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n$ where $\mathcal{W}_n = \{W = Y \sim f(\sim V): V \in \mathcal{V}_n\}$ is a σ-Q base for $\mathcal{U}$, the topology of Y. Let $y \in U \in \mathcal{U}$. Then there exists $V \in \mathcal{V}$ such that $f^{-1}(y) \subset V \subset f^{-1}(U)$, and $y \in Y \sim f(\sim U) \subset U$.

Under compact open maps, primitive bases in a T_3 space [W3], BCO's [W4], and PCB's [P3], are known to be preserved. It is known that uniform bases [A6], developable spaces [W8], σ-DJ bases, and σ-PF bases are not preserved; under these maps, as previously mentioned, the minimal class of T_1 spaces containing all metric spaces and closed under open compact mappings is known as Arhangel'skiǐ's class MOBI. A member of the class MOBI has a point-countable base and, if T_3, it has a BCO, but it may not be developable, metacompact or have a G_δ-diagonal [C4], even if it is Tychonoff. Hence WUB's are not preserved under compact open maps. It is unknown whether members of the class MOBI are QD, [B2]. For a discussion of the class analogous to the class MOBI that contains all developable spaces, see [N2]. See [Gl] for results on other types of open maps.

11. Examples

The countable ordinals satisfy BCO but not γ, $\delta\theta$-B, or WUB. The Sorgenfrey line [H11] satisfies σ-Q but does not have a primitive base, $\delta\theta$-base, or WUB. Corson and Michael [C8] have a hereditary paracompact T_3 Lindelöf space with a σ-disjoint base, an OB, and a WUB, which is not metrizable and hence does not have a BCO. The product of this space with the reals is not OB. Slaughter's [S4] finite-to-one closed image of the Michael line, being hereditarily paracompact and having a σ-point finite base, has a σ-DJ and is not BCO and, since it does not have a G_δ diagonal, it does not have a WUB [H8].

Heath [H6] has constructed an example of a $T_{3\frac{1}{2}}$ space with a UB that doesn't have σ-DJ base or a σ-LC base. We have mentioned the existence, under certain set theoretic assumptions, of a T_3 hereditarily Lindelöf PCB space (hereditarily separable BCO-sapce) which is not metrizable and hence does not have a primitive base (does not have a $\delta\theta$-base). The author's [A16] extension of an example of Miščenko is a hereditarily paracompact T_2-space with a point-countable base that does not have a BCO. Reed [R3] has constructed a Moore space without a PCB, and one with a PCB that does not have a UB. Gruenhage [G3] has an example of a PCB, OB space which is not a θ-space. Hodel [H12] has shown that an example of a Moore space of Heath's [H3] that is not quasimetrizable, does not satisfy γ. Kofner [K1] has constructed an example of a quasimetric space that does not have a σ-Q base. Gruenhage has constructed an example of σ-LCB, σ-DJ space which is not developable, since it is not perfect.

12. Some Questions

1) For all base axioms such that countably compact T_3 + base axiom $\Rightarrow$ metrizable, is it true that $T_3 + \beta$ + collectionwise normal + base axiom $\Rightarrow$ metrizability? It is true for γ-spaces and developable spaces but what about QD, $\delta\theta$-B, and PCB spaces?

2) Does there exist a reasonable costratifiable base

axiom weaker than all the axioms in this paper?

3) Is a collectionwise normal T_3 space with a σ-LCB, metrizable? Are σ-LCB spaces also QD-spaces?

4) Does there exist a normal nonmetrizable Moore space without using any axioms of set theory beyond the axiom of choice, as conjectured by Rudin [R5]?

5) In regard to ortho-bases there are many unsolved problems. Here are some, essentially from [L1]. Is every collectionwise normal space with an OB, paracompact? Related to this is the question of whether every countably compact space with an OB is compact. Given an OB space, does $\gamma \to$ QM $\to \sigma^*$-space (with θ-refinable added)? Is there a model of set theory in which every normal space with an ortho-base is paracompact?

6) In regard to WUB we have the following questions from [H7]. Is every first countable space with a WUB, QD? Does every WUB developable space without isolated points have a UB? In [D1] it is asked if there is an example using only ZFC of a normal Moore space with a WUB that has no PCB.

7) Are QM-spaces preserved under perfect maps? Related to this question, is every γ-space quasimetrizable? See [F7] for the multiple source of these questions. Are γ and QM spaces preserved under compact open maps [G1]?

8) Are any of the following bases or spaces preserved under perfect mapping (a) QD [B11], (b) $\delta\theta$-B, (c) θ, or (d) PRIM? But, of more importance, can one set up a unified theory for perfect mappings of costratifiable base axioms?

9) In regard to compact open mappings the question of preservation has been raised for the first three bases in

the previous question and (d) σ-Q bases, (e) OB-spaces, and (f) WUB [H7]. In particular, is the class MOBI, QD [B2]? See [G1] for other open mapping questions.

Note: in regard to mapping questions it is sometimes difficult to assign a proposer because such natural questions may have multiple sources. Where we know a source of a question we will indicate it, though there may be other sources.

13. Acknowledgements and Conclusions

The author wishes to acknowledge that certain survey papers such as Burke and Lutzer [B13], Gitting [G1], M. E. Rudin [R5], and the reviews by G. M. Reed were extremely helpful in preparing this manuscript. The author benefitted from conversations or (and) correspondence with J. Chaber, W. G. Fleissner, P. Fletcher, G. Gruenhage, R. E. Hodel, H. J. Junilla, W. F. Lindgren, P. J. Nyikos and H. Wicke.

Many of the base axioms are of relatively minor importance and their future in general topology will depend on the development of an interesting mapping theory. Some spaces of major importance are developable spaces, BCO's and PCB's; developable spaces have proved to be a stimulus in topology for over a half century and have, particularly in recent years, provided a gateway for the use of set theory in mathematics. The mapping theory for BCO's and PCB's is very interesting. If you exclude the normal Moore space conjecture, one of the most interesting questions is whether γ = QM. The reader should bear in mind that some sections of this paper will be obsolete by the time of publication, due to the activity in this field. The author had to make many changes while writing due to new results.

14. Postscript

Professor E. W. Chittenden, a very important mathematician in the early history of general topology, died on June 16, 1977 at the age of 91. He was probably best known for his classic paper [C5] "On the equivalence of écarte and voisinage" the results from which the Alexandroff-Urysohn metrization theorem is an easy consequence. In this paper Chittenden showed, as conjectured by Fréchet in 1910, that every Frechet V-space is metrizable. In the V-space definition, the triangle axiom is replaced by the formally weaker condition that there exists a positive function f, with $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$, such that if $d(x,y) < \varepsilon$ and $d(y,z) < \varepsilon$, then $d(x,z) < f(\varepsilon)$.

Appendix

Since the survey was submitted for publication many new results have been obtained.

1. In regard to perfect maps, Burke (notices A.M.S. 1978) has shown that T_2 QD's and primitive bases are preserved under perfect maps. Kofner (to appear in Pac. J. Math.) has shown that QM's are also preserved under these maps. He also had obtained the result on σ-Q bases earlier than the author (Theorem 10(g)).

2. Burke showed that a θ-refinable σ-LCB T_3-space is developable.

3. More results have been obtained on the normal Moore space problem by Rudin and Starbird, Nyikos, Shelah and Reed.

4. Wicke (notices A.M.S. 1978) has studied costrati-

fiable bases that generalize both θ-spaces and spaces with PCB and one that also generalizes δθ-bases.

References

A1. Abernathy, *On characterizing certain classes of first countable spaces by open mapping*, Pac. J. Math. 53 (1974), 319-326.

A2. P. S. Aleksandrov, *On metrization of topological spaces*, Bull. Acad. Pol. Sci. Math. 8 (1960), 589-595 (Russian English Summary).

A3. ________, *On some results in the theory of topological spaces in the last 25 years*, Russian Math. Surveys 15 (1960), 127-134.

A4. ________ and P. Urysohn, *Memoire sur les espace topologiques compacts*, Verh. Akad. Wetensch. Amsterdam, 14 (1929), 1-96.

A5. ________, *Une condition necessaire and suffisante pour qu'un espace {L} soit un class {D}*, C.R. Ac. Sc. 177 (1923), 1274-1276.

A6. K. Alster and R. Pol, *Moore spaces and collectionwise Hausdorff property*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. 23 (1975), 1189-1192 (MR 52#15383).

A7. A. V. Arhangel'skiǐ, *Mappings and spaces*, Russian Math. Surveys 21 (1966), 115-162.

A8. ________, *On mappings of metric spaces*, Soviet Math. Dokl. 3 (1962), 953-956.

A9. ________, *Open and near open mappings. Connections between spaces*, Trans. Moscow Math. Soc. 15 (1966), 204-250.

A10. ________, *Some metrization theorems*, Uspehi Math. Nauk 18 (1963), 139-143.

A11. N. Aronszsajn, *Uber Die Bogenverknüpung in Topologischen Räumen*, Fund. Math. 15 (1930), 228-241.

A12. C. E. Aull, *Point-countable bases and quasi-developments, General topology and its relations to modern analysis and algebra III*, Proceedings of the Third Prague Topological Symposium 1971. Academic Press

(1972), 47-50.

A13. ________, *Quasi-developments and δθ-bases*, J. of Lond. Math. Soc. 9 (1974), 197-204.

A14. ________, *Some base axioms involving enumerability*, Proc. of Topology Conference, Kanpur, India, 1968. Academic Press (1971), 55-61.

A15. ________, *Some properties involving base axioms and metrizability*, Topo 72, Springer Verlag Lecture Notes #378, pp. 42-45.

A16. ________, *Topological spaces with a σ-point finite base*, Proceedings AMS 29 (1971), 411-416.

A17. ________, *A semistratifiable is developable iff it has a δθ-base*, Abstract 74T-G86, Notices Amer. Math. Soc. 21 (1974), A-504.

B1. H. R. Bennett and E. S. Berney, *Spaces with a σ-minimal base*, Topology Proceedings II (1977), 1-10.

B2. H. R. Bennett, *On Arhangel'skiǐ's class MOBI*, Proc. Amer. Math. Soc. 26 (1970), 178-180.

B3. ________, *On quasi-developable spaces*, Gen. Top. & Appl. 1 (1971), 253-262.

B4. ________ and E. S. Berney, *On certain generalizations of developable spaces*, Gen. Top. Appl. 4 (1974), 43-50.

B5. H. R. Bennett and D. J. Lutzer, *A note on weak θ-refinability*, Gen. Top. & Appl. 2 (1972), 49-54.

B6. ________, *Ordered spaces with σ-minimal bases*, Topology Proceedings II (1977), 371-382.

B7. R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. 3 (1951), 175-186.

B8. C. R. Borges, *On stratifiable spaces*, Pac. J. Math. 17 (1966), 1-16.

B9. ________, *A survey of M_i-spaces, open questions and partial results*, Gen. Top. & Appl. 1 (1971), 79-84.

B10. ________ and D. Lutzer, *Characterizations and mappings of M_i-spaces*, Topology Conference VPI, Springer-Verlag Lecture Notes in Mathematics #375, pp. 34-40.

B11. D. K. Burke, *Preservation of certain base axioms under a perfect mapping* (to appear).

B12. ________, R. Engelking, and D. Lutzer, *Hereditarily closure-preserving collections and metrization*, Proc. Amer. Math. Soc. 51 (1975), 483-488.

B13. D. K. Burke and D. J. Lutzer, *Recent advances in the theory of generalized metric spaces*, Topology Proceedings of the Memphis State Topology Convergence, Marcel Dekker, New York, 1976.

C1. J. Ceder, *Some generalizations of metric spaces*, Pac. J. Math. 11 (1961), 105-125.

C2. J. Chaber, *Metacompactness in the class MOBI*, Fund. Math. 91 (1974), 211-217.

C3. ________, *On point-countable collections and monotonic properties*, Fund. Math. 94 (1977), 209-219.

C4. ________, M. M. Čoban, and K. Nagami, *On monotonic generalizations of Moore spaces, Čech complete spaces and* p*-spaces*, Fund. Math. 83 (1974), 107-119.

C5. E. W. Chittenden, *On the equivalence of écarte and voisinage*, Trans. Amer. Math. Soc. 18 (1917), 161-166.

C6. ________, *On the metrization problem and related problems in the theory of abstract sets*, Bull. Amer. Math. Soc. 33 (1927), 13-34.

C7. ________ and A. D. Pitcher, *On the theory of developments of an abstract class in relation to the calcul functionnel*, Trans. Amer. Math. Soc. 20 (1919), 213-233.

C8. Corson and Michael, *Metrizability of certain countable unions*, Illinois J. Math. 8 (1964), 351-360.

C9. G. Creed, *Concerning semistratifiable spaces*, Pac. J. Math. 32 (1970), 47-54.

D1. S. W. Davis, G. M. Reed, and M. L. Wage, *Further results on weakly uniform bases*, Houston J. Math. 2 (1976), 57-63.

F1. V. V. Fedorčuk, *Ordered sets and the product of topological spaces* (Russian), Vestnik Moskov. Univ. Ser. I Mat. Meh. (4) 21 (1966), 66-71.

F2. V. V. Filippov, *Preservation of the order of a base under a perfect mapping*, Soviet Math. Dokl. 9 (1968), 1005-1007.

F3. W. G. Fleissner, *Separation properties in Moore spaces*, Fund. Math. 98 (1978), 279-286.

F4. ________, *Normal Moore spaces in the constructible universe*, Proc. Amer. Math. Soc. 46 (1974), 244-248.

F5. ________, *When normality implies collectionwise Hausdorff*, Thesis Univ. of Cal., Berkeley, Cal., 1974.

F6. ________ and G. M. Reed, *Paralindelöf spaces and σ-locally countable bases*, Topology Proceedings II (1977), 89-110.

F7. P. Fletcher and W. F. Lindgren, *Locally quasi-uniform spaces with countable bases*, Duke Math. J. 41 (1974), 231-240.

F8. ________, *Orthocompactness and strong Čech completeness in Moore spaces*, Duke Math. J. 39 (1972), 753-766.

F9. ________, *θ-spaces*, Gen. Top. & Appl. 9 (1978), 139-153.

F10. ________, *Transitive quasiuniformities*, J. Math. Anal. App. 39 (1972), 397-405.

G1. R. F. Gittings, *Open mapping theory, Set theoretic topology*, Academic Press, New York, 1977.

G2. J. W. Green, *Moore-closed and locally Moore-spaces, Set theoretic topology*, Academic Press, New York, 1977.

G3. G. Gruenhage, *A note on quasimetrizability*, Can. J. Math. 29 (1977), 360-366.

G4. ________, *Stratifiable spaces are* M_2, Topology Proceedings I (1976), 221-226.

H1. S. Hanai, *On open mappings, II*, Proc. Jap. Acad. 37 (1961), 233-238.

H2. R. W. Heath, *A non-pointwise paracompact Moore space with a point countable base*, Notices Amer. Math. Soc. 10 (1963), 649-650.

H3. ________, *A postscript to a note on quasi-metric spaces*, Notices Amer. Math. Soc. 19 (1972), A-338, Abstract 72T-G22.

H4. ________, *On open mappings and certain spaces satisfying the first axiom of countability*, Fund. Math. 57 (1965), 91-96.

H5. ________, *On spaces with point-countable bases*, Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys. 13 (1965), 393-395.

H6. ________, *Screenability, pointwise paracompactness and metrization of Moore spaces*, Canad. J. Math. 16 (1964), 763-770.

H7. ________ and W. F. Lindgren, *Weakly uniform bases*, Houston J. Math. 2 (1976), 85-90.

H8. E. R. Hedrick, *On properties of a domain for which the derived set is closed*, Trans. Amer. Math. Soc. 12 (1911), 285-294.

H9. R. E. Hodel, *Metrizability of topological spaces*, Pac. J. Math. 55 (1974), 441-459.

H10. ________, *Some results in metrization theory, 1950-72*, Topology Conference, Virginia Polytechnic Institute and State University, Springer Verlag Lecture Notes in Mathematics #375 (1974), 120-136.

H11. ________, *Spaces defined by sequences of open covers which guarantee that certain sequences have cluster points*, Duke Math. J. 39 (1972), 253-263.

H12. ________, *Spaces characterized by sequences of covers which guarantee that certain sequences have cluster points*, Proceedings of the University of Houston Point Set Topology Conference 1971, Houston (1972), 105-114.

H13. W. N. Hunsaker and W. F. Lindgren, *Spaces with σ-disjoint bases*, Gen. Top. Appl. 8 (1978), 229-232.

J1. F. B. Jones, *Concerning normal and completely normal spaces*, Bull. Amer. Math. Soc. 43 (1937), 671-677.

J2. H. J. Junnila, *Neighbornets*, Pac. J. Math. 76 (1978), 83-108.

K1. Y. A. Kofner, *On Δ-metrizable spaces*, Math. Notes 13 (1973), 168-174.

L1. W. F. Lindgren and P. J. Nyikos, *Spaces with bases satisfying certain order and intersection properties*, Pac. J. Math. 66 (1976), 455-476.

L2. D. J. Lutzer, *On quasi-uniform bases*, Proceedings of the University of Oklahoma Topology Conference 1972, 104-117.

M1. H. W. Martin, *A note on the metrization of γ-spaces*, Proc. Amer. Math. Soc. 57 (1976), 332-336.

M2. E. Michael, *On Nagami's Σ-spaces and other related* *matters*, Proceedings of the Washington State Topology Conference 1970, 13-19.

M3. A. Miščenko, *Spaces with point-countable bases*, Soviet Mathematics 3 (1962), 855-858.

M4. E. H. Moore, *Introduction to a form of general analysis*, Yale Univ. Press, New Haven 1910.

M5. R. L. Moore, *Foundations of point set theory*, AMS Colloq. Publ. 13 (1932; revised edition 1962).

M6. K. Morita, *A survey of the theory of* M-*spaces*, Gen. Top. & Appl. 1 (1971), 49-55.

M7. ________ and S. Hanai, *Closed mapping and metric spaces*, Proc. Jap. Acad. 32 (1956), 10-14.

N1. K. Nagami, *Minimal class generated by open compact and perfect mappings*, Fund. Math. 78 (1973), 227-264.

N2. ________, *Paracompactness and strong screenability*, Nagoya Math. J. 8 (1955), 83-88.

N3. ________, *Σ-spaces*, Fund. Math. 65 (1969), 169-192.

N4. J. Nagata, *Problems on generalized metric spaces, II*, Proc. Houston Topology Conf. (1971), 42-52.

N5. S. Ǐ. Nedev and M. M. Čhoban, *On the theory of* 0-*metrizable spaces II*, Vestnik Moskow, Univ. Ser. I. Mat. Meh., (2) 27 (1972), 10-17.

N6. L. J. Norman, *A sufficient condition for quasi-metrizability of a topological space*, Portugal. Math. 26 (1967), 207-211.

N7. P. J. Nyikos, *Some surprising base properties in topology*, *Studies in topology*, Academic Press, New York, 1975, 427-450.

N8. ________, *Some surprising base properties in topology II*, *Set theoretic topology*, Academic Press, New York, 1977, 277-305.

O1. A. Okuyama, *A survey of the theory of σ-spaces*, Gen. Top. & Appl. 1 (1971), 57-63.

O2. A. Ostaszewski, *On countably compact, perfectly normal spaces*, J. Lond. Math. Soc. 14 (1976), 505-516.

P1. T. M. Phillips, *A note on monotonic ortho-bases* (to appear).

P2. V. Ponomarev, *Axioms of countability and continuous* *mappings*, Bull. Polon. Sci. Ser. Math., Astr., Phys. 8 (1960), 127-133 (Russian).

P3. T. C. Przymusinski, *Normality and separability of Moore spaces, Set theoretic topology*, Academic Press, New York, 1977, 325-338.

P4. ________ and F. D. Tall, *The undecidability of the existence of a non-separable Moore space satisfying the countable chain condition*, Fund. Math. 85 (1974), 291-297.

R1. G. M. Reed, *Concerning first countable spaces II*, Duke Math. 40 (1973), 677-682.

R2. ________, *Concerning first countable spaces III*, Trans. Amer. Math. Soc. 210 (1975), 169-177.

R3. ________, *On the existence of point countable bases in Moore spaces*, Proc. Amer. Math. Soc. 45 (1974), 437-440.

R4. ________ and P. L. Zenor, *Preimages of metric spaces*, Bull. Amer. Math. Soc. 80 (1974), 879-880.

R5. M. E. Rudin, *Lectures on set-theoretic topology*, Providence, 1975.

S1. B. M. Scott, *Toward a product theory for orthocompactness, Studies in topology*, Academic Press, New York, 1975, 517-537.

S2. T. Shiraki, *A note on spaces with a uniform base*, Proc. Jap. Acad. 47 (1971), Supp. II, 1036-1041.

S3. M. Sion and G. Zelmer, *On quasi-metrizability*, Canad. J. Math. 19 (1967), 1243-1249.

S4. F. G. Slaughter, *A note on perfect images of spaces having a G_δ-diagonal* (preprint).

S5. J. C. Smith and L. Krajewski, *Expandability and collectionwise normality*, Trans. Amer. Math. Soc. 160 (1971), 437-451.

S6. A. H. Stone, *Metrizability of decomposition spaces*, Proc. Amer. Math. Soc. 7 (1956), 690-700.

T1. F. D. Tall, *On the existence of normal metacompact Moore spaces which are not metrizable*, Canad. J. Math. 26 (1974), 1-6.

T2. F. D. Tall, *Set-theoretic consistency results and topological theorems concerning the normal Moore space con-*

jecture and related problems, Thesis, University of Wisconsin, Madison, Wisconsin, 1969.

T3. R. Traylor, *Concerning metrizability of pointwise paracompact spaces*, Can. J. Math. 16 (1964), 407-411.

T4. J. W. Tukey, *Convergence and uniformity in topology*, Annals of Mathematical Studies, No. 2, Princeton, 1940.

W1. M. Wage, *A collectionwise Hausdorff non-normal Moore space*, Can. J. Math. 28 (1976), 632-634.

W2. H. H. Wicke, *Complete mappings in base of countable order theory, Set theoretic topology*, Academic Press, New York, 1977, 383-412.

W3. ________ and J. Worrell, *A characterization of primitive bases*, Proc. Amer. Math. Soc. 50 (1975), 443-450.

W4. ________, *Characterizations of developable topological spaces*, Canad. J. Math. 17 (1965), 820-830.

W5. ________, *Open continuous mappings of spaces having bases of countable order*, Duke Math. J. 34 (1967), 255-272.

W6. ________, *Point-countability and compactness*, Proc. Amer. Math. Soc. 55 (1976), 427-431.

W7. ________, *The local implies global characteristic of primitive sequences*, Topology Proceedings of the Memphis State Topology Conference, Marcel Dekker, New York, 1976, 269-282.

W8. ________, *The regular open continuous images of complete metric spaces*, Pac. J. Math. 23 (1967), 621-625.

W9. J. M. Worrell, *Local separable Moore spaces, Set-theoretic topology*, Academic Press, New York, 1977, 413-436.

W10. ________, *Upper semicontinuous decompositions of developable spaces*, Proc. Amer. Math. Soc. 16 (1965), 485-490.

Virginia Polytechnic Institute and State University

Blacksburg, Virginia 24061