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by

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## FUNDAMENTAL DIMENSION OF FIBERS OF APPROXIMATE FIBRATIONS

P. F. Duvall<sup>1</sup> and L. S. Husch

A map  $f: E \rightarrow B$  between metric spaces has the *approximate homotopy lifting property* (AHLP) for the space  $X$  if for each open cover  $\varepsilon$  of  $B$  for each homotopy  $H: X \times [0,1] \rightarrow B$  with a lifting  $h: X \rightarrow E$  such that  $fh(x) = H(x,0)$ , there exists a homotopy  $\tilde{H}: X \times [0,1] \rightarrow E$  such that  $\tilde{H}(x,0) = h(x)$  and  $f\tilde{H}(x,t)$  and  $H(x,t)$  are  $\varepsilon$ -close for all  $(x,t) \in X \times [0,1]$ . A map  $f: E \rightarrow B$  between ANR's is an *approximate fibration* if  $f$  has the AHLP for all spaces  $X$ . This concept was defined by Coram and Duvall [2] who showed that approximate fibrations satisfied many properties of Hurewicz fibrations if one used shape theoretic concepts instead of their homotopy theoretic analogues. In particular, if  $B$  is path connected, all fibers of an approximate fibration have the same shape. They were motivated by the work of Lacher [8] who showed that cell-like mappings between ANR's satisfied the AHLP for polyhedra. Coram and Duvall [3] have shown that this implies AHLP for all spaces and hence cell-like mappings between ANR's are approximate fibrations. A natural question which arose was whether the converse were true; since Hurewicz fibrations are approximate fibrations, some additional hypotheses were needed: if  $f: M \rightarrow N$  is a monotone approximate fibration between closed manifolds of the same dimension, is  $f$  a cell-like mapping? A related question suggested by the thesis of R. Goad [6] is

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that if  $f: M \rightarrow N$  is a monotone approximate fibration between closed manifolds and if  $\dim M = \dim N + 1$ , does the fiber of  $f$  have the shape of the 1-sphere? In this note, we show that the answer to both of these questions is yes. Our main result is the following:

*Theorem 1. Let  $f: M \rightarrow N$  be a proper approximate fibration from a manifold without boundary to a polyhedron with fiber  $F$ ; then  $Fd(F) = \dim M - \dim N$ . ( $Fd(F)$  is the fundamental dimension [1] or shape dimension of  $F$ ; i.e.,  $Fd(F) = \inf\{\dim X: \text{shape } F = \text{shape } X \text{ and } X \text{ is a metric compactum}\}$ .)*

*Corollary 2. Let  $f: M \rightarrow N$  be a proper monotone approximate fibration from a manifold without boundary to a polyhedron with fiber  $F$ .*

*a) If  $\dim M = \dim N$  then  $f$  is cell-like and, hence cellular if  $n \geq 5$  or  $n = 3$  and  $M$  contains no false 3-cells.*

*b) If  $\dim M = \dim N + 1$ , then  $F$  has the shape of  $S^1$ .*

*c) If  $\dim M = \dim N = i$ ,  $i = 2, 3$ , and if  $F$  is shape 1-connected, then  $F$  has the shape of the  $i$ -sphere.*

J. Hollingsworth has indicated to the authors that he has a proof of part a) of Corollary 2.

*Corollary 3. Suppose that  $f: M \rightarrow N$  is a proper approximate fibration between connected  $n$ -manifolds without boundary,  $n \geq 5$  or  $n = 3$  and  $M$  contains no false 3-cells. If  $\varepsilon$  is an open cover of  $N$ , then there is a covering projection  $p: M \rightarrow N$  such that  $f$  and  $p$  are  $\varepsilon$ -close.*

*Proof.* Choose base points  $m_0$  and  $n_0$  such that  $f(m_0) = n_0$  and let  $\rho: \tilde{N} \rightarrow N$  be the covering projection corresponding to

image  $f_*(\pi_1(M, m_0))$ . Then  $f$  lifts to a map  $\tilde{f}: M \rightarrow \tilde{N}$ . It is not difficult to check that  $\tilde{f}$  is an approximate fibration. We wish to show that  $\tilde{f}$  is monotone. Suppose not. Let  $X_1$  and  $X_2$  be distinct components of  $F_0 = \tilde{f}^{-1}(\tilde{f}(m_0))$  and let  $\mathcal{U}$  be a neighborhood of  $F_0$  such that  $X_1$  and  $X_2$  lie in distinct path components of  $\mathcal{U}$ . Assume without loss of generality that  $m_0 \in X_1$ , let  $q \in X_2$  be any point, and let  $\alpha: I \rightarrow M$  be a path from  $m_0$  to  $q$ . Since  $f\alpha$  is a loop based at  $n_0$  which represents an element of  $f_*\pi_1(\tilde{N})$ , there is a loop  $\beta$  based at  $m_0$  such that  $f\alpha$  and  $f\beta$  are homotopic rel( $n_0$ ). It follows from the AHLF that  $m_0$  and  $q$  can be joined by a path in  $\mathcal{U}$ , a contradiction.

By Corollary 2 and [10] it follows that  $\tilde{f}$  can be approximated by homeomorphisms, so that  $p\tilde{f} = f$  can be approximated by covering projections.

Theorem 1 follows essentially from two recently proved results of Husch and Nowak, respectively:

*Theorem 4 [7]. Let  $f: M \rightarrow N$  be an approximate fibration from a closed manifold to a polyhedron with fiber  $F$ . If  $F$  has the shape of a finite complex and if  $\dim M - \dim N \geq 6$ , then  $F \times T^{n+1}$  has the shape of a closed manifold of dimension  $m + 1$  where  $m = \dim M$  and  $n = \dim N$ .*

$T^r$  is the product of  $r$  1-spheres.

*Theorem 5 [9]. If  $X$  is a continuum which has the shape of a CW-complex, then  $Fd(X \times Y_1 \times Y_2 \times \cdots \times Y_n) = Fd(X) + n$  for all continua  $Y_1, Y_2, \dots, Y_n$  such that  $Fd(Y_i) = 1$ ,  $i = 1, 2, \dots, n$ .*

*Proof of Theorem 1.* Let  $f: M \rightarrow N$  be an approximate fibration with fiber  $F$  and suppose  $\dim M = m$  and  $\dim N = n$ . By [2],  $F$  is an FANR and by [5],  $F \times S^1$  has the shape of a finite complex. Let  $r = \max\{1, 6 - (m - n)\}$  and define  $f: M \times T^r \rightarrow N$  to be the composition of the projection  $M \times T^r \rightarrow M$  and  $f$ . It is easily checked that  $\tilde{f}$  is an approximate fibration and, hence, by Theorem 4,  $F \times T^{r+n+1}$  has the shape of a closed manifold of dimension  $m + r + 1$ . Therefore  $\text{Fd}(F \times T^{r+n+1}) = m + r + 1$ ; by Theorem 5,  $\text{Fd}(F) = m - n$ .

Since  $F$  is an FANR, there exists a CW complex  $K$  such that  $\text{shape } K = \text{shape } F$  with  $\dim K = \text{Fd}(F)$  if  $\text{Fd}(F) \neq 2$  and  $\dim K = 3$  if  $\text{Fd}(F) = 2$  [4]. Hence  $K \times T^{r+n+1}$  has the homotopy type of a closed manifold. By Theorem 2.5 of [11],  $K$  is a Poincaré complex of formal dimension  $m - n$ . Corollary 2 now follows from Theorem 4.2 of [11].

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