TOPOLOGY PROCEEDINGS
Volume 3, 1978
Pages 53–57

http://topology.auburn.edu/tp/

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by

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Topology Proceedings

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ISSN:	0146-4124

FUNDAMENTAL DIMENSION OF FIBERS OF APPROXIMATE FIBRATIONS

P. F. Duvall[1] and L. S. Husch

A map f: E → B between metric spaces has the *approximate homotopy lifting property* (AHLP) for the space X if for each open cover ε of B for each homotopy H: X × [0,1] → B with a lifting h: X → E such that fh(x) = H(x,0), there exists a homotopy $\tilde{H}$: X × [0,1] → E such that $\tilde{H}(x,0) = h(x)$ and $f\tilde{H}(x,t)$ and H(x,t) are ε-close for all (x,t) ∈ X × [0,1]. A map f: E → B between ANR's is an *approximate fibration* if f has the AHLP for all spaces X. This concept was defined by Coram and Duvall [2] who showed that approximate fibrations satisfied many properties of Hurewicz fibrations if one used shape theoretic concepts instead of their homotopy theoretic analogues. In particular, if B is path connected, all fibers of an approximate fibration have the same shape. They were motivated by the work of Lacher [8] who showed that cell-like mappings between ANR's satisfied the AHLP for polyhedra. Coram and Duvall [3] have shown that this implies AHLP for all spaces and hence cell-like mappings between ANR's are approximate fibrations. A natural question which arose was whether the converse were true; since Hurewicz fibrations are approximate fibrations, some additional hypotheses were needed: if f: M → N is a monotone approximate fibration between closed manifolds of the same dimension, is f a cell-like mapping? A related question suggested by the thesis of R. Goad [6] is

[1] Research supported in part by an N.S.F. grant.

that if f: $M \to N$ is a monotone approximate fibration between closed manifolds and if dim M = dim N + 1, does the fiber of f have the shape of the 1-sphere? In this note, we show that the answer to both of these questions is yes. Our main result is the following:

Theorem 1. *Let* f: $M \to N$ *be a proper approximate fibration from a manifold without boundary to a polyhedron with fiber* F*; then* Fd(F) = dim M - dim N. (Fd(F) is the fundamental dimension [1] or shape dimension of F; i.e., Fd(F) = inf{dim X: shape F = shape X and X is a metric compactum}.)

Corollary 2. *Let* f: $M \to N$ *be a proper monotone approximate fibration from a manifold without boundary to a polyhedron with fiber* F.

a) If dim M = dim N *then* f *is cell-like and, hence cellular if* $n \geq 5$ *or* n = 3 *and* M *contains no false* 3-*cells.*

b) If dim M = dim N + 1, *then* F *has the shape of* S^1.

c) If dim M = dim N = i, i = 2,3, *and if* F *is shape* 1-*connected, then* F *has the shape of the* i-*sphere.*

J. Hollingsworth has indicated to the authors that he has a proof of part a) of Corollary 2.

Corollary 3. *Suppose that* f: $M \twoheadrightarrow N$ *is a proper approximate fibration between connected* n-*manifolds without boundary,* $n \geq 5$ *or* n = 3 *and* M *contains no false* 3-*cells. If* ε *is an open cover of* N, *then there is a covering projection* p: $M \to N$ *such that* f *and* p *are* ε-*close.*

Proof. Choose base points m_0 and n_0 such that $f(m_0) = n_0$ and let $\rho: \tilde{N} \to N$ be the covering projection corresponding to image $f_*(\pi_1(M,m_0))$. Then f lifts to a map $\tilde{f}: M \to \tilde{N}$. It is not difficult to check that $\tilde{f}$ is an approximate fibration. We wish to show that $\tilde{f}$ is monotone. Suppose not. Let X_1 and X_2 be distinct components of $F_0 = \tilde{f}^{-1}(\tilde{f}(m_0))$ and let $\mathcal{U}$ be a neighborhood of F_0 such that X_1 and X_2 lie in distinct path components of $\mathcal{U}$. Assume without loss of generality that $m_0 \in X_1$, let $q \in X_2$ be any point, and let $\alpha: I \to M$ be a path from m_0 to q. Since $f\alpha$ is a loop based at n_0 which represents an element of $f_*\pi_1(\tilde{N})$, there is a loop β based at m_0 such that $f\alpha$ and $f\beta$ are homotopic rel(n_0). It follows from the AHLP that m_0 and q can be joined by a path in $\mathcal{U}$, a contradiction.

By Corollary 2 and [10] it follows that $\tilde{f}$ can be approximated by homeomorphisms, so that $p\tilde{f} = f$ can be approximated by covering projections.

Theorem 1 follows essentially from two recently proved results of Husch and Nowak, respectively:

Theorem 4 [7]. *Let* $f: M \to N$ *be an approximate fibration from a closed manifold to a polyhedron with fiber* F. *If* F *has the shape of a finite complex and if* $\dim M - \dim N \geq 6$, *then* $F \times T^{n+1}$ *has the shape of a closed manifold of dimension* $m + 1$ *where* $m = \dim M$ *and* $n = \dim N$.

T^r is the product of r 1-spheres.

Theorem 5 [9]. *If* X *is a continuum which has the shape of a* CW-*complex, then* $Fd(X \times Y_1 \times Y_2 \times \cdots \times Y_n) = Fd(X) + n$ *for all continua* $Y_1, Y_2, \cdots, Y_n$ *such that* $Fd(Y_i) = 1$, $i = 1, 2, \cdots, n$.

Proof of Theorem 1. Let $f: M \to N$ be an approximate fibration with fiber F and suppose $\dim M = m$ and $\dim N = n$. By [2], F is an FANR and by [5], $F \times S^1$ has the shape of a finite complex. Let $r = \max\{1, 6 - (m - n)\}$ and define $\tilde{f}: M \times T^r \to N$ to be the composition of the projection $M \times T^r \to M$ and f. It is easily checked that $\tilde{f}$ is an approximate fibration and, hence, by Theorem 4, $F \times T^{r+n+1}$ has the shape of a closed manifold of dimension $m + r + 1$. Therefore $Fd(F \times T^{r+n+1}) = m + r + 1$; by Theorem 5, $Fd(F) = m - n$.

Since F is an FANR, there exists a CW complex K such that shape K = shape F with $\dim K = Fd(F)$ if $Fd(F) \neq 2$ and $\dim K = 3$ if $Fd(F) = 2$ [4]. Hence $K \times T^{r+n+1}$ has the homotopy type of a closed manifold. By Theorem 2.5 of [11], K is a Poincaré complex of formal dimension $m - n$. Corollary 2 now follows from Theorem 4.2 of [11].

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