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HYPERSPACES WITH THE KAPPA TOPOLOGY

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1. Introduction

In previous papers ([1], [2], [3]) the author has examined properties of semi-continuous multifunctions and spaces with the lambda topology (Kuratowski [4]). In 1959 V. I. Ponomarev ([5]) applied the so-called Kappa topology to the family of closed nonvoid subsets of a compact Hausdorff space, achieving several interesting results. The aim of this paper is to generalize the κ-topology to families of subsets not necessarily closed.

2. Foundations

Let R be a binary relation on X to $\mathcal{C}$, a cover of X, defined as follows: xRC iff $x \in C$, where $x \in X$, $C \in \mathcal{C}$. R is obviously equivalent to a multifunction mapping X to X. Furthermore let R_+ and R_- be mappings from $P(X) \to P(\mathcal{C})$ such that $R_+(A) = \{C \in \mathcal{C} \mid C \subseteq A\}$ and $R_-(A) = \{C \in \mathcal{C} \mid C \cap A \neq \emptyset\}$ (see [1]). R_+ and R_- are used to define topologies on the family $\mathcal{C}$. The λ-topology, generated by sets of the form $R_-(G)$, has been described extensively in [1]. The κ-topology, on the other hand, is generated by all sets of the form $R_+(G)$, where G is open in X. It is easy to see that this family is not only a subbasis but indeed a basis for the κ-topology. Also it is the smallest topology for $\mathcal{C}$ for which R is closed, i.e. whenever M is closed in X, then R(M) is closed in $\kappa\mathcal{C}$. We shall denote by κX the space of all closed subsets of X and by κP the space of all subsets of X, with the κ-topology.

3. Some Properties of the κ-Topology

Lemma. If $\mathcal{C}$ contains all singleton subsets of X, *then* X *is second countable provided* $\kappa\mathcal{C}$ *is second countable.*

Proof. Let $\{R_+(G_i)\}_{i=1}^{\infty}$ be a countable base for $\kappa\mathcal{C}$, where G_i is open in X. Let $x \in G$, G open in X. Then $\{x\} \in R_+(G)$ and $\{x\} \in R_+(G_i) \subseteq R_+(G)$ for some i. Thus $x \in G_i \subseteq G$.

Proposition 1. *Let* X *be a* T_1 *space. If* κX *is second countable, then* X *is compact.*

Proof. We know X is second countable. Let H be a countably infinite subset of X without cluster points. H is closed and discrete in the relative topology. Hence κH considered as a subspace of κX is also second countable. Let $\{G_\alpha\}_\alpha$ be a countable basis for κH and let $E \in \kappa H$. Obviously $E \in R_+(E)$ and $R_+(E)$ is open in κH. Hence there exists $\alpha(E)$ so that $E \in G_{\alpha(E)} \subseteq R_+(E)$. Let E and F be distinct elements of κH. Since either $E \notin R_+(F)$ or $F \notin R_+(E)$, we have $G_{\alpha(E)} \neq G_{\alpha(F)}$. But H has an uncountable number of subsets, so that $\{G_\alpha\}_\alpha$ cannot be a countable basis for κH, a contradiction.

Proposition 2. *Let* $\mathcal{C}$ *be a family of subsets of* X *containing all singletons.* X *is separable iff* $\kappa\mathcal{C}$ *is separable.*

Proof. Assume X is separable and let A be a countable dense subset of X, say $A = \{a_1, a_2, \cdots\}$. Let $\mathcal{A} = \{\{a_1\}, \{a_2\}, \cdots\}$. If $\mathcal{B}$ is an open subset of $\kappa\mathcal{C}$, say $\mathcal{B} = \bigcup_\alpha R_+(G_\alpha)$, then $\mathcal{B} \cap \mathcal{A} \neq \emptyset$ so that $\mathcal{A}$ is dense in $\kappa\mathcal{C}$. Conversely, let $\mathcal{A} = \{A_1, A_2, \cdots\}$ be a countable dense subset of $\kappa\mathcal{C}$. From each A_i we choose some a_i and form $A = \{a_1, a_2, \cdots\}$. Let 0 be open in X. Then $R_+(0)$ is open in $\kappa\mathcal{C}$ and thus $R_+(0) \cap \mathcal{A} \neq \emptyset$. Suppose $A_\kappa \in \mathcal{A} \cap R_+(0)$. Then $A_\kappa \subseteq 0$, and since $a_\kappa \in A_\kappa$, $A \cap 0 \neq \emptyset$. Hence X is separable.

Proposition 3. *Let* $\mathcal{C}$ *be a cover of* X *such that* $\emptyset \notin \mathcal{C}$ *and for each* $x \in X$, $\{x\} \in \mathcal{C}$. *Then* $\kappa\mathcal{C}$ *has isolated points iff* X *has isolated points.*

Proof. Assume X has no isolated points. Let A be an isolated point of $\kappa\mathcal{C}$. Hence $\{A\} = R_+(G)$ for some G open in X and it follows that $A \subseteq G$. Let a, a' be distinct elements of A, then $\{a\}$, $\{a'\}$ are distinct elements of $R_+(G)$, so that $A = \{a\} = \{a'\}$--a contradiction. This contradicts the assumption that X has no isolated points. Conversely, if $\kappa\mathcal{C}$ has no isolated points and $\{x\}$ is open in X, then $R_+\{x\} = \{\{x\}\}$ is open in $\kappa\mathcal{C}$, a contradiction.

Proposition 4. *Let* $\mathcal{A} = \{A_\alpha\}_\alpha$ *be a connected subset of* $\kappa\mathcal{C}$, A_α *connected in* X *for all* α. *Then* $A = \bigcup_\alpha A_\alpha$ *is connected.*

Proof. Assume A is not connected, i.e. $A = (G \cap A) \cup (H \cap A)$ where G and H are open in X, $G \cap A \neq \emptyset$, $H \cap A \neq \emptyset$ and $G \cap H \cap A = \emptyset$. Let $A_\alpha \in \mathcal{A}$. Since $A_\alpha \subset A$ and A_α is connected, either $A_\alpha \subset G$ or $A_\alpha \subset H$. Thus $(R_+(G) \cap \mathcal{A}) \cup (R_+(G) \cap \mathcal{A}) = \mathcal{A}$. Also $R_+(G) \cap \mathcal{A} \neq \emptyset$, $R_+(H) \cap \mathcal{A} \neq \emptyset$ and their intersection is empty. This contradicts the assumption that $\mathcal{A}$ is connected.

Corollary. If $\mathcal{C}$ *is a cover of* X *by connected sets and*

$\kappa\mathcal{C}$ *is connected, then* X *is connected.*

We now consider the family $\mathcal{L}$ of all open subsets of X with the κ-topology. The binary relation R defined above is closed and lower semicontinuous in this case (see [1]). We give the following elementary results without proofs. Moreover we also note without proof that $(\mathcal{L},\subseteq,\cup,\cap)$ is a complete lattice and that for each $G \in \mathcal{L}$, $R_+(G)$ is the smallest $\kappa\mathcal{L}$-open set about G.

Proposition 5.

(i) $\kappa\mathcal{L}$ *is connected, compact, locally connected.*

(ii) If $\phi \in \mathcal{A}$ *and* $\mathcal{A}$ *is dense in* $\kappa\mathcal{L}$*, then* $\cup\mathcal{A}$ *is dense in* X.

(iii) If A *is an open dense subset of* X*, then* $\{A \cap 0 \mid 0$ *is open in* $X\}$ *is dense in* $\kappa\mathcal{L}$.

It can be easily shown that X is a fixed point of any continuous map from $\mathcal{L}$ onto $\mathcal{L}$. We conclude this paper by establishing that even a continuous map that is not a surjection has a fixed point.

Proposition 6. *Let* $(X,\mathcal{L})$ *be a topological space and let* $f\colon \mathcal{L} \to \mathcal{L}$. *Then* $f\colon(\mathcal{L},\kappa\mathcal{L}) \to (\mathcal{L},\kappa\mathcal{L})$ *is continuous if, and only if,* $f\colon(\mathcal{L},\subseteq) \to (\mathcal{L},\subseteq)$ *is order preserving.*

Proof. Suppose first that f is continuous and let A and B be members of $\mathcal{L}$ such that $A \subseteq B$. Then $A \in \{G \in \mathcal{L} \mid G \subseteq B\}$, and this set is the smallest $\kappa\mathcal{L}$-open set about B. Since $\mathcal{B} = \{G \in \mathcal{L} \mid G \subset f(B)\}$ is a $\kappa\mathcal{L}$-open set about f(B) and since f is continuous we have that $f^{-1}(\mathcal{B})$ is a $\kappa\mathcal{L}$-open set about

B. Hence $A \in f^{-1}(\mathcal{B})$. In other words $f(A) \in \mathcal{B}$ and $f(A) \subseteq f(B)$.

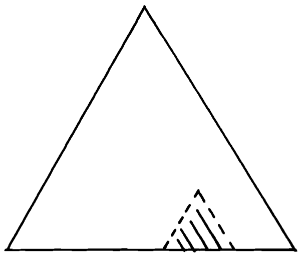
Now suppose that f is an order-preserving function. In order to show that $f:(\mathcal{L}, \kappa\mathcal{L}) \to (\mathcal{L}, \kappa\mathcal{L})$ is continuous, it suffices to show that for each $G \in \mathcal{L}, f^{-1}(R_+(G)) \in \kappa\mathcal{L}$. Let $G \in \mathcal{L}$ and let $A \in f^{-1}(R_+(G))$. If $B \in \mathcal{L}$ and $B \subseteq A$, then $f(B) \subseteq f(A) \subseteq G$ so that $f(B) \in R_+(G)$ and $B \in f^{-1}(R_+(G))$. Thus $A \in \{B \in \mathcal{L} | B \subseteq A\} = R_+(A) \subseteq f^{-1}(R_+(G))$. It follows that $f^{-1}(R_+(G)) \in \kappa\mathcal{L}$.

Corollary. Every continuous map from $(\mathcal{L}, \kappa\mathcal{L})$ *into* $(\mathcal{L}, \kappa\mathcal{L})$ *has a fixed point.*

Proof. As is well known, a lattice $\mathcal{L}$ is complete if, and only if, every order-preserving function from $\mathcal{L}$ into $\mathcal{L}$ has a fixed point.

4. Example of a κ-Space

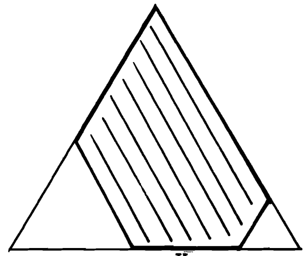
Let $X = [0,1]$ and let $\mathcal{C}$ be the family of all nonvoid, closed, connected subsets of X. With each $C = [a,b] \in \mathcal{C}$ associate the point $(\frac{a+b}{2}, \frac{b-a}{2}\sqrt{3})$. This yields a 1-1 correspondence of $\mathcal{C}$ onto the triangle with vertices (0,0), (1,0), $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. We shall topologize the triangle to make it homeomorphic with $\kappa\mathcal{C}$. Let G be open and connected in X, F closed and connected in X. Figure 1 (resp. 2) shows a typical basic open (resp. closed) subset of $\kappa\mathcal{C}$ (shaded areas plus heavy lines). If $P \in \kappa\mathcal{C}$ with $P \notin R_-(K)$ (Figure 2), then every open set containing $R_-(K)$ also contains P. Hence $\kappa\mathcal{C}$ is not regular (also not normal).



G

$R_+(G)$

Figure 1



K

$R_-(K)$

Figure 2

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