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In 1927 Kuratowski [12, p. 262] defined a continuum M to be of *type* λ if M is irreducible and every indecomposable continuum in M is a continuum of condensation. If a continuum M is of type λ, then M admits a monotone upper semicontinuous decomposition to an arc with the property that each element of the decomposition has void interior relative to M [13, Theorem 3, p. 216].

In 1933 Knaster and Mazurkiewicz [8] defined a continuum M to be λ-*connected* if for every pair p,q of points of M, there exists a continuum of type λ in M that is irreducible between p and q. They pointed out that λ-connectivity is a natural generalization of α-connectivity (arcwise connectivity) and gave two examples to show that unlike α-connectivity, λ-connectivity is not a continuous invariant. The domain in each of their examples is not planar.

Knaster and Mazurkiewicz [8, p. 90] raised the question of whether there exist counterexamples to the invariance of λ-connectivity under continuous transformations in the plane. In this paper I prove that if M is a λ-connected plane continuum and f is a continuous function of M into the plane, then f[M] is λ-connected.

The following intermediate property (weaker than α-connectivity but stronger than λ-connectivity) is defined in the last section of [8].

A continuum M is δ-*connected* if for each pair p,q of points of M, there exists a hereditarily decomposable continuum in M that is irreducible between p and q. The closure of any ray in E^3 (Euclidean 3-space) that limits on a disk is a λ-connected continuum that is not δ-connected. Every hereditarily unicoherent λ-connected continuum is δ-connected. It follows from Theorem 2 of this paper that δ-connectivity and λ-connectivity are equivalent properties for plane continua.

In 1972 I [1] proved that every δ-connected nonseparating plane continuum has the fixed-point property. Krasinkiewicz gave another proof of this theorem in [9].

There exists a ray P in E^3 such that P limits on a disk and the closure of P is a continuous image of the topologist's sine curve. Hence δ-connectivity is not a continuous invariant. However, I [4] proved that if M is a δ-connected continuum and f is a continuous function of M into the plane, then f[M] is δ-connected.*

Unfortunately, I [1,3,4,5,6, and 7] was unaware of Knaster and Mazurkiewicz's article [8] and called δ-connected continua λ-connected. In 1974 Krasinkiewicz [10, Theorem 3.2] proved that every hereditarily unicoherent continuum that is

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*The proof of Theorem 3 of [4] can be simplified considerably by replacing line 30 of page 280 through line 22 of page 282 with the following:

"element of V_1 that joins q_2 to a_1, and (2) q_2 is the last point of $[y_1,q_1]$ that can be joined to a_1 by an element of V_1. Define $K_1 = [p_1,a_1] \cup L_1 \cup [q_2,q_1]$.

Let $Z_1 = K_1$. Note that Z_1 is a continuum in $S^2 - G_1$ that contains $\{p_1,q_1\}$."

a continuous image of a δ-connected continuum is hereditarily decomposable. Although Krasinkiewicz said he was following Knaster and Mazurkiewicz [8], he also called δ-connected continua λ-connected. The second example of Knaster and Mazurkiewicz [8] shows that Krasinkiewicz's theorem does not hold for λ-connected continua. In this example the product of the pseudo-arc and a circle is projected onto the pseudo-arc. In [11] Krasinkiewicz proved several other interesting theorems for δ-connected continua that do not hold for λ-connected continua.

Let M be a plane continuum. A subcontinuum L of M is a *link* in M if L is either the boundary of a complementary domain of M or the limit of a convergent sequence of complementary domains of M. The following characterization of δ-connected plane continua is established in [3, Theorem 2].

Theorem 1. *A plane continuum* M *is* δ*-connected if and only if each link in* M *is hereditarily decomposable.*

An indecomposable subcontinuum I of a continuum M is *terminal* in M if there exists a composant C of I such that each subcontinuum of M that meets both C and M - I contains I.

Theorem 2. *If a plane continuum* M *is* λ*-connected, then* M *is* δ*-connected.*

Proof. According to Theorem 1, it suffices to show that every link in M is hereditarily decomposable. Suppose there exists a link in M that contains an indecomposable continuum I. It follows from [2, Theorem 2] and [4, Theorem 1] that I is terminal in M. Hence there exists a composant C of I such that each subcontinuum of M that meets C and M - I contains I. Let p and q be points of C and I - C, respectively.

Since M is λ-connected, there exists a continuum K of type λ in M that is irreducible between p and q. Since K is a decomposable continuum in M that meets C and I - C, K meets M - I. Therefore K contains I, and this contradicts the fact that K is a continuum of type λ irreducible between p and q. Hence every link in M is hereditarily decomposable.

Theorem 3. *Every λ-connected plane continuum that does not separate the plane has the fixed-point property.*

Proof. Since every δ-connected nonseparating plane continuum has the fixed-point property [1], this theorem follows immediately from Theorem 2.

Theorem 4. *A plane continuum* M *is λ-connected if and only if* M *cannot be mapped continuously onto Knaster's chainable indecomposable continuum with one endpoint.*

Proof. This follows from [5, Theorem 2] and Theorem 2.

Theorem 5. *If* M *is a λ-connected plane continuum and* f *is a continuous function of* M *into the plane, then* f[M] *is λ-connected.*

Proof. By Theorem 2, M is δ-connected. Hence f[M] is δ-connected [4, Theorem 5]. Therefore f[M] is λ-connected.

Still unanswered is the following:

Question. Is every continuous image of every λ-connected plane continuum λ-connected?

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