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# A NONSTANDARD APPROACH TO S-CLOSED SPACES

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## 1. Introduction

In [20], Gödel remarks that simplification facilitates discovery, and for this reason we may expect a bright future for nonstandard analysis. It has been amply demonstrated that nonstandard topology is a useful tool in eliminating certain pathological behavior inherent within the standard model as well as producing considerable economy of effort when obtaining standard results.

The major purpose of this paper is to introduce a new monad, the S-monad, which is nuclear but not filter base determined and which is capable of characterizing T. Thompson's [25] concept of the S-closed space, as well as improving on many of his results. In particular, we show that $(X,\tau)$ is extremally disconnected iff for each $p \in X$ either $\mu_\alpha(p) \subset \mu S(p)$ or $\mu_\theta(p) = \mu S(p)$ or $\mu(p) \subset \mu S(p)$. Moreover, X is S-closed iff $*X = \cup\{\mu S(x) | x \in X\}$. Using these results we improve somewhat upon various theorems in [25]. For example, we show that if X is nearly-compact [resp. quasi-H-closed] and extremally disconnected, then X is S-closed. If X is almost regular and S-closed, then X is nearly-compact and extremally disconnected. If X is weakly-Hausdorff and S-closed, then X is H-closed and extremally disconnected. If X is weakly-Hausdorff and extremally disconnected, then X is a dense subspace of the S-closed Fomin extension $\sigma(X)$. Using nonstandard mapping theory, we also show that for the class of all weakly-Hausdorff S-closed extensions of a weakly-Hausdorff, extremally disconnected space X the Fomin extension, $\sigma(X)$, is the almost-projective maximum. By reworking a proof of Rudolf's, it is shown that an almost-continuous r.o. proper map from a space X into a quasi-H-closed space Y has an almost-continuous extension to the generalized absolute closure $\tau X$ of X. Finally, examples are given of various well known maps which have the r.o. proper condition and an example is constructed of a Hausdorff S-closed space which is not compact.

## 2. Preliminaries

Throughout this paper, we assume that $\mathcal{M} = (\mathcal{U}, \in, pr, ap)$ is the standard set-theoretic superstructure constructed by Machover and Hirschfeld [13] even though any appropriate superstructure will suffice. We need only consider the nonstandard extension $*\mathcal{M} = (*\mathcal{U}, *\in, *pr, *ap)$ to be an enlargement. We assume that the reader is familiar with the basic concepts and methods associated with nonstandard topology [13] [20]. We shall use much of the notation as found in [7] and [13]. Moreover, throughout this paper the symbols $(X,\tau)$ and $(Y,T)$ will denote topological spaces. Recall that for $(X,\tau)$, the monad $\mu(p)$, $\alpha$-monad $\mu_\alpha(p)$ and $\theta$-monad $\mu_\theta(p)$ at a point $p \in X$ are defined as follows:
$\mu(p) = \cap\{*G \mid p \in G \in \tau\}$, $\mu_\alpha(p) = \cap\{*(\mathrm{int}_X \mathrm{cl}_X G) \mid p \in G \in \tau\}$ and $\mu_\theta(p) = \cap\{*(\mathrm{cl}_X G) \mid p \in G \in \tau\}$, where if $A \in *\mathcal{U}$, then $*A = \{x \mid [x \in *\mathcal{U}] \wedge [x *\in A]\}$. For basic properties of these monads, we refer the reader to references [6] [7] [8] [13] and [20].

Singal and Mathur [23] call a space $(X,\tau)$ *nearly-compact* if each open cover $\mathcal{C}$ has a finite subset, say $\{G_1,\cdots,G_n\}$ such that $X = \cup\{\mathrm{int}_X \mathrm{cl}_X G_i \mid i=1,\cdots,n\}$. Moreover, X is *almost-regular* [22] if for each regular-closed $F \subset X$ and $p \in X - F$ there exist disjoint open G, H such that $p \in G$ and $F \subset H$. We show in [6] [7] [8] that X is nearly-compact [resp. quasi-H-closed [18]] iff $*X = \cup\{\mu_\alpha(x) \mid x \in X\}$ [resp. $*X = \cup\{\mu_\theta(x) \mid x \in X\}$] and X is almost-regular iff $\mu_\alpha(p) = \mu_\theta(p)$ for each $p \in X$. A space $(X,\tau)$ is *weakly-Hausdorff* [24] if each $p \in X$ is the intersection of every regular-closed set containing it iff $\mu_\alpha(p) \cap X = \{p\}$ for each $p \in X$. T. Thompson [25] calls $(X,\tau)$ S-*closed* if each cover $\mathcal{S}$ of X by semiopen sets [2] has a finite subset, say $\{S_1,\cdots,S_n\}$, such that $X = \cup\{\mathrm{cl}_X S_i \mid i=1,\cdots,n\}$, where $S \subset X$ is semiopen if there exists $G \in \tau$ such that $G \subset S \subset \mathrm{cl}_X G$.

## 3. S-closed Spaces

*Definition* 3.1. For $(X,\tau)$, let $SO(X)$ denote the set of all semiopen subsets of X. For each $p \in X$, let the S-monad, $\mu S(p) = \cap\{*(\mathrm{cl}_X S) \mid p \in S \in SO(X)\}$.

It is not difficult to show that for each $p \in X$, $\mu S(p) = \cap\{*(\mathrm{cl}_X G) \mid [p \in \mathrm{cl}_X G] \wedge [G \in \tau]\}$. Application of Theorems 4.1 and 4.3 in [8] yield the following characterization.

*Theorem* 3.1. *A space* $(X,\tau)$ *is* S-*closed iff* $*X = \cup\{\mu S(x) \mid x \in X\}$.

*Theorem* 3.2. *For* $(X,\tau)$, *let* $G \in \tau$ *and* X *be* S-*closed. Then* $*(\mathrm{cl}_X G) \subset \cup\{\mu S(x) \mid x \in \mathrm{cl}_X G\}$.

*Proof.* Let $G \in \tau$ and $\mathcal{S} \subset SO(X)$. Then $\mathcal{S} \cup \{X - cl_X G\} = \mathcal{G}$ is a cover of X by semiopen subsets of X. Hence there exists a finite subset of $\mathcal{G}$, say $\{B_1, \cdots, B_n, X - cl_X G\}$, such that

$$X = cl_X B_1 \cup \cdots \cup cl_X B_n \cup (cl_X(X - cl_X G)).$$

Since $cl_X(X - cl_X G) \cap G = \emptyset$, then $cl_X G \subset cl_X B_1 \cup \cdots \cup cl_X B_n$. Application of Theorems 4.1 and 4.3 in [8] imply that $*(cl_X G) \subset \cup\{\mu S(x) | x \in cl_X G\}$ and proof is complete.

*Remark* 3.1. A set $W \subset *X$ is said to be SA-*compact* if $W \subset \cup\{\mu S(x) | x \in A\}$.

In what follows, we do not assume that an extremally disconnected space is Hausdorff and observe that for each $p \in X$, $\mu S(p) \subset \mu_\theta(p)$. Let $RO(X) = \{G | [G \subset X] \wedge [G \text{ is regular-open}]\}$.

*Theorem* 3.3. *For* $(X,\tau)$, *the following are equivalent.*

*(i)* X *is extremally disconnected.*

*(ii) For each* $p \in X$, $\mu_\alpha(p) \subset \mu S(p)$.

*(iii) For each* $p \in X$, $\mu S(p) = \mu_\theta(p)$.

*(iv) For each* $p \in X$, $\mu(p) \subset \mu S(p)$.

*Proof.* (i) implies (ii). Assume that X is extremally disconnected. Let $p \in cl_X G$, $G \in \tau$. Then $cl_X G \in \tau$. Hence $int_X cl_X(cl_X G) = int_X cl_X G = cl_X G$. Thus $cl_X G$ is regular-open. Hence, since $\mu_\alpha(p) = \cap\{*U | p \in U \in RO(X)\}$, we have that $\mu_\alpha(p) \subset \mu S(p)$.

(ii) implies (i). Assume that for each $p \in X$, $\mu_\alpha(p) \subset \mu S(p)$. Let $G \in \tau$ and assume that $p \in cl_X G$. Since $\mu(p) \subset \mu_\alpha(p) \subset \mu S(p) \subset *(cl_X G)$, we have that $cl_X G \in \tau$.

(i) implies (iii). Let $(X,\tau)$ be extremally disconnected and $p \in cl_X G$, $G \in \tau$. Since $cl_X G \in \tau$, then $cl_X G \in \{cl_X G \mid p \in G \in \tau\}$. Thus $\mu_\theta(p) \subset \mu S(p)$.

(iii) implies (ii) and (iii) implies (iv) are obvious.

(iv) implies (ii). Let $p \in cl_X G$, $G \in \tau$. Then $\mu(p) \subset *(cl_X G)$ implies that there exists some $H \in \tau$ such that $p \in H \subset cl_X G$. Consequently, $p \in int_X cl_X H \subset cl_X G$ implies that $\mu_\alpha(p) \subset *(cl_X G)$. Hence $\mu_\alpha(p) \subset \mu S(p)$ and this completes the proof.

We are now in a position to improve upon Theorems 5, 6, 7 in [25].

*Theorem* 3.4. *If* $(X, \tau)$ *is nearly-compact* [resp. quasi-H-closed] *and extremally disconnected, then* X *is* S-*closed.*

*Proof*. By application of Theorem 3.3, we have that $*X = \cup\{\mu_\alpha(x) \mid x \in X\} = \cup\{\mu S(x) \mid x \in X\}$. In like manner, if X is quasi-H-closed we obtain the result by application of Theorem 3.3.

*Theorem* 3.5. *If* X *is almost-regular and* S-*closed, then* X *is extremally disconnected and nearly-compact.*

*Proof*. Assume X is not extremally disconnected. Then there exists $G \in RO(X)$ such that $cl_X G - G \neq \emptyset$ and $X - cl_X G \neq \emptyset$. Let $p \in cl_X G - G$. Then $\mu(p) \cap *G \neq \emptyset$. Application of Theorem 3.2 implies that there exists some $q \in cl_X G$ such that $\mu(p) \cap *G \cap \mu S(q) \neq \emptyset$. If $q \in cl_X G - G$, then $q \in X - G$ implies, since $X - G$ is regular-closed and $\mu S(q) \subset *(X - G)$, that $\mu(p) \cap *G \cap *(X - G) \neq \emptyset$. This contradiction yields $\mu(q) \subset *G$. Almost-regularity and $G \in RO(X)$ imply that $\mu_\alpha(q) = \mu_\theta(q) \subset *G$. Since $p \in G$, then $\mu_\theta(q) \cap \mu(p) = \emptyset$.

This contradicts $\mu(p) \cap \mu S(q) \neq \emptyset$. Hence X is extremally disconnected. Theorem 3.3 implies that X is nearly-compact and the proof is complete.

*Corollary* 3.5.1. *Let* $(X,\tau)$ *be almost-regular. Then* X *is* S-*closed iff* X *is nearly-compact and extremally disconnected iff* $(X,\tau_s)$ *is regular, compact and extremally disconnected.*

*Theorem* 3.6. *If* $(X,\tau)$ *is weakly-Hausdorff and* S-*closed, then* X *is* H-*closed and extremally disconnected.*

*Proof*. Assume that X is weakly-Hausdorff and distinct $p,q \in X$. Hence there exists some $G' \in RO(X)$ and F such that $X - F \in RO(X)$, $p \in G'$, $q \in F$ and $G' \cap F = \emptyset$. Thus $\mu_\alpha(p) \cap \mu S(q) = \emptyset$. Assume that X is not extremally disconnected. Then there exists $G \in RO(X)$ such that $cl_X G - G \neq \emptyset$ and $X - cl_X G \neq \emptyset$. Let $p \in cl_X G - G$. Then $\mu(p) \cap {}^*G \neq \emptyset$ implies that $\mu_\alpha(p) \cap {}^*G \neq \emptyset$. Theorem 3.2 implies that there exists some $q \in cl_X G$ such that $\mu_\alpha(p) \cap {}^*G \cap \mu S(q) \neq \emptyset$. Weakly-Hausdorff implies that $p = q$. However, as in the proof of Theorem 3.5, we have that $q \in \mu(q) \subset {}^*G$. This contradiction implies that X is extremally disconnected. Since for each $p \in X$, $\mu S(p) = \mu_\theta(p)$, then Theorem 1.5 in [6] yields that X is Hausdorff. We know that a Hausdorff space is H-closed iff ${}^*X = \cup\{\mu_\theta(x) \mid x \in X\}$ and the proof is complete.

*Corollary* 3.6.1. *If* $(X,\tau)$ *is weakly-Hausdorff, then for distinct* $p,q \in X$, $\mu_\alpha(p) \cap \mu S(q) = \emptyset$.

The following result is obtained by immediate application of Corollary 3.6.1 and Theorem 1.5 [6].

*Theorem* 3.7. *If* $(X,\tau)$ *is weakly-Hausdorff and extremally disconnected, then* X *is Hausdorff (and Urysohn).*

*Corollary* 3.7.1. *If* $(X,\tau)$ *is weakly-Hausdorff, nearly-compact [resp. quasi-H-closed] and extremally disconnected, then* X *is* S-*closed and Hausdorff.*

*Corollary* 3.7.2. *Let* $(X,\tau)$ *be weakly-Hausdorff. Then* X *is* S-*closed iff* X *is quasi*-H-*closed and extremally disconnected iff* $(X,\tau_s)$ *is compact Hausdorff and extremally disconnected.*

*Theorem* 3.8. *If* $(X,\tau)$ *is an* S-*closed, first countable, almost-regular space, then* X *is finite.*

*Proof.* Assume that X is infinite. From Theorem 3.5 we have that X is nearly-compact and extremally disconnected. Consider the semiregularization, $\tau_s$ (i.e. the topology generated by RO(X)). Then $(X,\tau_s)$ is a first countable, regular compact space and S-closed by Theorems 3.1 and 3.3. Thompson's Theorem 3 [25] now implies that X is finite and the proof is complete.

*Corollary* 3.8.1. *If* $(X,\tau)$ *is infinite,* S-*closed and almost-regular, then* X *is uncountable.*

*Proof.* Simply consider $\tau_s$. Then $\tau_s$ is a compact, regular, S-closed topology for X. The result follows from the Corollary to Theorem 4 in [25].

One of the basic examples of an S-closed space cited by Thompson [25] is that of $\beta(N)$. If X is Hausdorff and extremally disconnected, then in [9] Iliadis and Fomin show that the extension $\sigma(X)$ is H-closed and extremally disconnected.

*Theorem* 3.9. *If* X *is weakly-Hausdorff and extremally disconnected, then* X *is a dense subspace of the* S*-closed, Hausdorff, almost-regular space* $\sigma(X)$.

As a final proposition in this section, we obtain a partial converse to the corollary to Theorem 5 in [25], where $\mathcal{I}(X)$ denotes the set of all isolated elements of X.

*Theorem* 3.10. *Let* $(X,\tau)$ *be a noncompact, weakly-Hausdorff, extremally disconnected space. If* $\sigma(X)$ *is compact, then* $\sigma(X) = \beta(X)$ *and* $X - \mathcal{I}(X)$ *is compact.*

*Proof*. By Theorem 6.2 in [19] there exists a continuous surjection from $\sigma(X)$ onto $\beta(X)$ since X is Tychonoff. Thus $\sigma(X) = \beta(X)$. Lemma 5 in [11] implies that $X - \mathcal{I}(X)$ is compact.

*Corollary* 3.10.1. *Let* $(X,\tau)$ *be a noncompact, weakly-Hausdorff, extremally disconnected space. If* $\sigma(X)$ *is compact, then* $\sigma(X) = \beta(X)$ *and* $\mathcal{I}(X)$ *is an infinite set.*

*Remark* 3.2. Katětov has shown in [11] that if X is *any* discrete space, then $\sigma(X) = \beta(X)$. This extends the corollary to Theorem 5 in [25].

In Thompson's fundamental paper [25] the only explicit types of S-closed spaces given are compact. The following is an example of a noncompact, Hausdorff, S-closed space.

*Example* 3.1. Let $(X,\tau)$ be an infinite discrete space and $\mathcal{U}$ the set of all nonprincipal (free) ultrafilters on X. Let $Y = X \cup \mathcal{U}$ and the topology T on Y be generated by $\tau$ and all sets of the form $F \cup \{\mathcal{F}\}$, where $F \in \mathcal{F} \in \mathcal{U}$. Assume p, q $\in$ X. Then {p}, {q} $\in$ T. Now assume that p $\in$ X, and $\mathcal{F} \in \mathcal{U}$. Then there exists some F $\in \mathcal{F}$ such that p $\notin$ F. Hence $\{p\} \cap (F \cup \{\mathcal{F}\}) = \emptyset$. If distinct $\mathcal{F}$ and $\mathcal{G}$ are members of $\mathcal{U}$, then there exists $F \in \mathcal{F}$, $G \in \mathcal{G}$ such that $F \cap G = \emptyset$. Consequently, the open neighborhoods of $\mathcal{F}$ and $\mathcal{G}$; $F \cup \{\mathcal{F}\}$, $G \cup \{\mathcal{G}\}$, respectively, are disjoint. Thus Y is Hausdorff. In order to show that Y is extremally disconnected, let G $\in$ T and assume that $p \in cl_Y G - G$. Clearly, if p $\in$ X, then p is an interior point in $cl_Y G$. Hence assume that p $\notin$ X. Hence $p = \mathcal{F} \in \mathcal{U}$. Let $A \cup \{\mathcal{F}\}$, $A \in \mathcal{F}$, be an open neighborhood of $\mathcal{F}$. Then $(A \cup \{\mathcal{F}\}) \cap G \neq \emptyset$. Since $\mathcal{F} \notin G$, then $A \cap G \neq \emptyset$ for each $A \in \mathcal{F}$. Moreover, $(G \cap X) \cap A \neq \emptyset$ for each $A \in \mathcal{F}$. Hence $G \cap X \in \mathcal{F}$ and $G \cup \{p\} = G \cup ((G \cap X) \cup \{p\}) \in T$. Thus $cl_Y G \supset G \cup \{p\} \in T$ implies that $cl_T G \in T$. Obviously, (Y,T) is noncompact. Now let $\mathcal{F} \in \mathcal{U}$. Thus $\{\mathcal{F}\} \notin T$ since no finite intersection of elements of $\mathcal{F}$ is the empty set. Therefore, since no element of $\mathcal{U}$ is isolated, we have that $\mathcal{I}(Y) = X$. Consequently, $Y - \mathcal{I}(Y) = \mathcal{U}$. However, since in the induced topology, $\mathcal{U}$ is an infinite discrete subspace of Y, then $\mathcal{U}$ is noncompact. By application of Theorem 3.10, we now obtain that $\sigma(X)$ is a noncompact, Hausdorff, S-closed space.

## 4. Mapping Theory

We briefly investigate various well known maps which preserve S-closedness. Let (Y,T) be an arbitrary space. A map f: $(X,\tau) \to (Y,T)$ is *almost-open* [4] if the image of each regular-open subset of X is open. The map f is W-*almost-open* [14] if $f^{-1}[cl_Y V] \subset cl_X(f^{-1}[V])$ for each V $\in$ T. Observe that if f is almost-open in the sense of Wilansky [26], then it is W-almost-open. Also Example 1 in [14] is a continuous, W-almost-open map which is not open. A map f: $(X,\tau) \to (Y,T)$ is almost-continuous [resp. $\theta$-continuous [9], weakly-$\theta$-continuous [6]] if for each $p \in X$ and each open neighborhood V of f(p) there exists an open neighborhood G of p such that $f[G] \subset \mathrm{int}_Y \mathrm{cl}_Y V$ [resp. $f[\mathrm{cl}_X G] \subset \mathrm{cl}_Y V$, $f[G] \subset \mathrm{cl}_Y V$] iff for each $p \in X$, $*f[\mu(p)] \subset \mu_\alpha(f(p))$ [resp. $*f[\mu_\theta(p)] \subset \mu_\theta(f(p))$, $*f[\mu(p)] \subset \mu_\theta(f(p))$ [6]]. Weakly-$\theta$-continuous mappings are also known as *weakly-continuous* mappings [15]. Clearly, an open map is almost-open as well as W-almost-open. Moreover, continuity on X implies almost-continuity implies $\theta$-continuity implies weak-$\theta$-continuity and none of these implications are, in general, reversible. Example 1.3 in [4] is that of a non-trivial, noncontinuous, almost-continuous, almost-open mapping.

*Theorem* 4.1. *A W-almost-open, weakly-$\theta$-continuous map on* X *is almost-continuous.*

*Proof.* Let f: $(X,\tau) \to (Y,T)$ be weakly-$\theta$-continuous. It is known that for $V \in T$, $\mathrm{cl}_X(f^{-1}[V]) \subset f^{-1}[\mathrm{cl}_Y V]$ [14]. Since f is W-almost-open, then $f^{-1}[\mathrm{cl}_Y V] \subset \mathrm{cl}_X(f^{-1}[V])$ implies that $\mathrm{cl}_X(f^{-1}[V]) = f^{-1}[\mathrm{cl}_Y V]$. Hence the inverse image of each regular-closed set in Y is closed in X. This global condition implies that f is almost-continuous as is well known and the proof is complete.

*Theorem* 4.2. *Let* f: $(X,\tau) \to (Y,T)$ *be almost-open and almost-continuous. Then for each* $p \in X$, $*f[\mu S(p)] \subset \mu S(f(p))$.

*Proof.* It is known [4] that the inverse image of a regular-closed set is regular-closed. Thus let $f(p) \in cl_Y V$, $V \in T$. Then $p \in f^{-1}[cl_Y V] = cl_X H$ for some $H \in \tau$. Hence $p \in *(cl_X H) = *f^{-1}[*(cl_Y V)]$. Consequently, $\mu S(p) \subset *f^{-1}[*(cl_Y V)]$. Hence $*f[\mu S(p)] \subset *(cl_Y V)$. Since $cl_Y V$ is arbitrary, then $*f[\mu S(p)] \subset \mu S(f(p))$.

*Corollary* 4.2.1. *Let* $f: (X,\tau) \to (Y,T)$ *be almost-open and almost-continuous.*

*(i) Then the image of an* SA-*compact* $W \subset *X$ *is* $Sf[A]$-*compact.*

*(ii) If* f *is a surjective map and* X *is extremally disconnected, then* Y *is extremally disconnected.*

*Corollary* 4.2.2. *The concept of* S-*closedness is a topological invariant.*

*Theorem* 4.3. *Let* $f: (X,\tau) \to (Y,T)$ *be* W-*almost-open and weakly-*$\theta$*-continuous. Then for each* $p \in X$, $*f[\mu S(p)] \subset \mu S(f(p))$.

*Proof*. Let $f(p) \in cl_Y V$, $V \in T$. Then $p \in f^{-1}[cl_Y V]$ and $f^{-1}[int_Y cl_Y V] = G \in \tau$ since f is almost-continuous. Hence, since $f^{-1}[V] \subset G$ and f is W-almost-open, then

$$cl_X(f^{-1}[V]) \subset cl_X G \subset cl_X(f^{-1}[cl_Y V]) \subset cl_X(f^{-1}[V])$$

implies that $cl_X G = cl_X(f^{-1}[cl_Y V])$. Almost-continuity implies that $cl_X(f^{-1}[cl_Y V]) = f^{-1}[cl_Y V]$. Thus $p \in cl_X G \subset f^{-1}[cl_Y V]$. Hence $\mu S(p) \subset *f^{-1}[\mu S(f(p))]$ implies that $*f[\mu S(p)] \subset \mu S(f(p))$.

*Corollary* 4.3.1. *Let* $f: (X,\tau) \to (Y,T)$ *be* W-*almost-open and weakly-*$\theta$*-continuous.*

*(i) Then the image of an* SA-*compact* $W \subset *X$ *is* $Sf[A]$-*compact.*

*(ii) If* f *is surjective, and* X *is extremally disconnected, then* Y *is extremally disconnected.*

We now give two final results which tend to show the importance of the almost-continuous mappings throughout extension theory.

*Theorem* 4.4. *Let* $(X,\tau)$ *be weakly-Hausdorff and extremally disconnected. Assume that* $\mathcal{C}$ *is the class of all weakly-Hausdorff,* S*-closed extensions of* X. *Then for each* $Z \in \mathcal{C}$ *there exists an almost-continuous surjection* $F\colon \sigma(X) \to Z$ *such that* $F|X =$ *identity* (i.e. $\sigma(X)$ *is the almost-projective maximum in* $\mathcal{C}$).

*Proof*. In [3], Fomin has shown that for each $Z \in \mathcal{C}$ there exists a $\theta$-continuous surjection $F\colon \sigma(X) \to Z$ such that $F|X =$ identity on X. Since Z is almost-regular, then F is almost-continuous and this completes the proof.

In what follows $\tau X$ will denote the *generalized absolute closure* [12] of a space $(X,\tau)$. If X is Hausdorff, then $\tau X$ is the Katětov extension [10] [21]. Rudolf [21] calls a map $f\colon (X,\tau) \to (Y,T)$ r.o. *proper* if for each $q \in Y$ and each regular-open neighborhood U of q there exists an open neighborhood V of q such that $\mathrm{int}_X(f^{-1}[\mathrm{cl}_Y V]) \subset \mathrm{cl}_X(f^{-1}[U])$. *Observe, that an open or* W-*almost-open map is* r.o. *proper*. Moreover, if f is almost-open and almost-continuous, then Theorem 1.3(b) [4] implies that for $U \in RO(Y)$

$$\mathrm{int}_X(f^{-1}[\mathrm{cl}_Y U]) = f^{-1}[\mathrm{int}_Y \mathrm{cl}_Y U] = f^{-1}[U] \subset \mathrm{cl}_X(f^{-1}[U]).$$

Hence *if* $f\colon (X,\tau) \to (Y,T)$ *is almost-open and almost-continuous, then* f *is* r.o. *proper*. By reworking Theorem 2.1 in [21], we have the following major result.

*Theorem* 4.5. *Let* $f: (X,\tau) \to (Y,T)$ *be an almost-continuous* r.o. *proper map and* Y *quasi*-H-*closed. Then* f *has an almost-continuous extension* $\tau f: \tau X \to (Y,T)$, *which is unique if* Y *is Hausdorff.*

*Proof.* Rewriting Rudolf's proof to Theorem 2.1 in [21], we first note the quasi-H-closed implies that $\cap\{cl_Y U \mid U \in \mathcal{U}(\xi)\} \neq \emptyset$. Now change equation (2) as follows:

(2) $f^{-1}[int_Y cl_Y U_y] \in \xi$ for each $U_y \in T$.

Change the phrase "$f^{-1}[U_y] \in \xi$" to "$f^{-1}[int_Y cl_Y U_y] \in \xi$" for some $U_y \in T$. Using the r.o. proper condition, equation (2) is proved in the same manner as in the proof of Theorem 2.1 since almost-continuity implies that $U' = U - f^{-1}[cl_Y V_y]$ is open and dense in U. We don't need a Hausdorff property for Y unless we want a unique map. Simply define $\tau f(\xi)$ to be any $y \in \cap\{cl_Y U \mid U \in \mathcal{U}(\xi)\}$.

Now complete the proof in the following manner. To show that $\tau f$ is almost-continuous at $x \in X$, let $U_{f(x)} \in RO(Y)$ be a neighborhood of $f(x)$. Then $U_x = f^{-1}[U_{f(x)}]$ is open in X, hence in $\tau X$ and $\tau f[U_x] = f[U_x] \subset U_{f(x)}$. Now taking $int_Y cl_Y U_y$ for $\xi \in \tau X - X$ to be an arbitrary regular-open neighborhood of $y = \tau f(\xi)$, we have that $f^{-1}[int_Y cl_Y U_y] \in \xi$ by (2). So, $U_\xi = \{\xi\} \cup f^{-1}[int_Y cl_Y U_y]$ is an open neighborhood of $\xi$ in $\tau X$ for which

$$f[U_\xi] = f[U_\xi \cap X] \cup f[U_\xi - (\tau X - X)] =$$
$$f[f^{-1}[int_Y cl_Y U_y] \cup \tau f(\xi) \subset int_Y cl_Y U_y.$$

Hence $\tau f$ is almost-continuous

*Corollary* 4.5.1. *Let* f: $(X,\tau) \to (Y,T)$ *be almost-continuous and* Y *be* H-*closed Urysohn. Then there exists a unique almost-continuous extension* $\tau f: \tau X \to (Y,T)$ *of* f.

*Proof*. This follows from Theorem 3.4 in [21].

*Remarks*. Theorem 3.1 implies that $(X,\tau)$ is S-closed iff every cover by regular-closed sets has a finite subcover. Moreover, since "$\mu S$" determines a unique pretopological convergence structure, then S-closed spaces may be discussed in terms of the S-convergence of filter bases. Corollaries 3.5.1, 3.7.2 hold since extremally disconnected is a semiregular property and these results have applications to projective objects within certain interesting categories.

## References

[1] D. Cameron, *Properties of* C-*closed spaces*, Proc. Amer. Math. Soc. (to appear).

[2] S. Crossley and S. Hildebrand, *Semi-topological properties*, Fund. Math. 74 (1972), 233-254.

[3] S. Fomin, *Extensions of topological spaces*, Ann. Math. 44 (1943), 471-480.

[4] L. Herrington, *Some properties preserved by almost-continuous functions*, Boll. Un. Mat. Ital. (4) 10 (1974), 556-568.

[5] ________, *Properties of nearly-compact spaces*, Proc. Amer. Math. Soc. 45 (1974), 431-436.

[6] R. A. Herrmann, *A note on weakly* $\theta$-*continuous extensions*, Glasnik Mat. 10 (1975), 329-336.

[7] ________, *The* Q-*topology, Whyburn type filters and the cluster set map*, Proc. Amer. Math. Soc. 59 (1976), 161-166.

[8] ________, *A nonstandard generalization for perfect maps*, Z. Math. Logik Grundlagen Math. (3) 23 (1977), 223-236.

[9] S. Iliadis and S. Fomin, *The method of centered systems in the theory of topological spaces*, Uspehi Mat. Nauk

21 (1966), 47-76 = Russian Math. Surveys 21 (1966), 37-62.

[10] M. Katětov, *Über* H-*abgeschlossene und bikompakte Räume*, Časopis Pěst. Fys. 69 (1940), 36-49.

[11] ________, *On the equivalence of certain types of extensions of topological spaces*, Časopis Pěst. Mat. Fys. 72 (1947), 101-106.

[12] C. T. Liu, *Absolutely closed spaces*, Trans. Amer. Math. Soc. 130 (1968), 86-104.

[13] M. Machover and J. Hirschfeld, *Lectures on non-standard analysis*, Lecture Notes in Math., No. 94, Springer-Verlag, Berlin, 1969.

[14] T. Noiri, *On semi*-$T_2$-*spaces*, Ann. Soc. Sci. Bruxelles 90 (1976), 215-220.

[15] ________, *On weakly continuous mappings*, Proc. Amer. Math. Soc. 46 (1974), 120-124.

[16] ________, *On* S-*closed spaces*, Ann. Soc. Sci. Bruxelles 91 (1977), 189-194.

[17] ________, *On* S-*closed subspaces*, Rend. Nax. Lincei. (to appear).

[18] J. Porter and J. Thomas, *On* H-*closed and minimal Hausdorff spaces*, Trans. Amer. Math. Soc. 138 (1969), 159-170.

[19] J. Porter and C. Votaw, H-*closed extensions, II*, Trans. Amer. Math. Soc. 202 (1975), 193-209.

[20] A. Robinson, *Non-standard analysis*, North Holland, Amsterdam, 1966.

[21] L. Rudolf, $\theta$-*continuous extensions on* $\tau X$, Fund. Math. 74 (1972), 111-131.

[22] M. K. Singal and S. P. Arya, *On almost-regular spaces*, Glasnik Mat. 4 (1969), 89-99.

[23] M. K. Singal and A. Mathur, *On nearly-compact spaces*, Boll. Un. Mat. Ital. (4) 2 (1969), 702-710.

[24] T. Soundararajan, *Weakly Hausdorff spaces and the cardinality of topological spaces*, General topology and its relation to modern analysis and algebra III, Proc. Conf. Kanpur. 1968, 301-306, Academia, Prague, 1971.

[25] T. Thompson, S-*closed spaces*, Proc. Amer. Math. Soc. 60 (1976), 335-338.

[26] A. Wilansky, *Topics in functional analysis*, Lecture Notes in Math, No. 45, Springer-Verlag, Berlin, 1967.

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