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# PIXLEY-ROY TOPOLOGY

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## I. Introduction

In the spring of 1969, the annual topology conference was held at Auburn University. In one of the most elegant papers presented at that conference, Carl Pixley and Prabir Roy [PR] gave an entirely new construction of an important example in Moore space theory, a nonseparable Moore space which has countable cellularity, i.e., which satisfies the countable chain condition (CCC). The original example of a space of this type had been given by Mary Ellen Rudin in 1951 [$R_1$] and is certainly among the most complicated known Moore spaces. (Recently, an easier-to-understand description of Rudin's space has been given by Aarts and Lowen-Colebunders in [ALC].) Intervening years have demonstrated that Pixley and Roy actually discovered a counterexample machine. Their construction has been used by Przymusiński and Tall [PT], Alster and Przymusiński [AP], and by van Douwen, Tall and Weiss [vDTW] in the study of Moore spaces. More recently the construction has been studied as an end in itself in such papers as [vD], [$BFL_1$], [$R_2$]. My goal today is to tell you about the PR-construction itself, about a few results obtained in the last year and a half, and about several problems which remain open.

In today's terminology, Pixley and Roy described a hyperspace construction, i.e., a method for imposing a topology on certain families of subsets of a given topological space

$(X,\mathcal{T})$. The Pixley-Roy hyperspace of X, which I will denote by $\mathcal{F}[X]$, has as its underlying set the collection of all finite, nonempty subsets of X. If $F \in \mathcal{F}[X]$, then basic neighborhoods of F have the form

$$[F,V] = \{H \in \mathcal{F}[X] \mid F \subset H \subset V\}$$

where V is allowed to be any open subset of X which contains F.

It is important to point out that the topology of $\mathcal{F}[X]$ is different from the more classical hyperspace topology called the Vietoris topology. The Vietoris topology has fewer open sets than does the PR-topology. But the relationship is closer than that. In the terminology of Aarts, de Groot and McDowell [AGM] if X is regular, the Vietoris hyperspace of finite subsets of X is a cospace of $\mathcal{F}[X]$. (This observation is due independently to van Douwen [vD] and to Alster and Przymusinski [AP].)

The remainder of my talk today will break into three parts. First I'll tell you certain general results on $\mathcal{F}[X]$. Then I'll look at the problems of metrization and normality of $\mathcal{F}[X]$ where X will be taken from certain special classes, e.g., where X is a space of ordinals, or a subspace of the usual real line $\mathbb{R}$, or a generalized ordered space built on a separable linearly ordered set, or a Souslin line--because one must have a great deal of control of X in order to study such complex properties of $\mathcal{F}[X]$.

Certain subspaces of $\mathcal{F}[X]$ will receive particular attention: for each $n < \omega_0$, let $\mathcal{F}_n[X] = \{F \in \mathcal{F}[X] \mid \mathrm{card}(F) \leq n\}$. As you will learn from a talk later today [B], $\mathcal{F}_2[X]$ is a well known space (for $X \subset \mathbb{R}$) and the other $\mathcal{J}_n[X]$ give the PR-hyperspace a hierarchical structure which invites proofs by induction.

Undefined notation and terminology will follow [E]. Definitions of special spaces such as Moore spaces, $\sigma$-spaces, perfect spaces, etc., can be found in [BL]. When viewed as topological spaces, ordinals will always carry the usual open interval topology. Terminology and notation related to ordered sets and spaces will follow [L]. In particular, the symbol $\omega$ will denote the first infinite ordinal.

## 2. General Properties of PR-Hyperspaces

Throughout the rest of this talk, I will assume that the ground space X is at least $T_1$. Other restrictions will be mentioned when needed.

The space $\mathcal{J}[X]$ always has certain properties.

2.1 *Theorem* [vD]. *If* X *is* $T_1$ *then* $\mathcal{J}[X]$ *is zero-dimensional, completely regular (and Hausdorff), and every subspace of* $\mathcal{J}[X]$ *is metacompact.*

A construction which yields metacompact spaces that do not have much stronger properties may be useful in the study of two nice open questions:

1) Is it true that every member of Arhangel'skǐ's class Regular-MOBI (terminology as in [BL]) is quasi-developable? One step toward a solution of this problem is to decide whether open-compact mappings preserve weak $\theta$-refinability [BeL], given regularity of the domain of the mapping. And, surprisingly, we do not even know whether the image of a

regular metacompact space under an open-compact mapping is weakly $\theta$-refinable.

2) It is true that every metacompact space is the image of a paracompact space under an open-compact mapping? (This question is usually ascribed to Arhangel'skiĭ.)

At the opposite extreme, there are certain properties that $\mathcal{F}[X]$ never has.

2.2 *Theorem. If* X *is not discrete, i.e., if* X *has at least one limit point, then* $\mathcal{F}[X]$ *is* <u>*not*</u> *a Baire space.*

*Proof.* Let p be any limit point of X. Then, X being $T_1$, each neighborhood of p is infinite. If $\mathcal{F}[X]$ were a Baire space then its open subspace $\mathcal{Z} = [\{p\},X]$ would also be a Baire space. And yet $\mathcal{Z} = \cup\{\mathcal{Z} \cap \mathcal{F}_n[X] \mid n \geq 1\}$ even though each $\mathcal{Z} \cap \mathcal{F}_n[X]$ is a closed, nowhere dense subspace of $\mathcal{Z}$.

There are some interesting problems related to the Baire category property, but posing them must wait for a minute.

The topology of the ground space X determines some properties of $\mathcal{F}[X]$, as the next three theorems show.

2.3 *Theorem. Let* X *be a* $T_1$*-space. The following are equivalent:*

*a)* X *has countable pseudo-character;*

*b)* $\mathcal{F}_2[X]$ *has countable pseudo-character;*

*c)* $\mathcal{F}[X]$ *has countable pseudo-character;*

*d)* $\mathcal{F}[X]$ *is perfect;*

*e)* $\mathcal{F}[X]$ *is semi-stratifiable;*

*f)* $\mathcal{F}[X]$ *is a* $\sigma$*-space;*

*g)* $\mathcal{F}[X]$ *is a union of countably many closed discrete subspaces.*

*Proof*. We prove that a) implies g). The implications g) $\to$ f) $\to$ e) $\to$ d) $\to$ c) $\to$ b) are all well-known, and the implication b) $\to$ a) is easy.

For each n, let $\mathcal{E}_n = \{F \in \mathcal{F}[X] \mid \text{card}(F) = n\}$. Then $\mathcal{E}_n$, topologized as a subspace of $\mathcal{F}[X]$, is discrete-in-itself (although not closed in $\mathcal{F}[X]$, unless $n = 1$). For each $x \in X$ select a decreasing sequence $g(n,x)$ of open sets having $\cap\{g(x,x) \mid n \geq 1\} = \{x\}$. For each $F \in \mathcal{F}[X]$, let $k(F)$ be the first integer k such that if $x \neq y$ belong to F, then $y \notin g(k,x)$. Now define

$$\mathcal{E}_{nk} = \{F \in \mathcal{E}_n \mid k(f) = k\} \qquad \text{for each } n \geq 2$$

and $k \geq 1$. Because $\mathcal{F}[X] = \mathcal{E}_1 \cup (\cup\{\mathcal{E}_{nk} \mid n \geq 2,\ k \geq 1\})$, the proof will be complete once we show that each $\mathcal{E}_{nk}$ is closed in $\mathcal{F}[X]$. So suppose $T \in \mathcal{F}[X] - \mathcal{E}_{nk}$ where $n \geq 2$ and $k \geq 1$. If card $(T) \geq n$, then $[T,X]$ is a neighborhood of T which misses $\mathcal{E}_{nk}$ entirely, so suppose $\text{card}(T) < n$. Consider the open set $U = \cup\{g(k,t) \mid t \in T\}$. If the open neighborhood $[T,U]$ of T contains a point $F \in \mathcal{E}_{nk}$ then $T \subset F \subset U$. Index T and F as

$$T = \{x_1, x_2, \cdots, x_m\} \subset F = \{x_1, \cdots, x_m, x_{m+1}, \cdots, x_n\}$$

where $m = \text{card}(T)$. Because $m < n$ and $F \subset U$ there must be some $x_i \in T$ such that $g(k,x_i)$ contains two distinct elements of F, and that is impossible since $k(F) = k$. Therefore, $[T,U] \cap \mathcal{E}_{nk} = \phi$ and the proof is complete.

2.4 *Theorem. The following properties of a* $T_1$*-space are equivalent:*

*a)* X *is first-countable;*

*b)* $\mathcal{F}_2[X]$ *is first countable;*

*c)* $\mathcal{J}[X]$ *is a Moore space.*

*Proof.* That a) implies c) is due to van Douwen [vD], and the other implications are easy.

In the next result, the cardinal functions c,d nw and w denote cellularity, density, netweight and weight respectively (see [E]).

2.5 *Theorem. For any infinite* $T_1$*-space* X,

*a)* $nw(\mathcal{J}[X]) = card(X) \cdot nw(X)$

*b)* $w(\mathcal{J}[X]) = card(X) \cdot w(X)$

*c)* $c(\mathcal{J}[X]) \leq nw(X)$

*d)* $d(\mathcal{J}[X]) = card(\mathcal{J}[X]) = card(X)$.

*Proof.* The first three assertions are easy to prove and the fourth (also easy) appears in [vD].

Something is obviously lacking in part (c) of (2.5): one would like to compute $c(\mathcal{J}[X])$ exactly.

Now I'd like to pause to tie together some of that general theory in an example. The original Pixley-Roy space was $\mathcal{J}[\mathbb{R}]$. According to (2.4) and (2.1), $\mathcal{J}[\mathbb{R}]$ is a completely regular, metacompact Moore space. According to (2.5d), $\mathcal{J}[\mathbb{R}]$ is not separable, and yet, by (2.5c), $\mathcal{J}[\mathbb{R}]$ satisfies the countable chain condition. It follows that $\mathcal{J}[\mathbb{R}]$ is not metrizable, of course, but more is true: no dense subspace of $\mathcal{J}[\mathbb{R}]$ can be metrizable. It follows that the Moore space $\mathcal{J}[\mathbb{R}]$ cannot be densely embedded in any complete Moore space (see Proposition 1 of [PR]) and, even more, that $\mathcal{J}[\mathbb{R}]$ cannot be densely embedded in any Moore space satisfying the Baire Category theorem. (Combining Theorems 4 and 9 of [$Re_1$], one sees that if a Moore space Y can be densely embedded in a Moore space satisfying the Baire Category Theorem, then Y has a dense metrizable subspace. Alternatively, see the proof of 3.3.1 in [AL].)

Looking at the Moore space $\mathcal{J}[X]$ for various subspaces X of $\mathbb{R}$ (and for various generalized ordered spaces constructed on $\mathbb{R}$--see Section 3 for a definition) leads to the following questions: for which first-countable spaces X will the Moore space $\mathcal{J}[X]$ be completable? In particular, is there a generalized ordered space X constructed on $\mathbb{R}$ such that $\mathcal{J}[X]$ is completable but not metrizable? (An excellent survey of the ideas and tools in completability theory is given in [$\mathrm{Re}_1$].)

There are certain fairly general results which have been useful in studying the PR-hyperspaces of, for example, ordinals. However, these lemmas do not themselves involve special spaces and may have broader use, so I will include them in this section. The proofs are straightforward.

2.6 *Theorem. Suppose* $X = Y \cup Z$ *where* Y *and* Z *are open subspaces of* X. *Define a function* $u: \mathcal{J}[Y] \times \mathcal{J}[Z] \to \mathcal{J}[X]$ *by* $u(A,B) = A \cup B$. *Then the image of* u *is a closed and open subspace of* $\mathcal{J}[X]$ *and* u *is a continuous open mapping. In particular, if* Y *and* Z *are* <u>*disjoint*</u> *open subspaces of* X *then* $\mathcal{J}[X]$ *is homeomorphic to the topological sum (=disjoint union)* $\mathcal{J}[Y] \dot{\cup} \mathcal{J}[Z] \dot{\cup} (\mathcal{J}[Y] \times \mathcal{J}[Z])$.

2.7 *Theorem. Suppose* $g: X \to Y$ *is a closed, continuous, finite-to-one mapping from* X *onto* Y. *Define* $g^*: \mathcal{J}[Y] \to \mathcal{J}[X]$ *by the rule that* $g^*(F) = g^{-1}[F]$ *for each* $F \in \mathcal{J}[Y]$. *Then* $g^*$ *is a homeomorphism of* $\mathcal{J}[Y]$ *onto a subspace of* $\mathcal{J}[X]$.

2.8 *Theorem. For any spaces* X *and* Y, $\mathcal{F}[X] \times \mathcal{F}[Y]$ *can be embedded as a closed subspace of* $\mathcal{F}[X \times Y]$ *by the map* $p(A,B) = A \times B$.

We have already seen (2.3 and 2.4) that sometimes the properties of $\mathcal{F}[X]$ can be studied in terms of properties of the subspaces $\mathcal{F}_n[X]$, and Theorem 3.6 below will present a more spectacular example of that phenomenon. These results suggest the following question: What is the relationship between $\mathcal{F}[X]$ and the sequence $\langle \mathcal{F}_n[X] \rangle$ of its subspaces?

## 3. Metrizability of Special Pixley-Roy Spaces

In his paper [vD], van Douwen pointed out the inverse relation which seems to exist between properties of X and properties of $\mathcal{F}[X]$: nice spaces X have bad hyperspaces $\mathcal{F}[X]$, and vice versa. To illustrate that thesis, he observed that while $\mathcal{F}[\mathbb{R}]$ does not have even a dense metrizable subspace, both of the spaces $\mathcal{F}[[0,\omega_1)]$ and $\mathcal{F}[S]$, where S is the familiar Sorgenfrey line, are themselves metrizable. Van Douwen did not publish proofs for those last two assertions; there are now several proofs available (besides the ones which he had in mind).

3.1 *Theorem. Suppose* X *is first-countable and locally countable (i.e.,* X *can be covered by open, countable subsets). Then* $\mathcal{F}[X]$ *is metrizable.*

The proof of (3.1) rests on an easy relative Smirnov's metrization theorem: if Y is regular, metacompact and can be covered by open, separable metrizable subspaces, then Y is metrizable. In addition to showing that $\mathcal{F}[[0,\omega_1)]$ is metrizable, (3.1) shows that $\mathcal{J}[X]$ is metrizable even for such wierd spaces as the Ostaszewski line (which exists if $\diamondsuit$ holds--see [O]), the Kunen line (which exists under CH--see [JKR]) and the wierd lines constructed by van Douwen and Wicke [vDW].

There is another result which proves that $\mathcal{J}[[0,\omega_1)]$ is metrizable, namely

3.2 *Theorem* [$\mathrm{BFL}_1$]. *Let* $\alpha$ *be any ordinal and let* X *be any first-countable subspace of* $[0,\alpha)$. *Then* $\mathcal{J}[X]$ *is metrizable.*

The original proof (3.2) was an inductive one and was very messy. The current proof is much cleaner and shorter. Certain generalizations of (3.2) are available. For example:

3.3 *Theorem. Let* $\alpha$ *be any ordinal and let* X *be any first-countable subspace of* $[0,\alpha)$. *Then* $\mathcal{J}[X^2]$ *is metrizable.*

One proof of (3.3) proceeds by induction and uses a result which may be valuable in other contexts (e.g., for giving a quick induction-proof of (3.2)).

3.4 *Proposition. Suppose the point* $p \in Z$ *has a decreasing countable base* $\{V(n) \mid n \geq 1\}$ *of open and closed neighborhoods. For each* $n \geq 1$ *let* $Z_n = Z - V(n)$. *If each* $\mathcal{J}[Z_n]$ *is paracompact, then so is* $\mathcal{J}[Z]$.

*Proof.* Let $\mathcal{Z}_n = \mathcal{J}[Z_n]$. Each $\mathcal{Z}_n$ is a clopen, paracompact subspace of $\mathcal{J} = \mathcal{J}[Z]$. Let $\Phi$ be any open cover of $\mathcal{J}$. Since each $\mathcal{Z}_n$ is a clopen paracompact subspace of $\mathcal{J}$, there is a $\sigma$-locally finite (in $\mathcal{J}$) collection of open subsets of $\mathcal{J}$ which refines $\Phi$ and covers $\mathcal{J}[\cup\{Z_n \mid n \geq 1\}] = \mathcal{J}[Z - \{p\}]$.

Since the set $\{p\}$ can be covered by a single member of $\Phi$, it is enough to find a $\sigma$-locally finite (in $\mathcal{J}$) collection of $\mathcal{J}$-open sets which refines $\Phi$ and covers the set

$$\mathcal{J}^{\#} = \{F \in \mathcal{J} | p \in F \text{ and } \mathrm{card}(F) \geq 2\}.$$

For each $F \in \mathcal{J}^{\#}$ write $F' = F - \{p\}$ and let $n(F)$ be the first integer $k$ having $F' \in \mathcal{J}[Z_k] = \mathcal{Z}_k$. Let

$$\mathcal{J}^{\#}_m = \{F \in \mathcal{J}^{\#} | n(F) = m\}.$$

Obviously it will be enough to show that each collection $\mathcal{J}^{\#}_m$ can be covered by a $\sigma$-locally finite (in $\mathcal{J}$) open family which refines $\Phi$.

Fix $m$. For each $F \in \mathcal{J}^{\#}_m$ choose $\mathcal{G}(F) \in \Phi$ having $F \in \mathcal{G}(F)$. Then find an open set $G(F) \subset Z_m$ and a positive integer $l(F)$ such that

(i) $F' \subset G(F)$;

(ii) $V(l(F)) \cap Z_m = \phi$, i.e., $l(F) \geq m$;

(iii) $[F, G(F) \cup V(l(F))] \subset \mathcal{G}(F)$.

Let $\Phi'(m) = \{[F', G(F)] | F \in \mathcal{J}^{\#}_m\}$. Then $\Phi'(m)$ is an open cover of the paracompact space $\mathcal{J}[Z_m]$ so that there is a collection $\Psi'(m)$ which refines $\Phi'(m)$ and is a locally finite (in $\mathcal{Z}_m$) open (in $\mathcal{Z}_m$) covering of $\mathcal{Z}_m$. Then $\Psi'(m)$ is a locally finite open collection in $\mathcal{J}$. For each $\mathcal{B} \in \Psi(m)$ choose $F(\mathcal{B}) \in \mathcal{J}^{\#}_m$ having $\mathcal{B} \subset [F'(\mathcal{B}), G(F(\mathcal{B}))]$. Then $l(F(\mathcal{B}))$ and $\mathcal{G}(F(\mathcal{B}))$ are defined and $[F(\mathcal{B}), G(F(\mathcal{B})) \cup V(l(F(\mathcal{B})))] \subset \mathcal{G}(F(\mathcal{B}))$. For each $T \in \mathcal{B}$, choose an open set $W(T,\mathcal{B}) \subset Z_m$ containing $T$ and having $[T, W(T,\mathcal{B})] \subset \mathcal{B}$. Now define

$$\mathcal{U}(\mathcal{B}) = \cup\{[T \cup \{p\}, W(T(\mathcal{B}) \cup V(l(F(\mathcal{B})))] | T \in \mathcal{B}\}.$$

Then $\mathcal{U}(\mathcal{B}) \subset \mathcal{G}(F(\mathcal{B}))$ so that the collection $\Psi(m) = \{\mathcal{U}(\mathcal{B}) | B \in \Psi'(m)\}$ refines $\Phi$. Further, $\Psi(m)$ covers $\mathcal{J}^{\#}_m$. Finally, it is a straightforward matter (albeit tedious) to verify that $\Psi(m)$ is locally finite in $\mathcal{J}$, as required to complete the proof.

*Remark.* After the original version of this paper was completed, a significant sharpening of (3.2) was obtained. We [$BFL_2$] can prove:

3.5 *Theorem. Let* X *be any subspace of any ordinal* $[0,\alpha)$. *Then* $\mathcal{J}[X]$ *is ultraparacompact, i.e., each open cover of* $\mathcal{J}[X]$ *admits an open refinement which is pairwise disjoint.*

Now consider the situation when the ground space X is a subspace of, or a relative of, the real line. The fact that the PR-hyperspaces of the Sorgenfrey line and the Michael line are both metrizable (as van Douwen pointed out) can be deduced from a more general result in [$BFL_1$] which tells us exactly when $\mathcal{J}[X]$ will be metrizable for any generalized ordered space X built on a separable linearly ordered set. For sake of simplicity, let me describe the situation for generalized ordered spaces built from $\mathbb{R}$, the usual space of real numbers. Select four disjoint (possibly empty) subsets R, E, I, L having $\mathbb{R} = R \cup E \cup I \cup L$. With $\mathcal{E}$ denoting the usual topology of R, let $\mathcal{J}$ be the topology on $\mathbb{R}$ having the following base:

$\mathcal{E} \cup \{[x,y) \mid x \in R, y > x\} \cup \{(z,x] \mid x \in L, z < x\} \cup \{\{x\} \mid x \in I\}$.

We denote the resulting space $(\mathbb{R},\mathcal{J})$ by $GO(R,E,I,L)$, and we can then prove

3.6 *Theorem. Let* $X = GO(R,E,I,L)$. *Then the following* *are equivalent:*

*a)* $\mathcal{J}[X]$ *is metrizable;*

*b)* $\mathcal{J}_n[X]$ *is metrizable for each* $n \geq 1$*;*

*c)* $\mathcal{J}_2[X]$ *is metrizable;*

*d) the sets* R, E, I *and* L *satisfy:*

*(i)* E *is countable;*

*(ii) If* $S = R \cup L$ *is topologized as a subspace of* X, *then* R *and* L *are each* $F_\sigma$*-subsets of* S*;*

*(iii)* R *can be written as* $R = \cup\{R(n) \mid n \geq 1\}$ *in such a way that if* $x \in E \cap cl_X(R(m))$ *then for some* $y < x$, $(y,x) \cap R(m) = \phi$*;*

*(iv)* L *can be written as* $L = \cup\{L(n) \mid n \geq 1\}$ *in such a way that if* $x \in E \cap cl_X(L(n))$ *then for some* $z > x$, $(x,z) \cap L(n) = \phi$.

Applying (3.6) to specific spaces is usually an easy matter. For example, if S is the Sorgenfrey line than $S = GO(\mathbb{R},\phi,\phi,\phi)$ so (d) is trivially satisfied. And if M is the Michael line then $M = GO(\phi,\emptyset,\mathbb{P},\phi)$, where $\emptyset$ and $\mathbb{P}$ denote the sets of rational and irrational numbers respectively, and again (d) is vacuoisly satisfied. On the other hand, the "mixed Sorgenfrey line" $GO(\emptyset,\phi,\phi,\mathbb{P})$ has a non-metrizable PR-hyperspace.

Stating the corresponding theorem for generalized ordered spaces built on arbitrary separable linearly ordered sets is messier because of the potential existence of two many "jumps," i.e., points $x < y$ having $[x,y] = \{x,y\}$. A typical example of this phenomenon is the "Alexandroff double arrow" space, i.e., the lexicographically ordered set $A = [0,1] \times \{0,1\}$

endowed with the usual open interval topology. That $\mathcal{F}[A]$ is not metrizable follows from Theorem I of [$BFL_1$] or from Theorem 2.7, above, since the mapping $g$: $A \to [0,1]$ which collapses the points $(x,0)$ and $(x,1)$ to $x$ induces an embedding $g^*$: $\mathcal{F}[[0,1]] \to \mathcal{F}[A]$, and $\mathcal{F}[[0,1]]$ is non-metrizable.

*Remark*. The sharpening of (3.2) mentioned in (3.5) has an analogue for the metrization theorem given in (3.6). Let's say that a space is *ultrametrizable* if it has a base of open sets which can be written as the union of countably many subcollections, each of which is a disjoint open cover of the entire space. It is proved in [$BFL_2$] that if X is a generalized ordered space constructed from $\mathbb{R}$ and satisfying d) of (3.6), then X must be ultrametrizable.

Having disposed of the PR-hyperspace of spaces related to separable linearly ordered sets, one might reasonably turn to subspaces of a Souslin line, i.e., a compact, connected, non-separable linearly ordered space which satisfies the countable chain condition. Everyone knows that whether or not such things exist depends on your set theory: under (V = L) they do; under Martin's Axiom plus $\omega_1 < c$ they do not. In her talk later today, Mary Ellen Rudin will present her solution of the following problem: Let X be a subspace of a Souslin line; give sufficient conditions for $\mathcal{F}[X]$ to be metrizable. Details of the solution will appear in [$Ru_2$]. You will see that the problem really reduces to the situation studied in (3.6).

## 4. Normality of Special PR Spaces

Some of the most elegant work involving the PR hyperspace has dealt with problems of normality. In their paper [PT], Przymusiński and Tall proved

4.1 *Theorem. Let* X *be an uncountable separable metric space. If each finite power of* X *is a* Q-*set then* $\mathcal{J}$[X] *is normal.*

To say that a space X is a Q-*set* means that X is an uncountable separable metric space and every subset of X is an $F_\sigma$-set in X. This notion entered modern topology through the work of Bing and Heath who proved that there is a separable, normal nonmetrizable Moore space if and only if there is a Q-set.

Today one of the most interesting problems about Q-set topology is "if X is a Q-set, what about $X^n$?" In a recent paper [$Ru_2$], Mary Ellen Rudin gave a new proof of (4.1) and proved its converse as well. Therefore we have

4.2 *Theorem. Let* X *be an uncountable separable metric space. Then each finite power of* X *is a* Q-*set if and only if* $\mathcal{J}$[X] *is normal.*

Using the observation in (2.8) above, we can sharpen (4.2) to read:

4.3 *Theorem. Let* X *be an uncountable separable metric space. Then each finite power of* X *is a* Q-*set if and only if* $(\mathcal{J}[X])^\omega$ *is normal (and metacompact,* CCC*).*

*Proof.* If $(\mathcal{J}[X])^\omega$ is normal then so is its closed subspace $\mathcal{J}$[X] so that (4.2) applies. Conversely, suppose each power of X is a Q-set. According to (2.8) applied inductively,

$(\mathcal{J}[X])^n$ embeds as a closed subset in $\mathcal{J}[X^n]$ for each $n \in \omega$. But every finite power of $X^n$ is a Q-set and so $\mathcal{J}[X^n]$ is normal. But then so is $(\mathcal{J}[X])^n$. Now consider $(\mathcal{J}[X])^\omega$. It is a Moore space and hence is perfect. But now a theorem of Katětov [K] may be applied to conclude that $(\mathcal{J}[X])^\omega$ is normal.

Most of the early work on normality of $\mathcal{J}[X]$ used Martin's axiom plus $(\omega_1 < c)$. For example, that special axiom was used in the paper by Przymusiński and Tall [PT], but only to conclude that every uncountable separable metric space with cardinality $< c$ is a Q-set. Also, the argument in (4.3) circumvents one use of Martin's Axiom in the paper by Alster and Przymusiński [AP].

Obviously, if one intends to prove that every finite power of every Q-set is again a Q-set, it is enough to prove that the *square* of every Q-set is a Q-set. However, it is also possible to investigate the various finite powers of an individual Q-set. One way is given in our next theorem which is essentially proved in [PT] and $[Ru_2]$.

4.4 *Theorem. Let* X *be an uncountable separable metric space. The following are equivalent:*

*a)* $X^n$ *is a* Q-*set*

*b) if* $\mathcal{H}$ *and* $\mathcal{K}$ *are disjoint closed subsets of* $\mathcal{J}_n[X]$ *then these are disjoint open subsets* $\mathcal{U}$ *and* $\mathcal{V}$ *of* $\mathcal{J}[X]$ *with* $\mathcal{H} \subset \mathcal{U}$ *and* $\mathcal{K} \subset \mathcal{V}$.

Studying normality in separable Moore spaces led to the study of Q-sets. Studying countable paracompactness in separable Moore spaces led to a related notion, called

$\Delta$-sets [$Re_2$]. An uncountable separable metric space X is called a *$\Delta$-set* if whenever $S_1 \supset S_2 \supset \cdots$ are subsets of X having $\cap\{S_n | n \geq 1\} = \phi$ it is possible to find open subsets $G_n$ of X having $G_n \supset S_n$ and $\cap\{G_n | n \geq 1\} = \phi$. (The argument given by Przymusiński in [P, p. 334] shows that any $\Delta$-set must have cardinality $< c$. Then it follows that any $\Delta$-set can be embedded in $\mathbb{R}$. Hence the restriction that a $\Delta$-set be a subspace of $\mathbb{R}$, which is part of the original definition of $\Delta$-set, is superfluous.) Combining theorems of Reed [$Re_2$] and Przymusinski [P] yields the conclusion that there is a separable, countably paracompact nonmetrizable Moore space if and only if there is a $\Delta$-set.

It is obvious that any Q-set is a $\Delta$-set, but the converse is open (unless special axioms are assumed: for example, under Martin's Axiom plus $\omega_1 < c$, Q-sets and $\Delta$-sets are exactly the same things). A slight modification of the argument given in [PT] combined with half of the proof of (4.3) yields the expected result, namely

4.5 *Theorem. Suppose every finite power of* X *is a $\Delta$-set. Then* $(\mathcal{J}[X])^n$ *is countably paracompact for each* $n < \omega$.

That result reduces at least part of the $\Delta$-set vs. Q-set problem to considering special PR-spaces: if $\mathcal{J}[X]$ is normal whenever it is countably paracompact (for $X \subset \mathbb{R}$), then every finite power of X is a Q-set provided every finite power of X is a $\Delta$-set.

*Remark.* It does not appear to be known whether, if each finite power of a space Y is perfect and countably paracompact, it follows that $Y^\omega$ is countably paracompact. Thus, (4.5) contrasts with (4.3) in that one does not obtain countable paracompactness of $(\mathcal{J}[X])^\omega$ free.

It would be desirable to prove the converse of (4.5), namely that if $\mathcal{J}[X]$ is countably paracompact (for $X \subset \mathbb{R}$), then every finite power of X is a $\Delta$-set. It is easy to see that if $\mathcal{J}_2[X]$ is countably paracompact then X is a $\Delta$-set (use the representation of $\mathcal{J}_2[X]$ described in [B]) and it may be that the techniques of [$Ru_2$] can be modified to prove the desired converse. However there is some work to be done because, unlike normality, countable paracompactness is not known to be a hereditary property in the class of Moore spaces.

It would be of interest to know whether countable paracompactness is a hereditary property in the special Moore spaces obtained from the Pixley-Roy construction applied to subspaces of the real line. A solution of that problem would have some bearing on the Q-set vs. $\Delta$-set problem, as the next result shows.

4.6 *Theorem. The following assertions are equivalent:*

*a) if each finite power of* X *is a* $\Delta$*-set, then each finite power of* X *is a* Q*-set (and conversely);*

*b) if each finite power of* X *is a* $\Delta$*-set, then each subspace of* $\mathcal{J}[X]$ *is countably paracompact.*

*Proof.* That a) implies b) is trivial since perfect normality of $\mathcal{J}[X]$ is hereditary property and implies countable paracompactness. To prove that b) implies a), suppose that each finite power of X is a $\Delta$-set. Let S denote a convergent sequence $\{1/n: n \geq 1\} \cup \{0\}$ with the usual topology. It is known that a space Y is normal provided the product space $Y \times S$ is hereditarily countably paracompact, so consider the space $\mathcal{J}[X] \times S$. Certainly S can be embedded in the hyperspace $\mathcal{J}[S]$ so that, in the light of (2.8), $\mathcal{J}[X] \times S$ can be embedded in $\mathcal{J}[X] \times \mathcal{J}[S]$ which embeds in $\mathcal{J}[X \times S]$. Now consider $(X \times S)^n$ where $n < \omega_o$. Since $(X \times S)^n = X^n \times S^n$, we see that $(X \times S)^n$ is the product of a $\Delta$-set and a countable metric space. But it is easy to show that any such product must be a $\Delta$-set. Therefore, according to b), each subspace of $\mathcal{J}[X \times S]$ is countably paracompact so that the same is true of $\mathcal{J}[X] \times S$. Therefore $\mathcal{J}[X]$ must be normal so that Rudin's theorem (4.2), above, may be applied to show that each finite power of X is a Q-set.

*Added in Proof:* 1) Concerning OC mappings (cf. Section 2), Bennett has shown that each space $\mathcal{J}[X]$ is the image of a paracompact space under a continuous, open, compact mapping. 2) Concerning squares of Q-sets: Przymusinski has recently announced [Notices Amer. Math. Soc. 25 (1978), A-610] that if there is a Q-set, then there is a Q-set all of whose finite powers are also Q-sets.

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