LOCALLY CONNECTED GENERALIZED ABSOLUTE NEIGHBORHOOD RETRACTS

by

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In 1952 Hiroshi Noguchi [No, p. 20] defined the follow-
ing generalization of absolute neighborhood retracts.

Definition. A compact subset X of a metric space Y is
an *approximative neighborhood retract of* Y *in the sense of*
Noguchi if there is a neighborhood U of X in Y such that for
every positive real number ε there is a map r from U into X
such that $d(r|X, idX) < \varepsilon$ [Gml, p. 61], [Gm2, p. 9], [Gr, pp.
16-17], [Fi, p. 42].

Definition. A compact metric space X is an *approxima-*
tive absolute neighborhood retract in the sense of Noguchi
$(AANR_N)$ provided it is an approximative neighborhood retract
in the sense of Noguchi of any metric space in which it can
be embedded.

A further generalization which allows the neighborhood
to vary with ε as well as the map r was given in 1970 by
Michael Clapp. These spaces are called *approximative absolute*
neighborhood retracts in the sense of Clapp $(AANR_C)$ [Cl, p.
118], [Fi, p. 43].

The importance of these generalizations is that they
satisfy certain fixed point properties which are well known
for the case of ANR's. Such properties are as follows:

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[1]This paper appears as a part of the author's Ph.D. dis-
sertation done under Professor Leonard R. Rubin at the Uni-
versity of Oklahoma.

(1) Every null-homotopic map of an AANR$_C$ into itself has
a fixed point [Cl, Theorem 5.7, p. 126].

(2) Every acyclic AANR$_C$ has the fixed point property
[Cl, Theorem 5.6, p. 126].

In some sense, however, these generalizations of compact
ANR's are very far from the concept of an ANR. In particular,
if one wishes to study the properties of the image of a com-
pact ANR under a certain class of maps it is clear that the
image must at least be locally connected. Since there are,
nevertheless, trivial examples of AANR's which are not
locally connected, it may be profitable to study generalized
ANR's that must be locally connected.

The first step in this direction is a modification of the
definitions given above.

Definition. By a *surjective approximative retract* of a
metric space Y will be meant a compact subspace X of Y such
that for every positive real number ε there is a map r from
Y onto X such that r(X) = X and d(r|X,idX) < ε.

Definition. By a *surjective approximate neighborhood
retract of a metric space Y in the sense of Noguchi* will be
meant a compact subspace X which is a surjective approximate
retract of some neighborhood of X in Y.

Definition. By a *surjective approximate neighborhood
retract of a metric space Y in the sense of Clapp* will be
meant a compact subspace X of Y such that for every positive
real number ε there are a neighborhood U of X in Y and a
map r from U onto X such that r(X) = X and d(r|X,idX) < ε.

If a compact metric space is a surjective approximative
retract (a surjective approximative neighborhood retract in
the sense of Noguchi or Clapp) of every metric space in which
it can be embedded, then it is said to be a *surjective approxi-*
mative absolute retract (a surjective approximative absolute
neighborhood retract in the sense of Noguchi or Clapp).
These will be abbreviated as SAAR, $SAANR_N$, or $SAANR_C$. It is
clear that since any SAAR, $SAANR_N$, $SAANR_C$ X is assumed to be
compact, it may be embedded in the Hilbert cube Q. Since any
neighborhood of X in the locally connected space Q contains
a compact locally connected neighborhood of X, it is clear
that X is the image of a compact locally connected space
under an identification. Hence, X is locally connected. This
means that the class of SAAR, $SAANR_N$, or $SAANR_C$ is properly
contained in the class of AAR, $AANR_N$, or $AANR_C$ respectively.
Thus, the fixed point properties (1) and (2) still hold for
these new generalizations. Of course, it is important to
ask that the surjective approximate retracts have certain
properties in common with ANR's which the more general ver-
sions fail to have. In this direction an apparently differ-
ent type of generalization of ANR will be made.

Definition. A compact metric space X is said to be a
quasi-ANR (q-ANR) if for every positive real number ε there
exist an ANR A, a map f from X onto A, and a map g from A
onto X such that gf is ε-near idX (i.e. $d(gf(x),x) < \varepsilon$ for
all x in X). If the choice of A is independent of ε then the
space X will be called a *uniform q*-ANR.

If in the above definition the ANR's can be replaced

by AR's the space will be called a *quasi*-AR (q-AR).

The surprising property which the q-ANR's satisfy is contained in the following lemma which has been proved in the ANR-case by Lončar and Mardešić [Lo&Ma, Lemma 1].

Lemma 1. Suppose A *is a q-ANR and* f *is a component preserving map of a compact metric space* X *onto* A. *Then given a positive real number* ε *there is a positive real number* δ *such that if* g *is a* δ-*map from* X *onto a metric space* Y, *then there is a map* h *from* Y *onto* A *which satisfies* d(hg,f) < ε.

Proof. We will use the version of this lemma where A is an ANR. This is the version which has been proved in Lončar and Mardešić.

Let ε > 0 be given. Since A is a q-ANR there are a compact ANR B, a map φ from A onto B, and a map ψ from B onto A such that ψφ is ε/2-near idA. Since ψ is uniformly continuous there is a positive real number η such that if b_1 and b_2 belong to B with $d(b_1,b_2) < \eta$ then $d(\psi(b_1),\psi(b_2)) < \varepsilon/2$. By making sure that ψφ is nearer to the identity on A than half the minimum distance between distinct components of A, one can guarantee that φ preserves components. Thus, φf from X onto B may be assumed to be component preserving.

Since B is an ANR and φf is component preserving we may apply the lemma for the ANR case and obtain a δ corresponding to η. Now suppose g is a δ-map from X onto a metric space Y. By the ANR case there is a map Π from Y onto B such that Πg is η-near φf. Hence, ψΠg is a map from Y onto A such that ψΠg is ε/2-near ψφf, which in turn is ε/2-near f. Hence by

letting h = ψΠ we have a map from Y onto A satisfying the
conclusion of the lemma.

It is easy to see that this lemma is another characteri-
zation of a q-ANR. Suppose A is a compact metric space which
makes lemma 1 true. That is, if f from a compact metric
space X onto A is component preserving, then given $\varepsilon > 0$
there is a $\delta > 0$ such that for any δ-map of X onto a metric
space Y there is a map h from Y onto A which satisfies
$d(hg,f) < \varepsilon$. Then it is claimed that A must be a q-ANR. To
see this it is necessary only to observe that the map idA
from A onto A is component preserving, and that for every
$\delta > 0$ there is a δ-map from A onto some finite polyhedron B
which is also an ANR. Thus, results about polyhedra and ANR's
which depend on lemma 1 are results which hold for q-ANR's.
See [Lo&Ma, Theorem 1] and [Fo&Rg, section 3].

The connection between q-ANR's and SAANR$_C$'s will be
given in the following lemmas and theorem.

Lemma 2. *If* X *is a surjective approximative neighbor-*
hood retract of the Hilbert cube Q *in the sense of Clapp*
(Noguchi) then it is an SAANR$_C$ *(*SAANR$_N$*)* [Cl], [Fi].

Since the proof of this lemma is similar to the corre-
sponding proof for the other generalized ANR's it will be
omitted.

Lemma 4. *Every* q-ANR *is a* SAANR$_C$.

Proof. It may be assumed that the q-ANR X is contained
in Q. Let $\varepsilon > 0$ be given. Since X is a q-ANR there exist
an ANR A, a map f from X onto A, and a map g from A onto X

such that gf is ε-near idX. Since A is also a neighborhood

extensor there is a map φ into A which extends f to some

neighborhood U of X in Q. Now let r = gφ. Then r is a map

from U onto X such that r(X) = gφ(X) = gf(X) = X and

d(r|X,idX) = d(gf,idX) < ε. Thus, X is a surjective approxi-

mative neighborhood retract of Q in the sense of Clapp.

Therefore, by lemma 2 X is a SAANR$_C$.

The surprising thing about lemma 3 is that the converse

is also valid. To see that, we need the following lemmas.

Lemma **4.** *Let* X *be a* SAANR$_C$. *Then for every positive*

real number ε *there is a positive real number* δ *such that if*

f *from* X *onto some metric space* Y *is a* δ-map, *then there is*

a map g *from* Y *into* X *so that* gf *is* ε-near idX [No, p. 21].

Proof. Notice that the proof given in Noguchi is still

valid if the neighborhoods as well as the maps are allowed

to vary with ε.

Lemma **5.** *Let* Q *be the product of intervals* [0,1/i] *with*

the restriction of the Hilbert space metric as a metric. Sup-

pose X *is a locally connected compact metric space embedded*

in Q. *Then for every* η > 0 *there is a map* h *from* X *into* Q

such that

  (i) if C *is a non-degenerate component of* X, *then* h(C)

      contains N$_η$(C), *and*

  (ii) h *is* 3η-near *the embedding of* X [Lo&Ma, lemma 5, p.

      42].

Proof. The proof for this lemma may be found in the

Lončar and Mardešić paper.

Theorem 1. *A compact metric space is a q-ANR if and only if it is a* SAANR$_C$.

Proof. One way has already been proven in lemma 3.

Assume that X is a SAANR$_C$ contained in Q = $\Pi[0,1/i]$. Let $\varepsilon > 0$ be given. Since X is a SAANR$_C$ there are a neighborhood $N_{4\eta}(X)$ of X in Q and a map r from $N_{4\eta}(X)$ onto X such that r(X) = X and d(r|X,idX) < ε. Notice that $\eta > 0$ can be chosen so that if d(a_1,a_2) < 4η for elements in $N_{4\eta}(X)$, then d(r(a_1),r(a_2)) < ε. By lemma 4 there is a $\delta > 0$ such that if f from X into a metric space Y is a δ-map, then there is a map g from Y into X such that gf is η-near idX.

Since X is compact there is a δ-map f from X onto some polyhedron Y which is also an ANR. Hence, there is a map g from Y into X such that d(gf(x),x) < η for all x in X. Hence X is contained in N_η(g(Y)). Clearly η may be chosen so that gf is component preserving. Thus, on the degenerate point components {x} of X gf(x) = x. According to lemma 5 there is a map h from g(Y) into Q such that hg(Y) $\supset$ X, h fixes the point components of g(Y), and d(hg(y),g(y)) < 3η for all y in Y. It is clear that d(hgf(x),x) < 4η for all x in X. Thus, rhgf is a map from X into X. Now X $\supset$ rhgf(X) = rhg(Y) $\supset$ r(X) = X and d(rhgf(x),x) < 2ε for all x in X. Therefore, rhg is a map from Y onto X such that rhgf is 2ε-near idX. Hence, X is a q-ANR.

In the following some results about q-ANR's, q-AR's, and SAAR's will be given.

Definition. A compact metric space will be said to be *quasi-contractible* provided that for each $\varepsilon > 0$ there is a

homotopy F_t of X into itself so that F_0 is a constant map,
$F_1(X) = X$, and F_1 is ε-near idX.

Lemma 6. *A quasi-contractible* SAANR$_C$ *is a* SAAR *[C1, p.*
128].

Proof. It suffices to show that if the quasi-contracti-
ble SAANR$_C$ X is embedded in Q, then it is a surjective ap-
proximate retract of Q.

Let $\varepsilon > 0$ be given. Since X in Q is quasi-contractible
there is a homotopy F_t such that F_0 is constant, $F_1(X) = X$,
and $d(F_1(x),x) < \varepsilon/2$ for all x in X. Now since X is a SAANR$_C$
there exist a neighborhood U of X in Q and a map r from U
onto X such that $r(X) = X$ and $d(r|X,idX) < \varepsilon/2$. It is clear
that F_0 extends to a map of Q into U (again a constant map).

We now make use of the following lemma found in Borsuk
[Bk, p. 94]. *Let X be a closed subset of a metrizable space*
Z. Let U be a metric ANR. *Let F_t be a homotopy from X into*
U so that F_0 has an extension to all of Z. Then there is a
homotopy H_t from Z into U such that H_t extends F_t for all t.
In particular, for our neighborhood U found above there is an
extension H_1 of F_1 to all of Q. Consider rH_1 from Q into X.
Then $rH_1(X) = rF_1(X) = r(X) = X$ and $d(rH_1|X,idX) \le d(rF_1,F_1)$
$+ d(F_1,idX) < \varepsilon$. Hence, X is a surjective approximate re-
tract of Q.

By embedding a SAAR X in Q and using the fact that Q is
contractible the converse of lemma 6 follows. Thus, we have
the following theorem.

Theorem 2. *A compact metric space is a* SAAR *if and only*

if it is a quasi-contractible SAANR$_C$.

By following the proof of lemma 3 it can be seen that
every q-AR is a SAAR. This leads naturally to the question;
is every SAAR a q-AR? A partial answer is the following
theorem.

Theorem 3. Every AR-*like* SAAR *is a q-*AR.

Proof. Let X be an AR-like SAAR. Then clearly X is a
SAANR$_C$; hence, X is a q-ANR. Let ε > 0 be given. Then by
lemma 1 there is a δ > 0 such that if g is a δ-map from X
onto a metric space Y, then there is a map h from Y onto X
such that hg is ε-near idX. Since X is AR-like there is a
δ-map from X onto an AR A. Hence, there is an h from A onto
X such that hg is ε-near idX. This proves that X is a q-AR.

The Hawaiian earring H (figure 1) is an example of a
q-ANR which is not an ANR. The Hawaiian earring also fails
to be a SAANR$_N$; hence, it can not be either a q-AR or SAAR,
nor can it be quasi-contractible. It is interesting to ob-
serve that the cone over H, C(H), is a contractible (hence,
quasi-contractible) q-ANR. Furthermore, since C(f) from
C(H) onto C(A) is an ε-map whenever f from H onto A is, C(H)
is an AR-like SAAR; thus, by theorem 3 C(H) is a q-AR.
Since like H, C(H) fails to be locally contractible, C(H) is
not an AR.

Figure 1

H
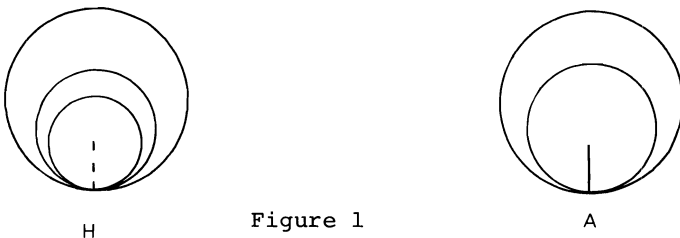
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References

Bk   K. Borsuk, *Theory of retracts*, Monografie Matematyczne
        Vol. 44, Polish Scientific Publishers, Warsaw, 1967.

Cl   M. H. Clapp, *On a generalization of absolute neighbor-*
        hood retracts, Fund. Math. 70 (1971), 117-130.

Fi   A. S. Finbow, *Proximate approximation theory and*
        generalized ANR's, doctoral dissertation, University
        of Washington, 1975.

Fo&Rg   J. Ford and J. W. Rogers, *Refinable maps* (to appear in
        Colloq. Math.

Gml   A. Gmurczyk, *Approximative retracts and fundamental*
        retracts, Coll. Math. 23 (1971), 61-63.

Gm2   _____, *On approximative retracts*, Bull. Acad. Polon.
        Sci., Ser. Sci. Math. Astr. et Phys. 16 (1968), 9-14.

Gr   A. Granas, *Fixed point theorems for the approximative*
        ANR's, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astr.
        et Phys. 16 (1968), 15-19.

Lo&Ma   I. Lončar and S. Mardešić, *A note on inverse sequences*
        of ANR's, Glasnik Matematicki (23) 3 (1963), 146-164.

No   H. Noguchi, *A generalization of absolute neighborhood*
        retracts, Kodai Math. Sem. Rep., Tokyo 1 (1953), 20-22.

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