A FINITENESS CONDITION IN CG-SHAPE

by

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PARACOMPACTNESS IN UNIFORM SPACES

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1. Introduction

If (X,d) is a pseudometric space, a cover $\mathcal{G} = (G_\alpha)_{\alpha \in I}$ is said to be *Lebesgue* if there exists a real number $\delta > 0$ such that for all x in X, $B(x,\delta) \subset G_\alpha$ for some $\alpha \in I$. (Of course, $B(x,\delta) = \{y \in X: d(x,y) < \delta\}$.) Lebesgue covers have been extensively studied and shown themselves to be important in many areas, especially dimension theory (see [4], [7], [9], [10], [11]).

There are at least two ways to generalize Lebesgue covers to more general spaces. A cover $\mathcal{G}$ of a topological space can be called Lebesgue if there exists a neighborhood W of the diagonal of X such that $(W(x))_{x \in X}$ refines $\mathcal{G}$. This term was used in [1, p. 28]. On the other hand, Smith [10] called a cover $\mathcal{G}$ of a uniform space $(X, \mathcal{U})$ Lebesgue if there exists a $U \in \mathcal{U}$ such that $(U(x))_{x \in X}$ refines $\mathcal{G}$. So as to avoid confusion and because it seems more natural we will call these latter covers $\mathcal{U}$-*even*. Our motivation comes from Kelley [6] where a cover $\mathcal{G}$ of a topological space is called *even* if there exists a neighborhood W of the diagonal of X such that $(W(x))_{x \in X}$ refines $\mathcal{G}$. We will use the term *Lebesgue cover* only if we are in a pseudometric space.

In [11], Smith studied Lebesgue covers and $\mathcal{U}$-even covers. He ended this paper by posing two questions:

(1) In a metric space (X,d) does every Lebesgue cover have a locally finite Lebesgue refinement?

(2) In a uniform space does every U-even cover necessarily have a normal sequence of U-even covers that refine the given cover?

Recently Ščepin ([7]) answered the first question negatively. In this paper we answer the second question positively (see 3.2). We then derive some consequences of this result. We then define U-paracompact and its generalization and show some of the interesting properties of this concept.

2. Definitions and Elementary Properties

In general we use the notation and terminology as in [1] and [5].

If $\mathcal{G} = (G_\alpha)_{\alpha \in I}$ and $\mathcal{H} = (H_\beta)_{\beta \in J}$ are two families of subsets of X we write $\mathcal{G} \wedge \mathcal{H}$ for $\{G_\alpha \cap H_\beta : \alpha \in I$ and $\beta \in J\}$. If γ is a cardinal number we say that $\mathcal{G}$ *has power at most* γ in case the cardinal number of I is less than or equal to γ.

If X is a topological space, if F is a subset of X and if U is a neighborhood of the diagonal of X then we set $U(F) = \{y \in X: (x,y) \in U$ for some $y \in F\}$. If $F = \{x\}$ we write $U(x)$ instead of $U(\{x\})$. We define $U \circ U = U^2 = \{(x,y) \in X \times X:$ there exists $z \in X$ with $(x,z) \in W$ and $(z,y) \in W\}$.

If X is a topological space and if $n \in \underline{N}$ we say that a cover $\mathcal{G}$ is n-*even* if there exist neighborhoods $W_1, \cdots, W_n$ of the diagonal of X such that $W_i^2 \subset W_{i-1}$ for $i = 2, \cdots, n$ and $(W_1(x))_{x \in X}$ refines $\mathcal{G}$. If there exists a sequence $(W_n)_{n \in \underline{N}}$ of neighborhoods of the diagonal of X such that $W_n^2 \subset W_{n-1}$ for all $n \in \underline{N}$, $n \neq 1$, and $(W_1(x))_{x \in X}$ refines $\mathcal{G}$ we say that $\mathcal{G}$ is $\aleph_0$-*even*. If $n = 1$ we write "even" instead of "1-even" and note that this is the usual definition of even.

A sequence $(U_n)_{n \in \underline{N}}$ of open covers of a topological

space X is *normal* if for all $n \in \underline{N}$, $U_{n+1}*$ refines U_n (i.e.

$(st(U, U_{n+1}))_{U \in U}$ refines U_n). The cover $\mathcal{G}$ is *normal* if there

is a normal sequence $(U_n)_{n \in \underline{N}}$ of open covers of X such that

U_1 refines $\mathcal{G}$.

By a *uniformity* U on X we mean a non-empty collection

of subsets of X × X satisfying the usual axioms. If X is a

completely regular space we write U_o for the *universal uni-*

formity on X; i.e. the finest uniformity compatible with the

topology on X. If the universal uniformity is the collec-

tion of all neighborhoods of the diagonal of X we will say

that X is *strongly collectionwise normal*. It is known that

if X is paracompact then X is strongly collectionwise normal

and that strongly collectionwise normal implies collectionwise

normal (see [2] and [3]). Furthermore, in general, neither

of these implications can be reversed. In [1] Richard A. Alo

and the first author proved the following two results.

2.1 *Theorem. If $\mathcal{G}$ is an open cover of a topological*

space then $\mathcal{G}$ is normal if and only if $\mathcal{G}$ is $\aleph_o$-even.

2.2 *Theorem. A completely regular topological space is*

strongly collectionwise normal if and only if every even open

cover is normal.

A completely regular topological space is *almost compact*

if $|\beta X - X| \leq 1$. Clearly every compact space is almost com-

pact. There are many characterizations in the literature of

almost compact spaces. We list here the following character-

ization and refer the reader to [5].

2.3 *Theorem. A completely regular topological space* X *is almost compact if and only if* X *admits a unique uniformity.*

3. $\mathcal{U}$-even Covers

Before starting on our main results concerning $\mathcal{U}$-even covers, we state some elementary properties of these covers.

3.1 *Lemma. If* $(X, \mathcal{U})$ *is a uniform space and if* $\mathcal{G}$ *and* $\mathcal{H}$ *are* $\mathcal{U}$-even *covers of* X, *then the following statements hold.*

    (1) If $\mathcal{J}$ *is a cover of* X *and if* $\mathcal{G}$ *refines* $\mathcal{J}$ *then* $\mathcal{J}$ *is*
        $\mathcal{U}$-even.

    (2) The cover $\mathcal{G} \wedge \mathcal{H}$ *is* $\mathcal{U}$-even.

    (3) There is an open $\mathcal{U}$-even *cover that refines* $\mathcal{G}$.

    Proof. The simple proofs of (1) and (2) are omitted. To see that (3) holds, let $U \in \mathcal{U}$ such that $(U(x))_{x \in X}$ refines $\mathcal{G}$. Since $\mathcal{U}$ is a uniformity, $V = \text{int } U$ is an element of $\mathcal{U}$, so $(V(x))_{x \in X}$ is an open $\mathcal{U}$-even cover that refines $\mathcal{G}$.

We can now show the relationship between the "evenness" and "normality" of a given cover relative to a uniformity.

3.2 *Theorem. If* $(X, \mathcal{U})$ *is a uniform space and if* $\mathcal{G} = (G_\alpha)_{\alpha \in I}$ *is a cover of* X *then the following statements are equivalent:*

    (1) The cover $\mathcal{G}$ *is* $\mathcal{U}$-normal.

    (2) The cover $\mathcal{G}$ *is* $\mathcal{U}$-even.

    Proof. (1) *implies* (2). If $\mathcal{G}$ is $\mathcal{U}$-normal there exists a sequence of open covers $(\mathcal{G}_i)_{i \in \underline{N}}$ such that for all $i \in \underline{N}$ $\mathcal{G}_i$ is $\mathcal{U}$-even, $\mathcal{G}_{i+1}*$ refines $\mathcal{G}_i$ and $\mathcal{G}_1$ refines $\mathcal{G}$. So, in particular, there exists a U in $\mathcal{U}$ such that $(U(x))_{x \in X}$ refines $\mathcal{G}_1$. Since $\mathcal{G}_1$ refines $\mathcal{G}$, 3.1 shows that $\mathcal{G}$ is $\mathcal{U}$-even.

(2) *implies* (1). Since $\mathcal{G}$ is $\mathcal{U}$-even we can choose an open symmetric U_o in $\mathcal{U}$ such that $\mathcal{G}_o = (U_o(x))_{x \in X}$ refines $\mathcal{G}$. Since $\mathcal{U}$ is a uniformity there exists an open symmetric element U_1 in $\mathcal{U}$ such that $U_1^2 \subset U_o$. Let $\mathcal{G}_1 = (U_1(x))_{x \in X}$ and note that $\mathcal{G}_1$ Δ-refines $\mathcal{G}_o$ since $\text{st}(x, \mathcal{G}_1) \subset U_o(x)$. Furthermore $\mathcal{G}_1$ is $\mathcal{U}$-even. We now proceed by induction to construct covers $\mathcal{G}_n$ so that each $\mathcal{G}_n$ is a Δ-refinement of $\mathcal{G}_{n-1}$ by simply selecting neighborhoods $U_2, U_3, \cdots$ of the diagonal of X so that $U_i^2 \subset U_{i-1}$ for $i \geq 2$. We thus have a sequence $(\mathcal{G}_n)_{n \in \underline{N}}$ such that $\mathcal{G}_{n+1}$ is a Δ-refinement of $\mathcal{G}_{n-1}$. By [1, 1.20], $(\mathcal{G}_{2n})_{n \in \underline{N}}$ is a normal sequence of open covers and since each cover is $\mathcal{U}$-even the proof is complete.

Since $\mathcal{U}$-normal implies normal, we have the following corollary.

3.3 *Corollary.* *If* (X, $\mathcal{U}$) *is a uniform space and if* $\mathcal{G}$ *is a* $\mathcal{U}$-even *cover of* X, *then* $\mathcal{G}$ *is normal.*

As a result of 3.3 every $\mathcal{U}$-even cover has a locally finite cozero-set refinement, however, as was shown by Ščepin in [7] we may not in general choose this refinement to be $\mathcal{U}$-even.

Since every $\mathcal{U}$-even cover $\mathcal{G}$ of a uniform space (X, $\mathcal{U}$) is normal there is a continuous pseudometric associated with it. Using standard techniques it can be shown that this pseudometric is uniformly continuous. Conversely, if we have a uniformly continuous pseudometric associated with a normal cover $\mathcal{G}$ then $\mathcal{G}$ has a $\mathcal{U}$-normal refinement and hence is $\mathcal{U}$-even. Thus we have the following result.

3.4 *Theorem.* *If* (X, $\mathcal{U}$) *is a uniform space and if* $\mathcal{G}$ *is*
an open cover of X *then the following statements are equiva-*
lent:

    (1) The cover $\mathcal{G}$ *is* $\mathcal{U}$-*even.*

    (2) There exists a uniformly continuous pseudometric
        associated with $\mathcal{G}$.

We next observe that in a pseudometric space a given
cover is Lebesgue if and only if it is $\mathcal{U}$-even in the pseudo-
metric uniformity. This follows from the fact that sets of
the form $\{(x,y) \in X \times X: d(x,y) < \delta\}$ for all $\delta > 0$ are a
basis for this uniformity.

3.5 *Proposition.* *Suppose that* (X, d) *is a pseudometric*
space, that $\mathcal{U}$ *is the pseudometric uniformity and that* $\mathcal{G}$ *is a*
cover of X. *Then the following statements are equivalent:*

    (1) The cover $\mathcal{G}$ *is Lebesgue (in* (X, d)).

    (2) The cover $\mathcal{G}$ *is* $\mathcal{U}$-*even (in* (X, $\mathcal{U}$)).

We next show that in a pseudometric space every Lebesgue
cover is $\aleph_0$-even but the converse need not hold.

3.6 *Proposition.* *Suppose that* (X, d) *is a pseudometric*
space and that $\mathcal{G}$ *is an open cover of* X. *If* $\mathcal{G}$ *is Lebesgue then*
$\mathcal{G}$ *is* $\aleph_0$-*even.*

    Proof. By hypothesis $\mathcal{G}$ is Lebesgue so there exists
$\delta > 0$ such that $(B_d(x, \delta))_{x \in X}$ refines $\mathcal{G}$. Suppose that $\delta < 1$
and for each $n \in \underline{N}$ let $U_n = \{(x,y) \in X \times X: d(x,y) < \delta/2^n\}$.
Then $U_n^2 \subset U_{n-1}$ and $(U_1(x))_{x \in X}$ refines $\mathcal{G}$ so $\mathcal{G}$ is $\aleph_0$-even.

    3.7 *Example.* Let X be the open interval (0,1) with the

usual metric m and let $\mathcal{U}$ be the uniformity generated by m.

For each n $\in$ $\underline{N}$ let G_n = (1/n+2,1/n) and let $\mathcal{G}$ = $(G_n)_{n \in \underline{N}}$.

Then $\mathcal{G}$ is a countable star-finite normal open cover. Hence

by 2.1 $\mathcal{G}$ is $\aleph_o$-even. But $\mathcal{G}$ is not Lebesgue and therefore by

3.5, $\mathcal{G}$ is not $\mathcal{U}$-even.

Suppose that (X,m) is a pseudometric space and that $\mathcal{G}$

is an open cover. In a pseudometric space every open cover

has a locally finite cozero-set refinement and hence every

open cover is normal. It follows that $\mathcal{G}$ is $\aleph_o$-even. But as

we saw above $\mathcal{G}$ need not be $\mathcal{U}$-even (where $\mathcal{U}$ is the pseudo-

metric uniformity on X) or equivalently, $\mathcal{G}$ need not be

Lebesgue. Since $\mathcal{G}$ is normal there exists a continuous pseudo-

metric d associated with $\mathcal{G}$ such that $(B_d(x,1/2^2))_{x \in X}$ refines

$\mathcal{G}$. We now give sufficient conditions on d in order that $\mathcal{G}$ be

Lebesgue.

3.8 *Theorem.* *Suppose that* (X,m) *is a pseudometric space*

and that $\mathcal{G}$ is an open cover of X. *If there exists a continu-*

ous pseudometric d *associated with $\mathcal{G}$ such that* d $\leq$ m *then $\mathcal{G}$*

is Lebesgue (in (X,m)).

Proof. Since d is associated with $\mathcal{G}$, $(B_d(x,1/2^2))_{x \in X}$

refines $\mathcal{G}$. We show that $(B_m(x,1/2^2))_{x \in X}$ refines $\mathcal{G}$. If

y $\in$ $B_m(x,1/2^2)$ then m(x,y) < $1/2^2$. Therefore d(x,y) $\leq$ m(x,y)

< $1/2^2$ so y $\in$ $B_d(x,1/2^2)$ and hence $(B_m(x,1/2^2))_{x \in X}$ refines $\mathcal{G}$.

4. The Main Results

We are now ready to introduce the concepts of $\mathcal{U}$-para-

compact and $\mathcal{U}$-collectionwise normal in the context of uniform

spaces.

4.1 *Definition.*   Let $(X, \mathcal{U})$ be a uniform space.

(1) Suppose that γ is a cardinal number not necessarily infinite. We say that $(X, \mathcal{U})$ is $\mathcal{U}^{\gamma}$-*paracompact* if every open cover of power at most γ is $\mathcal{U}$-even. Furthermore, we say that $(X, \mathcal{U})$ is $\mathcal{U}^{\gamma}$-*collectionwise normal* if for every family $(F_{\alpha})_{\alpha \in I}$ of discrete closed subsets of X of power at most γ, there is a $U \in \mathcal{U}$ such that $(U(F_{\alpha}))_{\alpha \in I}$ is a pairwise disjoint family of subsets of X.

(2) We say that $(X, \mathcal{U})$ is $\mathcal{U}$-*paracompact* (respectively, $\mathcal{U}^{T}$-*paracompact*) if the space is $\mathcal{U}^{\gamma}$-paracompact for every (respectively, every finite) cardinal number γ. Furthermore, we say that $(X, \mathcal{U})$ is $\mathcal{U}$-*collectionwise normal* (respectively, $\mathcal{U}^{T}$-*collectionwise normal*) if the space is $\mathcal{U}^{\gamma}$-collectionwise normal for every (respectively, every finite) cardinal number γ.

Let us point out here that $\mathcal{U}^{\gamma}$-paracompact and $\mathcal{U}^{\gamma}$-collectionwise normal are of special significance in the case that γ is 2 or $\aleph_{0}$. In [8] we investigated the properties of even and n-even covers. In general, these were not $\mathcal{U}$-even. However, many of the techniques used there can be modified and generalized to give similar results. Wherever needed we will assume that this has been done.

In this section we will present several results showing the relationship between $\mathcal{U}^{\gamma}$-paracompact and $\mathcal{U}^{\gamma}$-collectionwise normal. In particular we will show that $\mathcal{U}^{T}$-paracompact, $\mathcal{U}^{2}$-paracompact, $\mathcal{U}^{T}$-collectionwise normal and $\mathcal{U}^{2}$-collectionwise normal are all equivalent. We will also show some of the connections to paracompact and present examples to show

that $\mathcal{U}^T$-paracompact and $\mathcal{U}$-paracompact are not equivalent
and that $\mathcal{U}$-paracompactness in one (in particular, the uni-
versal) uniformity does not imply $\mathcal{U}$-paracompactness in other
uniformities.

First we will present some elementary properties of
$\mathcal{U}^\gamma$-paracompactness and $\mathcal{U}^\gamma$-collectionwise normality and
examine them in some familiar classes of topological spaces.
At the outset let us observe the following which shows the
similarity to paracompactness on subspaces and in particular
on closed subspaces.

4.2 *Theorem.* *If* $(X,\mathcal{U})$ *is a* $\mathcal{U}^\gamma$-*paracompact uniform space,*
then every closed uniform subspace is $\mathcal{U}^\gamma$-*paracompact.*

Proof. Let S be a closed uniform subspace of X and let
$A = (A_\alpha)_{\alpha \in I}$ be an open cover of S of power at most γ. For
each $\alpha \in I$, there is an open subset B_α of X such that
$B_\alpha \cap S = A_\alpha$. Let $\mathcal{G} = (G_\alpha)_{\alpha \in I}$ be defined as follows: choose
β in I arbitrary, set $G_\alpha = B_\alpha$ if $\alpha \neq \beta$ and set $G_\beta = B_\beta \cup (X - S)$.
Then $\mathcal{G}$ is an open cover of X of power at most γ with the pro-
perty that $G_\alpha \cap S = A_\alpha$. By hypothesis X is $\mathcal{U}^\gamma$-paracompact,
so $\mathcal{G}$ is $\mathcal{U}$-even, i.e. there is a $U \in \mathcal{U}$ such that $(U(x))_{x \in X}$
refines $\mathcal{G}$. Let $V = U \cap (S \times S)$. Clearly $V \in \mathcal{U}|S$ and for each
x in S, $V(x) = \{y \in S: (x,y) \in V\} \subset U(x) \cap S \subset (G_\alpha \cap S) = A_\alpha$
for some $\alpha \in I$. Thus $(V(x))_{x \in X}$ refines A and S is $\mathcal{U}^\gamma$-para-
compact.

Note that in the preceding proof if γ is an infinite
cardinal number we can construct the cover $\mathcal{G} = (B_\alpha)_{\alpha \in I} \cup (X - S)$
so that A and $\mathcal{G}$ will have the same power.

4.3 *Corollary.* *If* $(X, \mathcal{U})$ *is a uniform space that is* $\mathcal{U}$*-paracompact (respectively,* $\mathcal{U}^2$*-paracompact; respectively,* $\mathcal{U}^T$*-paracompact) then every closed uniform subspace of X is* $\mathcal{U}$*-paracompact (respectively,* $\mathcal{U}^2$*-paracompact; respectively,* $\mathcal{U}^T$*-paracompact).*

4.4 *Theorem.* *If* $(X, \mathcal{U})$ *is a uniform space and if* γ *is a cardinal number then the following statements are equivalent:*

(1) *Every subspace of X is* $\mathcal{U}^\gamma$*-paracompact.*

(2) *Every open subspace of X is* $\mathcal{U}^\gamma$*-paracompact.*

Proof. We need only show (2) *implies* (1). Let S be a subspace of X and let $(A_\alpha)_{\alpha \in I}$ be an open cover of S of power at most γ. For all $\alpha \in I$ there exists B_α open in X such that $B_\alpha \cap S = A_\alpha$. Let $B = \cup_{\alpha \in I} B$ and note that $(B_\alpha)_{\alpha \in I}$ is an open cover of the open subspace B of power at most γ. Hence there exists $U \in \mathcal{U}|B \times B$ such that $(U(x))_{x \in B}$ refines $(B_\alpha)_{\alpha \in I}$ (on B). Let $V = U \cap (S \times S)$ and note that $V \in \mathcal{U}|S \times S$ such that $(V(x))_{x \in X}$ refines $(A_\alpha)_{\alpha \in I}$.

The previous results did not require γ to be an infinite cardinal but we now will restrict ourselves to the case where γ is not finite.

4.5 *Theorem.* *Suppose that* $(X, \mathcal{U})$ *is a uniform space. If* $(X, \mathcal{U})$ *is* $\mathcal{U}$*-paracompact then X is paracompact.*

Proof. If $\mathcal{G}$ is an open cover of X then by definition there exists an element U in $\mathcal{U}$ such that $(U(x))_{x \in X}$ refines $\mathcal{G}$. Since U is a neighborhood of the diagonal of X we have that every open cover is even and therefore X is paracompact.

The preceding result used the fact that a space is para-
compact if and only if every open cover is even. If γ is an
infinite cardinal number it does not appear to be known if X
is γ-paracompact if and only if every open cover of power at
most γ is even. However we do have the following.

   4.6 *Theorem. Suppose that* X *is a normal topological
space and that* γ *is an infinite cardinal number. If* X *is
γ-paracompact then every open cover of power at most γ is
even.*

   Proof. The result follows from the fact that in a normal
space every locally finite open cover is normal and hence
even.

   In the case that γ is $\aleph_0$ we can sharpen the above result
as follows.

   4.7 *Theorem. If* X *is a topological space then the fol-
lowing statements are equivalent:*

   (1) The space X *is countably paracompact and normal.*

   (2) Every countable open cover of X *is* $\aleph_0$*-even.*

   (3) Every countable open cover of X *is 3-even.*

   Proof. (1) *implies* (2). If $\mathcal{G}$ is a countable open cover
of X then, since X is countably paracompact, $\mathcal{G}$ has a locally
finite open refinement $\mathcal{H}$. Since X is normal $\mathcal{H}$ is normal and
therefore $\aleph_0$-even.

   (2) *implies* (3). This implication is clear.

   (3) *implies* (1). If $\mathcal{G}$ is a countable open cover then,
by (3), $\mathcal{G}$ is 3-even and therefore by [8, Theorem 3.7] $\mathcal{G}$ has
a locally finite even refinement and hence X is countably

paracompact.

We now show that X is normal. By [1, Theorem 11.7] it
will suffice to show that every countable locally finite
open cover $(G_n)_{n \in \underline{N}}$ has a refinement $(F_n)_{n \in \underline{N}}$ such that
cl $F_n \subset G_n$ for all n ∈ $\underline{N}$. But this follows immediately from
[8, Theorem 3.6].

Thus we again see the special role of $\aleph_o$ in the study
of properties of families of subsets of varying cardinalities
in topology. As contrasted with Theorem 4.7 the next theorem
shows that a converse to Theorem 4.6 can be obtained in the
class of strongly collectionwise normal spaces.

4.8 *Theorem.* *If* X *is a strongly collectionwise normal*
topological space and if γ *is an infinite cardinal number*
then the following statements are equivalent:

(1) *The space* X *is* γ-*paracompact.*

(2) *Every open cover of power at most* γ *is even.*

Proof. By 4.6, (1) *implies* (2). To prove (2) *implies*
(1), let $\mathcal{G}$ be an open cover of power at most γ. By (2) $\mathcal{G}$
is even and hence by 2.2 $\mathcal{G}$ is normal. Since a normal open
cover has a locally finite open refinement the proof is com-
plete.

4.9 *Corollary.* *If* X *is a completely regular topological*
space and if $\mathcal{U}_o$ *is the universal uniformity on* X *then* X *is*
$\mathcal{U}_o$-*paracompact if and only if* X *is paracompact.*

Proof. If X is $\mathcal{U}_o$-paracompact then, by 4.5, X is para-
compact. Conversely, if $\mathcal{G}$ is an open cover of the paracompact
space X, then $\mathcal{G}$ is even. So there exists a neighborhood W of

the diagonal of X such that $(W(x))_{x \in X}$ refines $\mathcal{G}$. But X is

paracompact so X is strongly collectionwise normal. Hence W

is an element of the universal uniformity. It follows that

X is $\mathcal{U}_o$-paracompact.

Since a compact space has a unique uniformity it follows

from the above and the fact that a compact space is paracom-

pact that every compact space is $\mathcal{U}$-paracompact. Actually a

paracompact almost compact space is $\mathcal{U}$-paracompact for every

(one!) admissible uniformity $\mathcal{U}$. In 4.13 we will show the

converse of this statement but first we present an example

which shows that neither paracompact nor $\mathcal{U}_o$-paracompact im-

plies $\mathcal{U}$-paracompact for an arbitrary uniformity.

4.10 *Example*. Suppose that X is an infinite discrete

topological space and that $\mathcal{U}_o$ is the universal uniformity.

Then X is paracompact and therefore by 4.3, X is $\mathcal{U}_o$-paracom-

pact. Moreover X is not almost compact so there is a uni-

formity $\mathcal{U}$ compatible with the topology on X such that $\mathcal{U} \neq \mathcal{U}_o$.

We show that X is not $\mathcal{U}$-paracompact.

Let $\mathcal{G} = (\{x\})_{x \in X}$ and observe that $\mathcal{G}$ is an open cover

of X. If $\mathcal{G}$ is $\mathcal{U}$-even then there is a $U \in \mathcal{U}$ such that

$(U(x))_{x \in X}$ refines $\mathcal{G}$. This implies that $U(x) = \{x\}$ and

$U = \{(x,x) \in X \times X : x \in X\}$ whence $\mathcal{U} = \mathcal{U}_o$, a contradiction.

Thus $\mathcal{G}$ is not $\mathcal{U}$-even so X is not $\mathcal{U}$-paracompact.

We will now sharpen the relationship between compact,

$\mathcal{U}$-paracompact and almost compact but first we must show the

relationship to completeness.

4.11 *Theorem.* *If* $(X, \mathcal{U})$ *is a* $\mathcal{U}$-*paracompact uniform space then* $(X, \mathcal{U})$ *is complete.*

Proof. Suppose that there exists a Cauchy filter $\mathcal{F}$ with no cluster point. Thus for all $x \in X$ there exists $U_x \in \mathcal{U}$ and $A_x \in \mathcal{F}$ such that $A_x \cap U_x(x) = \emptyset$. Let $\mathcal{A} = (U_x(x))_{x \in X}$. Then $\mathcal{A}$ is a cover of X. Since X is $\mathcal{U}$-paracompact there exists $V \in \mathcal{U}$ such that $(V(x))_{x \in X}$ refines $\mathcal{A}$. Now $\mathcal{F}$ is a Cauchy filter, hence there exists $F \in \mathcal{F}$ such that $F \times F \subset V$. Let $x \in F$ be fixed. Then $F \subset V(x)$. Also $(V(x))_{x \in X}$ refines $\mathcal{A}$ so there exists $y \in X$ such that $V(x) \subset U_y(y)$. Thus $F \subset U_y(y)$. Since $U_y(y) \cap A_y = \emptyset$ we have that $F \cap A_y = \emptyset$, a contradiction.

If $\mathcal{U}$ is the universal uniformity then 4.9 and the above yields the following known result.

4.12 *Corollary.* *A paracompact space is complete in its universal uniformity.*

4.13 *Theorem.* *A uniform space* $(X, \mathcal{U})$ *is compact if and only if it is* $\mathcal{U}$-*paracompact and almost compact.*

Proof. From the discussion after 4.9 we have that a compact space is $\mathcal{U}$-paracompact and almost compact. Conversely, if $(X, \mathcal{U})$ is $\mathcal{U}$-paracompact then $(X, \mathcal{U})$ is complete (4.11) and it is easy to see that a complete almost compact space is compact.

4.14 *Remark.* Clearly $\mathcal{U}$-paracompact implies $\mathcal{U}^{\gamma}$-paracompact for every infinite cardinal number γ and this implies $\mathcal{U}^{\aleph_0}$-paracompact, $\mathcal{U}^{T}$-paracompact, and $\mathcal{U}^{2}$-paracompact. Moreover, $\mathcal{U}^{T}$-paracompact does not in general imply $\mathcal{U}$-paracompact. However we will show that $\mathcal{U}^{T}$-paracompact and $\mathcal{U}^{2}$-paracompact

are equivalent. In order to do this we need to investigate the relationship between $\mathcal{U}^\gamma$-paracompactness and $\mathcal{U}^\gamma$-collection-wise normality in further detail so we now proceed to investigate $\mathcal{U}^\gamma$-collectionwise normal spaces.

4.15 *Theorem.* *Suppose that* $(X, \mathcal{U})$ *is a completely regular uniform space and that* γ *is a cardinal number. If* $(X, \mathcal{U})$ *is* $\mathcal{U}^\gamma$*-paracompact then* X *is* $\mathcal{U}^\gamma$*-collectionwise normal.*

Proof. Suppose that $\mathcal{J} = (F_\alpha)_{\alpha \in I}$ is a discrete family of closed subsets of X of power at most γ. For each $\alpha \in I$ let $G_\alpha = \cup_{\zeta \in I, \zeta \neq \alpha} F_\zeta$ and let $\mathcal{G} = (G_\alpha)_{\alpha \in I}$. Since $\mathcal{J}$ is discrete it follows that G_α is open for each $\alpha \in I$. Thus $\mathcal{G}$ is an open cover of X. Since X is $\mathcal{U}^\gamma$-paracompact there exists an open symmetric W_1 in $\mathcal{U}$ such that $(W_1(x))_{x \in X}$ refines $\mathcal{G}$. Furthermore, since $\mathcal{U}$ is a uniformity there exists an open symmetric W_2 in $\mathcal{U}$ such that $W_2^2 \subset W_1$. Let $\mathcal{W}_2 = (W_2(x))_{x \in X}$ and for each $\alpha \in I$ set $H_\alpha = \text{st}(F_\alpha, \mathcal{W}_2)$ and let $\mathcal{H} = (H_\alpha)_{\alpha \in I}$. We assert that $\mathcal{H}$ is a family of pairwise disjoint open sets such that $F_\alpha \subset H_\alpha$ for all $\alpha \in I$. Clearly H_α is open and $F_\alpha \subset H_\alpha$ for $\alpha \in I$. If $H_\alpha \cap H_\beta \neq \emptyset$ for $\alpha, \beta \in I$ then there exists $W_2(b)$ and $W_2(c)$ in $\mathcal{W}_2$ such that $W_2(b) \cap F_\alpha \neq \emptyset$, $W_2(c) \cap F_\beta \neq \emptyset$ and $a \in W_2(b)$ $\cap W_2(c)$ for some $a \in X$. Hence there exists $x \in W_2(b) \cap F_\alpha$ and $y \in W_2(c) \cap F_\beta$. But then $x \in W_2(b) \subset \text{st}(a, \mathcal{W}_2) \subset W_1(a)$. Since $(W_1(x))_{x \in X}$ refines $\mathcal{G}$ there exists $\delta \in I$ such that $W_1(a) \subset G_\delta$. It follows that $x \in G_\delta$ and therefore $x \notin F_\zeta$ if $\zeta \neq \delta$, whence $\delta = \alpha$. A similar argument shows that $\delta = \beta$. Thus if $H_\alpha \cap H_\beta \neq \emptyset$ then $\alpha = \beta$.

We will now show that in the finite case the situation

is quite special. Specifically in 4.18 we will show the

equivalence of U^2-paracompact, U^2-collectionwise normal,

U^T-paracompact, and U^T-collectionwise normal. We will do

this in two steps. First we have the following result.

   4.16 *Theorem.* *If* (X, U) *is a uniform space then the*

following statements are equivalent:

   (1) The space (X, U) *is* U^2-*paracompact.*

   (2) The space (X, U) *is* U^2-*collectionwise normal.*

   Proof. (1) *implies* (2). Suppose that $J = (F_1, F_2)$ is

a family of two discrete closed subsets of X. For i = 1,2

let G_i = X - F_i and let $G = (G_1, G_2)$. Then G is a binary

open cover of the U^2-paracompact space X hence there exists

a U $\in$ U such that $(U(x))_{x \in X}$ refines G. Furthermore U is a

uniformity so there exists a symmetric V $\in$ U such that $V^2 \subset$ U.

We need only show $V(f_1) \cap V(F_2) = \emptyset$. If y $\in$ $V(F_1) \cap V(F_2)$

then there exists an a $\in F_1$ and b $\in F_2$ such that (a,y) $\in$ V

and (b,y) $\in$ V, hence (a,b) $\in V^2 \subset$ U. Since $(U(x))_{x \in X}$ refines

G, U(b) $\subset G_1$. But then a $\in$ U(b) $\subset G_1$ = X - F_1, a contradic-

tion.

   (2) *implies* (1). Let $A = (A_1, A_2)$ be a binary open cover.

Set F_i = X - A_i, i = 1,2 and note that $F_1 \cap F_2 = \emptyset$. If

$F_1 = \emptyset$ or $F_2 = \emptyset$ the statement clearly holds. So assume each

is nonempty. By (2) there exists U $\in$ U such that $U(F_1) \cap$

$U(F_2) = \emptyset$. Since U is a uniformity there exists an open

symmetric V such that $V^2 \subset$ U. We assert that $(V(x))_{x \in X}$ re-

fines A. Suppose that there exists x $\in$ X such that V(x) $\not\subset A_1$

and V(x) $\not\subset A_2$. Then there exists $z_1 \in$ V(x) such that $z_1 \not\in A_1$

and $z_2 \in$ V(x) such that $z_2 \not\in A_2$. But then (x,z_1) $\in$ V and

$(x,z_2) \in V$ so $(z_1,z_2) \in V^2 \subset U$. Now $z_1 \notin A_1$ so $z_1 \in X - A_1$
$= F_1$ and therefore $z_2 \in U(z_1) \subset U(F_1)$. Also $z_2 \notin A_2$ so
$z_2 \in X - A_2 = F_2$ hence $z_2 \in F_2 \subset U(F_2)$. It follows that
$U(F_1) \cap U(F_2) \neq \emptyset$, a contradiction.

   4.17 *Corollary. If* $(X,\mathcal{U})$ *is* $\mathcal{U}^2$*-paracompact then* X *is*
normal.

   We can now show that in the finite case $\mathcal{U}^T$-paracompact
and $\mathcal{U}^T$-collectionwise normal are equivalent to each other and
to $\mathcal{U}^2$-paracompact and $\mathcal{U}^2$-collectionwise normal.

   4.18 *Theorem. If* $(X,\mathcal{U})$ *is a uniform space then the fol-*
lowing statements are equivalents:

   (1) The uniform space $(X,\mathcal{U})$ *is* $\mathcal{U}^2$*-paracompact.*

   (2) The uniform space $(X,\mathcal{U})$ *is* $\mathcal{U}^T$*-paracompact.*

   (3) The uniform space $(X,\mathcal{U})$ *is* $\mathcal{U}^2$*-collectionwise normal.*

   (4) The uniform space $(X,\mathcal{U})$ *is* $\mathcal{U}^T$*-collectionwise normal.*

   Proof. (1) *implies* (2). Let $A = (A_1, \cdots, A_n)$ be a
finite open cover of X. By 4.17, X is normal, hence by
[1, 11.7], there exists an open cover $B = (B_1, \cdots, B_n)$ such
that cl $B_i \subset A_i$ for $i = 1, \cdots, n$. For each i, let $F_i = X -$
cl B_i. Then $\mathcal{J}_i = (A_i, F_i)$ is a binary open cover of X and
hence by (1) there exists $U_i \in \mathcal{U}$ such that $(U_i(x))_{x \in X}$ refines
$\mathcal{J}_i$. Let $U = \cap_{i=1}^n U_i$ and note that $U \in \mathcal{U}$. We assert that
$(U(x))_{x \in X}$ refines A. For, if x is an element of X and if
$U(x)$ is not contained in A_i for all $i = 1, \cdots, n$ then, since
$(U_i(x))_{x \in X}$ refines $\mathcal{J}_i$, $U(x) \subset U_i(x) \subset F_i$ for all $i = 1, \cdots, n$.
Therefore, $U(x) \subset \cap_{i=1}^n F_i = \cap_{i=1}^n (X - \text{cl } B_i) = X - \cup_{i=1}^n \text{cl } B_i = \emptyset$,
a contradiction.

The implication (2) *implies* (3) follows from 4.15.
Clearly (3) *implies* (4). Finally, (4) *implies* (1) follows
from 4.16. The proof is now complete.

Finally, we give an example of a uniform space that is
U^T-paracompact but not U-paracompact.

4.19 *Example* of a uniform space (X, U) that is U^T-para-
compact but not U-paracompact.

Let X be the topological space of all ordinal numbers
less than the first uncountable ordinal number ω_1. One often
writes this space as $W(\omega_1)$. For a description of this space
see [1] or [5]. In particular X is almost compact but not
compact. Since X is almost compact it has a unique uniformity
which we denote by U. By 4.13, X is not U-paracompact.

To see that X is U^T-paracompact we show that X is
U^2-collectionwise normal. Suppose that F_1 and F_2 are two
(discrete) closed subsets of X. Since X is normal there
exists a real valued continuous function f from X into the
closed unit interval [0,1] such that $f(x) = 0$ if x is in F_1
and $f(x) = 1$ if x is in F_2. Let d be the continuous pseudo-
metric associated with f and let $U = \{(x,y) \in X \times X: d(x,y)$
$< 1/2\}$. Then U is an element of the universal uniformity
which must be in U and one easily shows that $U(F_1) \cap U(F_2) = \emptyset$.

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