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by

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In [R] Mary Ellen Rudin gave a very complicated example of a non-completable Moore Space. Carl Pixley and Prabir Roy [PR] presented an example which could be used in place of the example in [R]. Their example was obtained by constructing a certain hyperspace over the reals. This example is easy to describe. The techniques they used have been useful in constructing other examples and have been studied in their own right [PT], [vDTW], and [vD].

Given a topological space (X,τ), Pixley and Roy constructed the hyperspace

$$\mathcal{F}[X] = \{F \subseteq X \mid 0 < \text{card}(F) < \omega_0\}.$$

If F is in $\mathcal{F}[X]$ and U is a τ-open set containing F then

$$[F,U] = \{S \in \mathcal{F}[X] \mid F \subseteq S \subseteq U\}$$

is a basic neighborhood of the point F. The Pixley-Roy space associated with (X,τ) is $\mathcal{F}[X]$ topologized by this neighborhood system. (This notation follows [vD].) For each positive integer n let

$$\mathcal{F}_n[X] = \{F \in \mathcal{F}[X] \ \text{card}(F) \leq n\}$$

be topologized as a subspace of $\mathcal{F}[X]$. It was pointed out in [vD] that if (X,τ) is first countable then $\mathcal{F}[X]$ is a metacompact Moore Space.

Let $(Y,<)$ be a linearly ordered set and let λ be the associated open interval topology. Let A, B, and C be disjoint, possibly empty, subsets of Y and let τ be a topology on Y having $\lambda \cup \{[x,y[\mid x \in A, x < y\} \cup \{]x,y] \mid x < y, \in B\} \cup$

$\{\{x\} | x \in C\}$ as a base. The space (Y, τ) is a generalized ordered space (=GO-space [L]) on $(Y, <)$ and can be denoted by $GO_Y(A,B,C)$.

Associated with each GO-space $X = GO_Y(A,B,C)$ is a topological space called the Heath plane of X. Let

$$H(X) = \{(x,y) \in Y^2 | x \leq y\}.$$

Topologize H(X) as follows:

a) each point $(x,y) \in H(X)$ with $x < y$ is isolated;

b) each point $(x,x) \in H(X)$ with $x \in C$ is isolated;

c) each point $(x,x) \in H(X)$ with $x \in A$ has neighborhoods of the form $\{x\} \times [x,y[$ where $x < y$;

d) each point $(x,x) \in H(X)$ with $x \in B$ has neighborhoods of the form $]y,x] \times \{x\}$ where $y < x$;

e) each point $(x,x) \in H(X)$ with $x \in Y - (A \cup B \cup C)$ has neighborhoods of the form $(]y,x] \times \{x\}) \cup (\{x\} \times [x,z])$ where $y < x < z$.

If x is an endpoint of Y then only the relevant half of the neighborhood is used.

Notice if $X = GO_Y(A,B,C)$ is a first countable space, then a countable neighborhood base at each point can be specified and H(X) is a metacompact Moore Space.

In [H], R. W. Heath gave an example in the study of Moore spaces which is often called the Heath V-space or the Heath plane. If R denotes the set of real numbers and $X = GO_R(\emptyset,\emptyset,\emptyset)$, then, after a 45° rotation, H(X) is Heath's plane.

Theorem [BFL]. *Let* $(Y,<)$ *be a linearly ordered set whose usual interval topology* λ *is separable and let* $X = (Y,\tau)$ *be a* GO-*space constructed on* Y. *Then the following are* *equivalent:*

(a) $\mathcal{J}[X]$ *is metrizable,*

(b) $\mathcal{J}_2[X]$ *is metrizable.*

(c) If $I = \{x \in Y \mid \{x\} \in \tau\}$,

$L = \{x \in Y-I \mid]\leftarrow,x] \in \tau\}$,

$R = \{x \in Y-I \mid [x,\rightarrow[\in \tau\}$, *and*

$E = Y-(I \cup R \cup L)$,

then

i) E *is countable,*

ii) R *and* L *are* F_σ*-subsets of* (S,τ_S) *where* $S = R \cup L$,

iii) R *can be written as* $R = \cup\{R_n \mid n \in \omega_0\}$ *in such a way so that if* $x \in E \cap cl_\tau(R_n)$ *then for some* $y < x,]y,x[\cap R_n = \emptyset$,

iv) L *can be written as* $L = \cup\{L_n \mid n \in \omega_0\}$ *in such a way that if* $x \in E \cap cl_\tau(L_n)$ *then for some* $z > x$, $]x,z[\cap L_n = \emptyset$

v) The set $K_0 = \{x \in Y \mid x$ *has a neighbor point in* Y *and neither* x *nor* x' *is* τ*-isolated}* *is countable.* (x *and* y *are neighbor points if* $]x,y[= \emptyset$.)

(d) H(X) *is metrizable.*

The equivalence of (b) and (d) indicates that Heath planes can be embedded in suitable Pixley-Roy spaces. This was independently observed by Jerry Vaughn.

Only proofs of (d) $\leftrightarrow$ (b) and (d) $\rightarrow$ (c, part i) are offered since the remaining parts are very long and delicate.

Proof. (d) $\leftrightarrow$ (b). If {a} is a member of $\mathcal{J}_2[X]$, let h({a}) = (a,a) and if {a,b} is a member of $\mathcal{J}_2[X]$ with a < b, let h({a,b}) = (a,b). Then h: $\mathcal{J}_2[x] \rightarrow H(X)$ is clearly a homeomorphism.

(d) $\rightarrow$ (c, part i). Since $(Y,<)$ is separable it is hereditarily separable [BL]. Thus any uncountable subset of (Y,λ) contains a λ-limit point of itself. Since (Y,λ) is separable it follows that both (Y,λ) and (Y,τ) are first countable. Let Q be any countable dense subset of (Y,λ).

Let $\mathcal{U}$ be any open cover of $H(Y,\tau)$ such that for each x in Y, the point (x,x) of $H(Y,\tau)$ is in only one member of $\mathcal{U}$. Let $U(x)$ be the unique member of $\mathcal{U}$ that contains (x,x). Since $H(Y,\tau)$ is metrizable, there is a locally finite open refinement $\mathcal{V}$ of $\mathcal{U}$. For each (x,x) in $H(Y,\tau)$ let $V(x)$ be a member of $\mathcal{V}$ that contains (x,x). Notice if $x_1 \neq x_2$ then $V(x_1) \neq V(x_2)$.

Suppose E was uncountable. For each x in E there exists some $q(x)$ in Q such that $x < q(x)$ and $\{x\} \times [x,q(x)]$ is contained in $V(x)$. Since Q is countable and E is uncountable there is some q_0 in Q such that $J = \{x \in E \mid q(x) = q_0\}$ is uncountable. Let p be a λ-limit point of J that is in J and let $x(1) < x(2) < \cdots$ be a sequence of elements of J that τ-converges to p. (Recall p is in E and, thus, is not τ-isolated.) But then each neighborhood of (p,p) in $H(Y,\tau)$ intersects $V(x(i))$ for infinitely many i. From this contradiction it follows that E is countable.

Pixley and Roy's original example was $\mathcal{J}[R]$. It is now easy to see that $\mathcal{J}[R]$ is non-metrizable since $R = GO_R(\emptyset,\emptyset,\emptyset)$ and card $(E) = c > \omega_0$. In [vD] it was stated without proof that $\mathcal{J}[X]$ was metrizable if X was either the Michael line M or the Sorgenfrey Line S. Michaels line $M = GO_R(\emptyset,\emptyset,P)$ where P denotes the set of irrationals. It is easy to see that H(M) is metrizable and, thus, $\mathcal{J}[M]$ is metrizable. Notice that if $X = GO_R(\emptyset,\emptyset,Q)$, where Q denotes the set of rationals, then X is metrizable but H(X) and, thus, $\mathcal{J}[X]$ is non-metrizable. The Sorgenfrey Line $S = GO_R(R,\emptyset,\emptyset)$. It is again easily seen that H(S) and, therefore, $\mathcal{J}[S]$ is metrizable. The space $Z = GO_R(Q,P,\emptyset)$ is known to be non-metrizable [EL]. Neither H(Z) nor $\mathcal{J}[Z]$ is metrizable.

In [vD] it was also announced that $\mathcal{J}[[0,\omega_1)]$ was metrizable. This result cannot be obtained from the work presented here since $[0,\omega_1)$ is not separable. This example leads to the obvious question whether or not separability can be dropped from the hypothesis of the theorem.

The results of this note with their proofs will appear in Fundamenta Mathematica in the paper "Metrizability of Certain Pixley-Roy Spaces" by H. R. Bennett, W. G. Fleissner and D. J. Lutzer.

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