TOPOLOGY PROCEEDINGS

Volume 3, 1978

Pages 335–345

http://topology.auburn.edu/tp/

SUBCLASSES OF p-SPACES AND STRICT p-SPACES

by

Raymond F. Gittings

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

SUBCLASSES OF p-SPACES AND STRICT p-SPACES

Raymond F. Gittings

1. Introduction

The purpose of this paper is to introduce and investigate certain concepts which are defined in terms of the way a space X is embedded in its Stone-Čech compactification, βX. The definitions of these concepts are motivated by characterizations obtained recently in [10].

The word "space" will always mean "completely regular T_1-space." Throughout, N denotes the set of positive integers and $\langle x_n \rangle$ denotes the sequence whose n^{th} term is x_n. If $A \subset X$ we denote the closure of A in X by $Cl_X A$.

2. Preliminaries and Definitions

If $\mathcal{U}$ is a collection of subsets of a space X and $x \in X$, we define

$$St^1(x,\mathcal{U}) = St(x,\mathcal{U}) = \cup\{U \in \mathcal{U}: x \in \mathcal{U}\}$$

and, for $k \geq 2$, we define

$$St^k(x,\mathcal{U}) = \cup\{U \in \mathcal{U}: U \cap St^{k-1}(x,\mathcal{U}) \neq \emptyset\}.$$

A sequence $\langle \mathcal{B}_n \rangle$ of subsets of a space X is said to be a *refining sequence*[(1)] if, for every $n \in N$, $\mathcal{B}_{n+1} < \mathcal{B}_n$ (i.e., $\mathcal{B}_{n+1}$ is a refinement of $\mathcal{B}_n$).

Let $\langle \mathcal{B}_n \rangle$ be a refining sequence of covers of a space X by sets open in βX. For each $k \in N$, consider the following

---

(1) This terminology should not be confused with the use of the term "refining set of coverings" to mean development in [0].

conditions on the sequence $\langle \mathcal{B}_n \rangle$:

(A_k) $\cap_{n=1}^{\infty} St^k(x,\mathcal{B}_n) \subset X$ for each $x \in X$.

(B_k) $\cap_{n=1}^{\infty} St^k(x,\mathcal{B}_n) = \{x\}$ for each $x \in X$.

(C_k) For each $x \in X$ and $n \in N$, there exists $n(x) \in N$ such that $Cl_{\beta X} St^k(x,\mathcal{B}_{n(x)}) \subset St^k(x,\mathcal{B}_n)$.

(D_k) For each $x \in X$ and $n \in N$, there exists $n(x) \in N$ such that $Cl_{\beta X} St^{k+1}(x,\mathcal{B}_{n(x)}) \subset St^k(x,\mathcal{B}_n)$.

A space X with such a sequence $\langle \mathcal{B}_n \rangle$ is a Moore space if $\langle \mathcal{B}_n \rangle$ satisfies (B_1) and (C_1), a metrizable space if $\langle \mathcal{B}_n \rangle$ satisfies (B_k) and (C_k) for any $k \geq 2$, and a paracompact M-space if $\langle \mathcal{B}_n \rangle$ satisfies (A_k) and (D_k) for any $k \in N$ [10]. Conversely, for any Moore space, metrizable space or paracompact M-space there exists such a sequence $\langle \mathcal{B}_n \rangle$ satisfying the appropriate conditions.

Because of the above characterizations, it is natural to consider the following concepts. A space X is called a p_k-*space* if there exists such a sequence $\langle \mathcal{B}_n \rangle$ for X satisfying (A_k). If in addition, the sequence $\langle \mathcal{B}_n \rangle$ satisfies (C_k), then X is called a *strict* p_k-*space*. Actually, the concept of a p_1-space is equivalent to that of a p-space [1] and the concept of a strict p_1-space is equivalent to that of a strict p-space [6]. If $j > k$, it is clear that every p_j-space is a p_k-space. Unfortunately, we do not know if a strict p_j-space is a strict p_k-space whenever $j > k$. However, the following concept, which is at least formally weaker than that of a strict p_k-space, satisfies the desired implication for $j > k$. A space X is called a *weak strict* p_k-*space* if there exists a sequence $\langle \mathcal{B}_n \rangle$ of covers of X by sets open in βX such that $\cap_{n=1}^{\infty} Cl_{\beta X} St^k(x,\mathcal{B}_n) \subset X$. Moreover, this weaker concept is sufficient for the results that follow and is, in fact, the primary concept studied in this paper.

The following implications among the concepts discussed above are immediate from the definitions. (The value of k is assumed to be fixed.)

$$\begin{array}{ccc}
\text{paracompact M-space} & & \text{locally compact space} \\
\downarrow & & \Big| \\
\text{strict } p_k\text{-space} & & \Big| \\
\downarrow & & \downarrow \\
\text{weak strict } p_k\text{-space} & \longrightarrow & p_k\text{-space}
\end{array}$$

The following internal characterization of a weak strict p_k-space will be useful in the next section.

Theorem 2.1. *A space* X *is a weak strict* p_k*-space if and only if there exists a sequence* $\langle \mathcal{G}_n \rangle$ *of open covers of* X *satisfying the following conditions:*

(a) $P_x = \bigcap_{n=1}^{\infty} Cl_X St^k(x, \mathcal{G}_n)$ *is a compact subset of* X *for each* $x \in X$.

(b) if $P_x \subset U$, *an open subset of* X, *there exists a* $j \in N$ *such that* $Cl_X St^k(x, \mathcal{G}_j) \subset U$.

Proof. Suppose $\langle \mathcal{B}_n \rangle$ is a sequence of covers of X illustrating that X is a weak strict p_k-space. We may assume that $\mathcal{B}_{n+1} < \mathcal{B}_n$ for each $n \in N$. For each $n \in N$, let $\mathcal{G}_n = \{B \cap X: B \in \mathcal{B}_n\}$ and note that $St^k(x, \mathcal{G}_n) = St^k(x, \mathcal{B}_n) \cap X$ for each $x \in X$ and $n \in N$. Using this fact and the fact that $\bigcap_{n=1}^{\infty} Cl_{\beta X} St^k(x, \mathcal{B}_n) \subset X$, the reader can easily show that $\bigcap_{n=1}^{\infty} Cl_X St^k(x, \mathcal{G}_n) = \bigcap_{n=1}^{\infty} Cl_{\beta X} St^k(x, \mathcal{B}_n)$. Hence $P_x = \bigcap_{n=1}^{\infty} Cl_X St^k(x, \mathcal{G}_n)$ is a compact subset of X. The fact that $\langle \mathcal{G}_n \rangle$ satisfies condition (b) follows in the usual manner; i.e.

as in the proof of [6, Theorem 2.2].

Conversely, suppose $\langle \mathcal{G}_n \rangle$ is a sequence of open covers of X satisfying conditions (a) and (b) of the theorem. For each $n \in N$, let $\mathcal{B}_n = \{B \text{ open in } \beta X\colon B \cap X \in \mathcal{G}_n\}$ and note that $St^k(x,\mathcal{G}_n) = St^k(x,\mathcal{B}_n) \cap X$. Let $x \in X$, $y \in \beta X - X$ and W be an open set in βX with $P_x \subset W \subset Cl_{\beta X}W \subset \beta X - \{y\}$. By condition (b), there is an $n \in N$ such that $P_x \subset Cl_X St^k(x,\mathcal{G}_n) \subset W \cap X \subset Cl_{\beta X}W \cap X$. Since $St^k(x,\mathcal{G}_n) \subset Cl_{\beta X}W$, it follows that $St^k(x,B_n) - Cl_{\beta X}W$ is an open subset of $\beta X - X$ and therefore must be empty. Thus, $St^k(x,\mathcal{B}_n) \subset Cl_{\beta X}W$ and so $Cl_{\beta X}St^k(x,\mathcal{B}_n) \subset Cl_{\beta X}W \subset \beta X - \{y\}$. Hence, $y \notin Cl_{\beta X}St^k(x,\mathcal{B}_n)$. Because y was an arbitrary element of $\beta X - X$, it follows that $\bigcap_{n=1}^{\infty} Cl_{\beta X}St^k(x,\mathcal{B}_n) \subset X$.

The internal characterizations for p_k-spaces and strict p_k-spaces are contained in Theorems 2.2 and 2.3. As the technique used to prove these results is similar to that used in Theorem 2.1 (also see [4, Theorem 1.3] and [6, Theorem 2.2]), the proofs are left to the reader.

If $\mathcal{U}$ is a collection of subsets of a space X and $x \in X$, a k-tuple $\langle U(1),\cdots,U(k)\rangle$ with $U(i) \in \mathcal{U}$ for $i = 1,2,\cdots,k$, such that $x \in U(1)$ and $U(i) \cap U(i+1) \neq \emptyset$ for $i = 1,2,\cdots,k-1$ is called a k-*chain at* x *from* U.

Theorem 2.2. *A space* X *is a* p_k-*space if and only if there exists a sequence* $\langle \mathcal{G}_n \rangle$ *of open covers of* X *such that, for every* $x \in X$, *if* $\langle G_n(1),\cdots,G_n(k)\rangle$ *is a* k-*chain at* x *from* $\mathcal{G}_n$, *then*

(a) $C_x = \bigcap_{n=1}^{\infty} Cl_X(\bigcup_{i=1}^{k} G_n(i))$ *is compact.*

(b) if $C_x \subset U$, *an open subset of* X, *there exists a* $j \in N$

such that $\cap_{n=1}^{j} Cl_X(\cup_{i=1}^{k} G_n(i)) \subset U$.

Theorem 2.3. *A space* X *is a strict* p_k*-space if and only if there is a refining sequence* $\langle \mathcal{G}_n \rangle$ *of open covers of* X *satisfying the following conditions:*

(a') $P_x = \cap_{n=1}^{\infty} St^k(x, \mathcal{G}_n)$ *is a compact subset of* X *for each* $x \in X$.

(b') $\{\cap_{j=1}^{n} St^k(x, \mathcal{G}_j) : n \in N\}$ *is a basis of neighborhoods for the set* P_x.

Remark. The author does not know if a space X with a refining sequence of open covers satisfying (a) and (b) of Theorem 2.1 is a strict p_k-space. However, it can be shown that (a) of Theorem 2.1 and (b') of Theorem 2.3 are sufficient to show that X is a strict p_k-space.

We haven't introduced any terminology for a space with a sequence $\langle \mathcal{B}_n \rangle$ of covers satisfying condition (B_k). However, the following characterization is an easy consequence of the definitions of a p_k-space and of a space having a $G_\delta(k)$*-diagonal* [12][(1)].

Theorem 2.4. *A space* X *is a* p_k*-space with a* $G_\delta(k)$*-diagonal if and only if there exists a sequence* $\langle \mathcal{B}_n \rangle$ *of covers of* X *by sets open in* βX *satisfying condition* (B_k).

3. Weak Strict p_k-Spaces

In this section we show that the class of weak strict p_k-spaces ($k \geq 2$) is contained in the class of wM-spaces. In addition, we see that the concept of a weak strict p_2-space

---

(1) The concept of a $G\delta(1)$-diagonal coincides with the characterization of G_δ-diagonal obtained in [7].

posseses sufficient additional structure to provide some interesting results which are not valid for wM-spaces.

According to [11], a space X is a wM-*space* if there exists a sequence $\langle \mathcal{U}_n \rangle$ of open covers of X such that if $x_n \in St^2(x, \mathcal{U}_n)$, then the sequence $\langle x_n \rangle$ has a cluster point.

Theorem 3.1. *If* X *is a weak strict* p_k-*space* $(k \geq 2)$, *then* X *is a* wM-*space*.

Proof. It suffices to show that every weak strict p_2-space is a wM-space. Suppose $\langle \mathcal{U}_n \rangle$ is a sequence of open covers of X satisfying conditions (a) and (b) of Theorem 2.1 for k = 2. If $x_n \in St^2(x, \mathcal{U}_n)$, it is easy to see that the sequence $\langle x_n \rangle$ has a cluster point in the set P_x. Thus X is a wM-space.

The Niemytzki plane (see [16, Example 82]) is a strict p-space (hence, a weak strict p-space) which is not a wM-space. Thus, Theorem 3.1 is not true for k = 1. The space $[0,\Omega)$, where Ω is the first uncountable ordinal, is a wM-space which is not a weak strict p_k-space (for any $k \in N$). Thus, the converse of Theorem 3.1 is not true. However, we obtain a partial converse in Theorem 3.2.

A space X is called *isocompact* [2] if every closed countably compact subset of X is compact.

Theorem 3.2. *Every isocompact* wM-*space is a weak strict* p_k-*space* (*for any* $k \in N$).

Proof. Let X be a wM-space. Suppose $\langle \mathcal{U}_n \rangle$ is a sequence of open covers illustrating that X is a wM-space. We may assume that $\mathcal{U}_{n+1} < \mathcal{U}_n$ for all $n \in N$. It suffices to show that the sequence $\langle \mathcal{U}_n \rangle$ satisfies conditions (a) and (b) of Theorem

2.1. Let $\langle x_n \rangle$ be a sequence with $\langle x_n \rangle \subset P_x = \cap_{n=1}^{\infty} Cl_X St^k(x,\mathcal{U}_n)$ and note that, for each $n \in N$, $x_n \in Cl_X St^k(x,\mathcal{U}_n) \subset St^{k+1}(x,\mathcal{U}_n)$. It follows from [11, Lemma 2.5] that the sequence $\langle x_n \rangle$ has a cluster point. Since any such cluster point is surely in P_x, the set P_x is countably compact and thus compact because of the isocompactness of X. Hence, the sequence $\langle \mathcal{U}_n \rangle$ satisfies condition (a) of Theorem 2.1.

Suppose U is an open subset of X and $P_x \subset U$. If $Cl_X St^k(x,\mathcal{U}_n) \not\subset U$ for every $n \in N$, there is a sequence $\langle x_n \rangle$ such that $x_n \in Cl_X St^k(x,\mathcal{U}_n) - U$. As was noted in the proof of Theorem 3.1, the sequence $\langle x_n \rangle$ has a cluster point x_o and $x_o \in P_x \subset U$. Since this is impossible, we must have $Cl_X St^k(x,\mathcal{U}_n) \subset U$ for some n. Hence, the sequence $\langle \mathcal{U}_n \rangle$ satisfies condition (b) of Theorem 2.1 showing that X is a weak strict p_k-space.

Corollary 3.3. *If* X *is an isocompact space, the following are equivalent:*

(1) X *is a* wM-*space.*

(2) X *is a weak strict* p_2-*space.*

(3) X *is a weak strict* p_k-*space* ($k \geq 3$).

In [4], Burke shows that the class of p-spaces coincides with the class of strict p-spaces in the class of θ-refinable spaces. Since any locally compact nonmetrizable Moore space is a θ-refinable p_k-space (for any $k \in N$) and is not a weak strict p_k-space (for any $k \geq 2$), this coincidence does not hold if $k \geq 2$. However, we do have the following:

Theorem 3.4. *If* X *is a paracompact space, the* *following are equivalent:*

(1) X *is a* p_k*-space (for any* $k \in N$).

(2) X *is a strict* p_k*-space (for any* $k \in N$).

(3) X *is a weak strict* p_k*-space (for any* $k \in N$).

Proof. To show (1) → (2) it suffices to show that a paracompact p-space is a strict p_k-space (for any $k \in N$). A paracompact p-space is a paracompact M-space [1, 13] and thus has a refining sequence $\langle \mathcal{B}_n \rangle$ satisfying conditions (A_k) and (D_k) of the introduction [10]. Hence the sequence $\langle \mathcal{B}_n \rangle$ satisfies conditions (A_k) and (C_k) showing that X is a strict p_k-space.

To show (2) → (1) it suffices to show that a paracompact strict p-space is a strict p_k-space (for any $k \geq 2$). But this is immediate since a strict p-space is a p-space.

To show (3) → (1) it suffices to show that a paracompact weak strict p-space is a p_k-space (for any $k \in N$). But, by Corollary 3.3 a paracompact weak strict p-space is a weak strict p_k-space (for any $k \geq 2$) and hence a p_k-space (for any $k \in N$).

For the remaining results of this section, we need the following theorem due to Chaber [8].

Theorem 3.5. *A locally compact space* X *is* θ*-refinable if for every open cover* $\mathcal{U}$ *of* X *there exists a sequence* $\langle \mathcal{V}_n \rangle$ *of open covers of* X *and a cover* $\{A_n : n \in N\}$ *such that, for each* $n \geq 1$ *and* $x \in A_n$, $St(x, \mathcal{V}_n) \subset \mathcal{U}_x$ *for some finite subset* $\mathcal{U}_x$ *of* $\mathcal{U}$.

Theorem 3.6. *A locally compact weak strict* p*-space is*

θ-refinable.

Proof. Let X be a weak strict p-space and let $\langle \mathcal{U}_n \rangle$ be a sequence of open covers of X satisfying conditions (a) and (b) (for k = 1) of Theorem 2.1. Suppose $\mathcal{U}$ is an open cover of X. For each $n \in N$, let $A_n = \{x \in X\colon St(x, \mathcal{U}_n) \subset \cup \mathcal{U}_x$ for some finite subset $\mathcal{U}_x$ of $\mathcal{U}\}$. By conditions (a) and (b), $\{A_n\colon n \in N\}$ is a cover of X. It follows from Theorem 3.5 that X is θ-refinable.

The proof of the following corollary follows exactly as that of Corollary 2.2.2 in [8].

Corollary 3.7. *A locally compact, weak strict* p-*space with a* G_δ-*diagonal is a Moore space.*

For weak strict p_2-spaces we have the following results.

Theorem 3.8. *A locally compact, weak strict* p_2-*space is paracompact.*

Proof. By Theorem 3.6, such a space is θ-refinable and, by Theorem 3.1, a weak strict p_2-space is a wM-space. But a θ-refinable wM-space is paracompact [15, Theorem 5.1].

Corollary 3.9. *A locally compact weak strict* p_2-*space with a* G_δ-*diagonal is metrizable.*

Proof. This is an immediate consequence of Theorem 3.8 and the fact that every paracompact Moore space is metrizable [3].

As the example in [14, Section 4] shows, Theorem 3.8 does not necessarily hold for wM-spaces. In fact, the space Y in that example is not even isocompact.

In [5], Burke gave an example of a locally compact space with a G_δ-diagonal that is not a Moore space. Hence, a p_k-space with a G_δ-diagonal need not be a Moore space.

In [9], Cook and Gibson construct, for each $j > 1$, a locally compact nonmetrizable Moore space with the j-link property (= $G_\delta(j)$-diagonal property). Since every locally compact space is a p_k-space, those examples show that the analogue of Corollary 3.9 does not hold for p_k-spaces (for any $k \in N$).

References

0. A. V. Arhangel'skii, *Mappings and spaces*, Russian Math. Surveys 21 (1966), 115-162.

1. ________, *On a class of spaces containing all metric and all locally bicompact spaces*, Soviet Math. Dokl. 4 (1963), 1051-1055.

2. P. Bacon, *The compactness of countably compact spaces*, Pacific J. Math. 32 (1970), 587-592.

3. R. H. Bing, *Metrization of topological spaces*, Can J. Math. 3 (1951), 175-186.

4. D. K. Burke, *On* p-*spaces and* wΔ-*spaces*, Pacific J. Math. 35 (1970), 285-296.

5. ________, *A nondevelopable locally compact Hausdorff space with a* G_δ-*diagonal*, Gen. Top. Appl. 2 (1972), 287-291.

6. ________ and R. A. Stoltenberg, *A note on* p-*spaces and Moore spaces*, Pacific J. Math. 30 (1969), 601-608.

7. J. G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math. 11 (1961), 105-126.

8. J. Chaber, *On* θ-*refinability and strict* p-*spaces* (preprint).

9. H. Cook and L. Gibson, *The* j-*link property in Moore spaces*, Topology Proceedings 1 (1976), 305-310.

10. R. F. Gittings, *Characterizations of spaces by embeddings in* βX (to appear in Gen. Top. Appl.).

11. T. Ishii, *On* wM-*spaces. I*, Proc. Japan Acad. 46 (1970), 5-10.

12. ________, *On* wM-*spaces. II*, Proc. Japan Acad. 46 (1970), 11-15.

13. K. Morita, *Products of normal spaces with metric spaces*, Math. Ann. 154 (1964), 365-382.

14. ________, *Some properties of* M-*spaces*, Proc. Japan Acad. 43 (1967), 869-872.

15. T. Shiraki, M-*spaces, their generalizations and metrization theorems*, Sci. Rep. Tokyo Kyoiku Daigaku 11 (1971), 57-67.

16. L. A. Steen and J. A. Seebach, Jr., *Counterexamples in topology*, Holt, Rinehart and Winston, Inc. (1970).

Brooklyn College

Brooklyn, New York 11210