PROJECTIVE COVERS OF ORDINAL SUBSPACES

by

K. Kunen and L. Parsons

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0. History and Preliminaries

Gleason [G] introduced in 1958 the notion of projective cover for compact Hausdorff spaces. Later, Ponomarev [P1, P2, P3] and Strauss [S] independently obtained more general results. A detailed historical survey is given in [Wi2]. The projective cover, also known as the absolute, of a space X, denoted $E(X)$, is characterized as that extremally disconnected space which is the perfect, irreducible preimage of X. Its existence and uniqueness are established in the references, above, for regular topological spaces.

Many properties, particularly those of the compactness gender, are preserved from a regular topological space (here we assume all spaces to be completely regular) to its projective cover. Of import is that a space X has paracompact projective cover if and only if X is paracompact. The normality of a space is not necessarily preserved in its projective cover. Warren [Wa] showed that the projective cover of ω_1 is not normal. Malyhin [M] showed that if κ is a singular cardinal with uncountable cofinality, or regular but not inaccessible, then $E(\kappa)$ is not normal. In this paper we characterize those sets of ordinals that have normal projective cover.

The following useful theorems of Iliadis [1] are stated here for reference:

Theorem 0.1. If X *contains a dense set* D *of isolated*

points, then E(X) *is homeomorphic to the set of those ultra-*
filters on D *which converge to points of* X, *with the relative*
topology of βD.

   Theorem 0.2. *If* G *is open in* X, *and* X *is dense in the*
extremally disconnected space T, *then* $cl_T G$ *is open in* T.

   The following concepts play a significant role in our
results:

   Definition. A κ-Aronszajn tree is a tree of height κ
such that all levels are of cardinality less than κ, having
the additional property that no branch runs through the tree.

   Definition. A cardinal κ is said to be weakly compact
if and only if it is strongly inaccessible and there is no
κ-Aronszajn tree.

   Weakly compact cardinals, whose non-existence is con-
sistent with the axioms of ZFC, are, if they exist, quite
large. In fact, the set of inaccessible cardinals less
than a weakly compact cardinal is stationary in that car-
dinal. Weakly compact cardinals are treated in detail in
[D] and [Kl].

   Section 1 is devoted to an analysis of the projective
cover of ordinals and cardinals. We show that the projective
cover of an inaccessible cardinal is normal if and only if
the cardinal is weakly compact. Then, the results of Warren
and Malyhin, above, are extended as follows: If α is an
ordinal, then E(α) is normal if and only if α is a successor
or has countable cofinality or contains a weakly compact

final segment.

Section 2 considers ordinal subsets. If S is an infi-
nite set of ordinals, then a cofinite subset of S is dense
in some limit ordinal γ. We prove that E(S) is normal if
and only if for all limit ordinals $\alpha \leq \gamma$, the conditions S
is stationary in α and $\alpha \notin$ S imply E(α) is normal.

1. The Projective Cover of Ordinals

Theorem 1.1. *If κ is a regular cardinal, then E(κ) is*
the set of non-uniform ultrafilters on κ.

Proof. Use 0.1. The set of successor ordinals is dis-
crete and dense in κ. Every non-uniform ultrafilter p on
that set contains a set A whose supremum is less than κ.
The set $\text{cl}_\kappa A$ is compact, so p converges. Conversely, if p
converges to $\alpha < \kappa$, then p contains the successor ordinals
of $\alpha + 1$, and thus is non-uniform. The following remark
completes the proof.

Remark. If D is the set of successor ordinals of κ
and E is the set of ultrafilters of κ that converge to ordi-
nals less than κ, then the bijection $\phi : D \rightarrow \kappa$ defined by
$\phi(\alpha) = \alpha - 1$ for $\alpha > \omega$ and $\phi(\alpha) = \alpha$ for $\alpha < \omega$, induces a
homeomorphism $\phi^{\#} : E \rightarrow E(\kappa)$. It will often be convenient, as
above, to regard the projective cover of κ as the set of
ultrafilters of κ that converge to ordinals less than κ.

Theorem 1.2. *If $\text{cf} \kappa = \omega$, then E($\kappa$) is normal.*

Proof. From the hypothesis, κ is Lindelöf, the latter
property being preserved to the projective cover.

Lemma 1.3. *If* κ *is a cardinal and* cf $\kappa > \omega$, *then* $E(\kappa \cdot \kappa)$
is not normal.

Proof. Regard the ordinal $\kappa \cdot \kappa$ as the lexicographic
product $\kappa \cdot \kappa$, for notational convenience, topologized with
the order topology. (E.g., $\kappa \equiv (1,0)$). According to Theorem
0.1., $E(\kappa \cdot \kappa)$ is the collection of ultrafilters of $\kappa \cdot \kappa$ that
converge to ordinals less than $\kappa \cdot \kappa$.

Let $H = \{p \in E(\kappa \cdot \kappa): (\exists \beta < \kappa)(\forall \alpha < \kappa)[\{\beta\} \times (\kappa - \alpha) \in p]\}$.
Note that for each $p \in E(\kappa \cdot \kappa)$, there is a $\beta_p < \kappa$ for which
$(\beta_p \times \kappa) \in p$. We assume that β_p is the least such ordinal.
Let $K = \{p \in E(\kappa \cdot \kappa): \beta_p \times \beta_p \in p\}$.

The closures of H and K in $E(\kappa \cdot \kappa)$ are disjoint: Let
$p \in E(\kappa \cdot \kappa)$. The union of the sets $(\beta_p \times \beta_p)$ and $\beta_p \times (\kappa - \beta_p)$
is in p, so exactly one is in p. Thus, p is an element of
exactly one of cl H, cl K.

To show that the closures of H and K cannot be separated
by open sets of $E(\kappa \cdot \kappa)$, it suffices, in view of Theorem 0.1.,
to show that $cl_{\beta D}H \cap cl_{\beta D}K \neq \emptyset$, where D is the set $\kappa \cdot \kappa$ with
the discrete topology. The latter two sets are completely
separated if and only if they are separated by a partition
of D [GJ].

If $D' \subseteq D$ with cl $D' \supseteq$ cl H, then for each $\alpha < \kappa$, there
corresponds an ordinal ξ_α such that for $\beta > \xi_\alpha$, $(\alpha, \beta) \in D'$.
Otherwise, there is an $\alpha < \kappa$ and a sequence $\{\beta_\delta\}_{\delta < \kappa}$ such that
$(\alpha, \beta_\delta) \notin D'$ for each $\delta < \kappa$. Then choose an ultrafilter q
containing $\{(\alpha, \beta_\delta): \delta < \kappa\}$, and converging to $(\alpha+1, 0)$. We
have $q \in H$, and $q \notin$ cl D', a contradiction.

We now apply the hypothesis that cf $\kappa > \omega$. Choose
$\alpha_0 \in \kappa$ arbitrarily. If $\alpha_0, \ldots, \alpha_n$ are selected (assume α_n

is the largest), then define $\alpha_{n+1} = \alpha_n + \omega + \xi_{\alpha_n}$. From the

hypotheses, $\alpha_n < \alpha_{n+1} < \kappa$. Moreover, $\alpha = \sup\{\alpha_n : n \in \omega\}$

is less than κ.

   Consider the square $\alpha \times \alpha$. Let $D_n = \{(\alpha_n, \xi_{\alpha_n} + i) : i \in \omega\}$,

for each $n \in \omega$. By choice of α, $D_n \subset D' \cap (\alpha \times \alpha)$ for each

n. We may choose an ultrafilter q containing $\overset{\infty}{\underset{n=i}{\cup}} D_n$ for each

$i \in \omega$. Then $\beta_q = \alpha$ and $\alpha \times \alpha \in q$. So $q \in K \cap cl\ D'$. Thus,

D' cannot separate cl H and cl K and the theorem follows.

   Theorem 1.4. Let γ be a cardinal with uncountable

cofinality. Either of the following implies $E(\gamma)$ is not nor-

mal: (a) γ is singular, (b) γ is regular and not inacces-

sible.

   Proof. (a) Let $cf\ \gamma = \kappa > \omega$. We find an increasing

sequence of limit ordinals, $\{\xi_\alpha : \alpha < \kappa\}$, with $\underset{\alpha < \kappa}{\sup}\ \xi_\alpha = \gamma$.

We may assume that for each $\alpha < \kappa$, $|\{\beta : \xi_\alpha < \beta < \xi_{\alpha+1}\}| \geq \kappa$.

Define a perfect map $f : \gamma \rightarrow \kappa \cdot \kappa$ by induction as follows:

   $f(0) = 0$

   If λ is a limit ordinal, then put $f(\lambda) = \underset{\beta < \lambda}{\sup} f(\beta)$

   If $f(\delta)$ is defined and α is the least index for

   which $\delta < \xi_\alpha$, then put $f(\delta+1) = f(\delta) + 1$ if

   $f(\delta) < \kappa \cdot \alpha$ and put $f(\delta+1) = \kappa \cdot \alpha$ if $f(\delta) = \kappa \cdot \alpha$.

   The map f induces a perfect map $\tilde{f} : E(\gamma) \rightarrow E(\kappa \cdot \kappa)$. Since

$E(\kappa \cdot \kappa)$ is not normal (Lemma 1.3.), $E(\gamma)$ is not normal.

   The proof of (b) is given in [M]. We improve this re-

sult later.

   Theorem 1.5. If κ is a strongly inaccessible cardinal,

then $E(\kappa)$ is normal if and only if there is no κ-Aronszajn

tree.

The proof is deferred until later.

If $<$ is the partial order of a κ-Aronszajn tree, we write a)(b in case $a \not< b$ and $b \not< a$. If $Z_1, \ldots, Z_n$ are κ-Aronszajn trees, $\prod_{i=1}^{n} Z_i = \bigcup_{\alpha < \kappa} \prod_{i=1}^{n} \text{lev}_\alpha Z_i$. In the product tree $a < b$ if and only if $a^{(i)} < b^{(i)}$ for all $i = 1, \ldots, n$. Finally, a)!(b if and only if $a^{(i)})(b^{(i)}$ for all $i = 1, \ldots, n$.

Lemma 1.6. *If* $Z_1, \ldots, Z_n$ *are* κ-*Aronszajn trees, then* $\prod_{i=1}^{n} Z_i$ *is a* κ-*Aronszajn tree.*

Proof. Any branch through ΠZ_i would induce a branch through each Z_i.

More general results concerning product trees appear in [Wil].

Lemma 1.7. *Let* Z *be a* κ-*Aronszajn tree. Let* $S \subset Z$ *with* $|S| = \kappa$. *There is an* $S' \subset Z$ *for which each point of* S' *has* κ *successors in* S *and* $|S'| = \kappa$.

Proof. Fix $\xi < \kappa$. Consider $S_\xi = S \cap \bigcup_{\alpha \geq \xi} \text{lev}_\alpha Z$. The cardinal $|S_\xi| = \kappa$. Then $\pi_\xi[S_\xi] \subset \text{lev}_\xi Z$ and has cardinality less than κ. Thus, there is an $s_\xi \in \text{lev}_\xi Z$ which preceeds κ points of S_ξ. Let $S' = \{s_\xi : \xi < \kappa\}$.

Lemma 1.8. *Let* $\{Z_i : i \in n\}$ *be* κ-*Aronszajn trees and let* $Z = \prod_{i \in n} Z_i$. *If* $S \subset Z$ *with* $|S| = \kappa$, *then there exist* $a, b \in S$ *such that* a)!(b.

Proof. By induction. The lemma is trivially true for $n = 1$. Let $n > 1$. Apply the inductive hypothesis to S' of Lemma 1.7.: There are $r, s \in S'$ with $i \neq 0 \rightarrow r^{(i)})(s^{(i)}$. (Use Lemma 1.6.). Then r has κ successors in S. There must

be $b,b' > r$ (both in S) for which $b^{(0)}$)$(b'^{(0)}$. Let $a > s$ be above the level of b,b', with $a \in S$. Clearly, $i \neq 0$ implies $a^{(i)}$)$(b^{(i)}$ and $a^{(i)}$)$(b'^{(i)}$. Since a is above the level of b,b', one of the pairs a,b or a,b' satisfies the lemma.

Lemma 1.9. *Let T be a κ-Aronszajn tree, where κ is regular and uncountable. Let $\alpha_\beta \uparrow \kappa$ and suppose $r_\beta \subset \mathrm{lev}_{\alpha_\beta} T$, with $|r_\beta| = n < \omega$. There exist θ and ζ (without loss of generality $\theta < \zeta$) such that $\pi_{\alpha_\theta} [r_\zeta] \cap r_\theta = \emptyset$.*

Proof. By induction on n. Clearly, the theorem holds if $n = 1$: If the conclusion is false, then the r_β's form a path through the tree.

Assume that the theorem is true for $n \leq k$, and that $n = k + 1$. For each $\beta < \kappa$, let ξ_β be the least ordinal for which $|\pi_{\xi_\beta} [r_\beta]| = k + 1$. Then $\xi_\beta \leq \alpha_\beta$. Two cases arise:

   I. $\{\xi_\beta : \beta < \kappa\}$ is unbounded in κ.

   II. $\{\xi_\beta : \beta < \kappa\}$ is bounded in κ.

In Case I we define, inductively, an increasing function $\beta : \kappa \to \kappa$. Let $\beta(0) = 0$. Assume $\beta(\delta)$ has been selected. Since the set $\{\xi_\beta : \beta < \kappa\}$ is unbounded, choose $\beta(\delta+1) > \beta(\delta)$ so that $\xi_{\beta(\delta+1)} > \alpha_{\beta(\delta)}$, and let $S_\delta = \pi_{\alpha_{\beta(\delta)}} [r_{\beta(\delta+1)}]$. Then $|S_\delta| \leq k$ and $S_\delta \subset \mathrm{lev}_{\alpha_{\beta(\delta)}} T$. If $\beta(\delta)$'s are selected for all $\beta < \lambda$, choose $\beta(\lambda) \geq \sup_{\delta < \lambda} \beta(\delta)$. Then $\beta(\delta) \uparrow \kappa$ for $\delta < \kappa$. And for each $\delta < \kappa$, there is a $S_\delta \subset \mathrm{lev}_{\alpha_{\beta(\delta)}} T$. Without loss of generality, we may assume the $|S_\delta|$'s are equal. Then by the induction hypothesis, there are η and ε ($\eta < \varepsilon$) so that $\pi_{\alpha_{\beta(\eta)}} [S_\varepsilon] \cap S_\eta = \emptyset$. Let $\theta = \beta(\eta+1)$ and $\zeta = \beta(\varepsilon+1)$.

Claim: $\pi_{\alpha_\theta}[r_\zeta] \cap r_\theta = \emptyset$. Pf: Suppose $t \in \pi_{\alpha_\theta}[r_\zeta] \cap r_\theta$. Then $t \in \pi_{\alpha_{\beta(\eta+1)}}[r_{\beta(\epsilon+1)}] \cap r_{\beta(\epsilon+1)}$. Therefore,

$$\pi_{\alpha_{\beta(\eta)}}(t) \in \pi_{\alpha_{\beta(\eta)}}[\pi_{\alpha_{\beta(\eta+1)}}[r_{\beta(\epsilon+1)}]] \cap \pi_{\alpha_{\beta(\eta)}}[r_{\beta(\eta+1)}] =$$

$$\pi_{\alpha_{\beta(\eta)}}[r_{\beta(\epsilon+1)}] \cap S_\eta = \pi_{\alpha_{\beta(\eta)}}[\pi_{\alpha_{\beta(\epsilon)}}[r_{\beta(\epsilon+1)}]] \cap S_\eta =$$

$$\pi_{\alpha_{\beta(\eta)}}[S_\epsilon] \cap S_\eta = \emptyset. \quad \text{Contradiction.}$$ The claim follows, establishing the theorem in Case I.

Now consider Case II. The set $\{\xi_\beta : \beta < \kappa\}$ is bounded in κ. Then there is some γ such that $\gamma \gneqq \xi_\beta$ for each $\beta < \kappa$. $|\pi_\gamma[r_\beta]| = k + 1$ for each β such that $\alpha_\beta \geqq \gamma$. Consider the set of unordered $(k+1)$-tuples of distinct elements of $\text{lev}_\gamma T$. There are fewer than κ of them. Since each $r_\beta (\alpha_\beta \gneqq \gamma)$ can be associated with a unique member of this set, there is at least one $(k+1)$-tuple with κ r_β's so associated. (Here, we use the regularity of κ). I.e., there are $t_1, \ldots, t_{k+1}$, all distinct elements of $\text{lev}_\gamma T$, and an increasing function $\beta: \kappa \to \kappa$, with each $\alpha_{\beta(\delta)} \geqq \gamma$, so that

$$\pi_\gamma[r_{\beta(\delta)}] = \{t_1, t_2, \ldots, t_{k+1}\}.$$

Let $r_{\beta(\delta)}$ be ordered: $r_{\beta(\delta)} = \langle x_\delta^1, x_\delta^2, \ldots, x_\delta^{k+1} \rangle$ where $\pi_\gamma(x_\delta^i) = t_i$ for $i = 1, 2, \ldots, k+1$. Let $T_i = T$ for each i. As observed above, $\prod_{i=1}^{k+1} T_i$ is a tree, with the canonical partial ordering, and by Lemma 1.8., there are η and ϵ (without loss of generality $\eta < \epsilon$) such that $r_{\beta(\eta)}) ! (r_{\beta(\epsilon)}$.

Claim: $\pi_{\alpha_{\beta(\eta)}}[r_{\beta(\epsilon)}] \cap r_{\beta(\eta)} = \emptyset$. Pf: If $i \neq j$, then $\pi_{\alpha_{\beta(\eta)}}(x_\epsilon^i) \neq x_\eta^j$, because $\pi_\gamma(x_\epsilon^i) = t_i \neq t_j = \pi_\gamma(x_\eta^j)$. On the other hand, $\pi_{\alpha_{\beta(\eta)}}(x_\epsilon^i) \neq x_\eta^i$ because $r_{\beta(\eta)}) ! (r_{\beta(\epsilon)}$. This establishes the claim and the theorem follows.

Lemma 1.10. *Let κ be a regular uncountable cardinal,*

T *a κ-Aronszajn tree. If* F *is a family of pairwise disjoint finite subsets of* T *and* $|F| = \kappa$, *then there is a level* $\mu < \kappa$ *and an infinite* $G \subset F$ *such that* $\forall$ r $\in$ G[r $\subset \bigcup_{\xi > \mu} \mathrm{lev}_\xi T$] *and*

$\forall$ r,s $\in$ G[r $\neq$ s $\rightarrow$ π_μ[r] $\cap$ π_μ[s] $= \emptyset$].

Proof. Suppose that the conclusion is false. We will choose by induction a family of finite sets that contradicts Lemma 1.9. Choose a level μ_0 arbitrarily. Then there is a finite set $G_0 = \{r_1, \ldots, r_{n_0}\}$ which has the property that $\pi_{\mu_0}[r_i]$ are pairwise disjoint, but for any other r $\in$ F that is contained in $\bigcup_{\xi > \mu_0} \mathrm{lev}_\xi T$, $\pi_{\mu_0}[r]$ meets one of the $\pi_{\mu_0}[r_i]$'s. Next, if G_α and μ_α are selected, then define $\mu_{\alpha+1} = \sup \bigcup G_\alpha$, and define $G_{\alpha+1}$ as above. If γ is a limit ordinal, let $\mu_\gamma = \sup_{\beta < \gamma} \mu_\beta$. Define G_γ as above. Then the μ_α's strictly increase to κ.

Let $S_\alpha = \pi_{\mu_\alpha}[\bigcup G_\alpha]$. Since each S_α is finite and there are κ of them, we may assume without loss of generality that $|S_\alpha| = n$ for each $\alpha < \kappa$. But by the choice of the G_α's, if $\beta < \delta$, then $\pi_{\mu_\beta}[S_\delta] \cap S_\beta \neq \emptyset$. The family $\{G_\alpha\}$ yields the contradiction.

We now give the proof of Theorem 1.5.

Proof. ($\leftarrow$) Suppose E(κ) is not normal. We construct a κ-Aronszajn tree. Let H and K be disjoint closed sets of E(κ) which cannot be separated by a partition of κ. The sets H and K can surely be separated on $\mathrm{cl}_{E(\kappa)} \alpha$ for each $\alpha < \kappa$, since the latter is compact. (Here, regard α as a discrete subspace of E(κ)). Let T = {X $\subset \alpha$: X separates H and K at some level α and sup X = α}.

Put $X \in \text{lev}_\alpha T$ if and only if $\sup X = \alpha$ and X separates H, K on $\text{cl } \alpha$. Order T by the following scheme: $X_1 < X_2$ if and only if X_1 is an initial segment of X_2. Note that if X' is an initial segment of X, then X' separates H and K at level $\sup X'$. Thus, the ordering is a tree ordering. Moreover, since κ is inaccessible, $|\text{lev}_\alpha T| = 2^{|\alpha|} < \kappa$. The height of T must be κ since at each level α, H and K can be separated. The union of any branch through the tree would separate H and K.

($\rightarrow$)   In this direction we only use the uncountability and regularity of κ. The following quantities are defined:

T is the κ-Aronszajn tree that exists by hypothesis

D_ξ = the finite subsets of $\text{lev}_\xi T$

$D = \bigcup_{\xi < \kappa} D_\xi$

$A_\eta = \bigcup_{\alpha \leq \eta} D_\alpha$

$E(\kappa)$ is the set of non-uniform ultrafilters of D

y_η is an arbitrary element of $\text{lev}_\eta T$ $(\eta < \kappa)$

$P_\xi^\eta = \{d \in D_\xi : \pi_\xi(y_\eta) \in d\}$ $(P_\xi^\eta \subset D_\xi)$

$Q_\xi^\eta = D_\xi \backslash P_\xi^\eta \subset D_\xi$

$H_\eta = \bigcup_{\xi \leq \eta} P_\xi^\eta$

$K_\eta = \bigcup_{\xi \leq \eta} Q_\xi^\eta$

$H = \{H_\eta^c : \eta < \kappa\}$

$K = \{K_\eta^c : \eta < \kappa\}$

Each of H and K has the finite intersection property, and thus they may be regarded as closed sets in $E(\kappa)$. We shall show that these two disjoint closed sets cannot be separated by a partition of D. For the sake of contradiction,

assume that there is an $S \subset D$ with cl $S \supset K$, cl $S \cap H = \emptyset$.

Then by assumption, H and K are separated by the parti-tion $\{S, S^c\}$ of D. We define by induction an increasing function $\xi \colon \kappa \to \kappa$ and a sequence $\{a_{\xi(\alpha)} \colon \alpha < \kappa\}$ of finite subsets of κ, with $a_{\xi(\alpha)} \subset \kappa \backslash \xi(\alpha)$ and $D_{\xi(\alpha)} \cap S \subset \bigcup_{\eta \in a_{\xi(\alpha)}} P_{\xi(\alpha)}^\eta$.

First, fix $\xi \in \kappa$. Let p be an ultrafilter containing $D_\xi \cap S$. Since H is a filterbase, p must omit some $\bigcap_{i=1}^n H_{\eta_i}^c$.

Therefore, $\bigcup_{i=1}^n H_{\eta_i} \in p$, so for some j, $1 \leqq j \leqq n$, $H_{\eta_j} \in p$.

Thus, $P_\xi^{\eta_j} = H_{\eta_j} \cap D_\xi \in p$, and $p \in \text{cl}_{E(\kappa)} P_\xi^{\eta_j}$. It now follows that $\{\text{cl } P_\xi^\eta \colon \eta \geqq \xi\}$ is an open cover of the compact set $\text{cl}(D_\xi \cap S)$. There must exist a finite set a_ξ such that $\text{cl}(D_\xi \cap S) \subset \bigcup_{\eta \in a_\xi} \text{cl } P_\xi^\eta$. The set $a_\xi \subset \kappa \backslash \xi$. Necessarily,

$$D_\xi \cap S \subset \bigcup_{\eta \in a_\xi} P_\xi^\eta.$$

The induction goes as follows: Let $\xi(0) = 0$; if $\xi(\alpha)$ is selected, let $\xi(\alpha+1) > \sup\{a_{\xi(\alpha)}\}$, where $a_{\xi(\alpha)}$ is chosen as above. Let $\xi(\gamma) = \sup_{\alpha < \gamma} \xi(\alpha)$ if γ is a limit ordinal. Note that the $a_{\xi(\alpha)}$'s are disjoint.

Let $N_\alpha = \{y_\eta \colon \eta \in a_{\xi(\alpha)}\}$. The family $\{N_\alpha \colon \alpha < \kappa\}$ con-sists of pairwise disjoint non-empty subsets of T. Apply Lemma 1.10. There is a $\mu < \kappa$ and an infinite subfamily $\{N_{\alpha_1}, N_{\alpha_2}, \ldots\}$ with all N_{α_i}'s being contained above the level μ, such that $i \neq j$ implies $\pi_\mu[N_{\alpha_i}] \cap \pi_\mu[N_{\alpha_j}] = \emptyset$. Let

$$\gamma = \sup_{n \in \omega} \{\xi(\alpha_n)\}$$

(Note that $a_{\xi(\alpha_n)} \subset (\kappa \backslash \xi(\alpha_n) \cap \xi(\alpha_{n+1}))$.)

Claim 1. $A_\gamma \backslash S \subset \bigcup_{\eta \in b} K_\eta$ for some finite $b \subset \kappa$.

Pf: Recall that $A_\gamma = \bigcup\limits_{\xi \leq \gamma} D_\xi$. Let $p \in cl(A_\gamma \setminus S) \subset cl\ S^c$, so

$p \notin K$. There exist, then, $\eta_1,\ \ldots,\ \eta_n$ such that $\bigcap\limits_{i=1}^{n} K^c_{\eta_i} \notin p$.

Therefore, $\bigcup\limits_{i=1}^{n} K_{\eta_i} \in p$. For some j, $K_{\eta_j} \in p$, or $p \in cl\ K_{\eta_j}$.

It follows that $\{cl\ K_\eta : \eta < \kappa\}$ is an open cover of $cl(A_\gamma \setminus S)$.

So there is a finite $b \subset \kappa$ such that $cl(A_\gamma \setminus S) \subset \bigcup\limits_{\eta \in b} cl\ K_\eta =$

$cl \bigcup\limits_{\eta \in b} K_\eta$. Necessarily, $A_\gamma \setminus S \subset \bigcup\limits_{\eta \in b} K_\eta$. Claim 1 is established.

                    Let $c = b \setminus \gamma$

Claim 2. $c \neq \emptyset$. Pf: Suppose $b \subset \gamma$. Let η^* be the

largest ordinal of b. Choose ξ so that $\eta^* < \xi < \gamma$. Let

$x \in D_\xi$. If $x \in A_\gamma \setminus S$, then by the last claim, there is an

$\eta_0 \in b\ (\eta_0 \leq \eta^*)$ such that $x \in K_{\eta_0} \subset A_{\eta_0}$, which is impossible.

Therefore, $x \in S$, so $D_\xi \subset S$.

        Let $\eta_1,\ \ldots,\ \eta_n$ be given. We show that $\{D_\xi\} \cup$

$\{H^c_{\eta_1}, \ldots, H^c_{\eta_n}\}$ have non-empty intersection. First, we may

assume without loss of generality that $|lev_\xi T| \geq \omega$ and that

$\eta_i \geq \xi$ for all $i = 1,\ \ldots,\ n$, (for if $\eta_i < \xi$, then $D_\xi \subset H^c_{\eta_i}$).

Let $e \in lev_\xi T \setminus \{\pi_\xi(y_{\eta_i}) : i = 1,\ \ldots,\ n\}$. Then $\{e\} \in H^c_{\eta_i}$ for

each $i = 1,\ \ldots,\ n$ and $\{e\} \in D_\xi$.

        There then is an ultrafilter p containing $\{H^c_\eta : \eta < \kappa\} \cup$

$\{D_\xi\}$. The point p is in H, and $p \in cl\ D_\xi \subset cl\ S$. But $cl\ S$

was assumed to miss H. This contradiction establishes

Claim 2.

        Let $J = \{j : \pi_\mu(y_\eta) \in \pi_\mu[N_{\alpha_j}]$ for some $\eta \in c\}$. Clearly,

J is a finite set. Let $J = \{k_1,\ \ldots,\ k_n\}$ and let k be the

largest member of J (put $k = 0$ if $J = \emptyset$). Let $i > k$, $\eta \in c$,

$\eta' \in a_{\xi(\alpha_i)}$.

Claim 3. $\pi_\mu(y_\eta) \neq \pi_\mu(y_{\eta'})$. Pf: $\eta' \in a_{\xi(\alpha_i)}$ implies

$y_{\eta'} \in N_{\alpha_i}$. Therefore, $\pi_\mu(y_{\eta'}) \in \pi_\mu[N_{\alpha_i}]$. But $\pi_\mu(y_\eta) \notin$

$\pi_\mu[N_{\alpha_i}]$ because $i \notin J$. Claim 3 now follows.

Choose m so that $\xi(\alpha_m) > \max(b \cap \gamma)$, or m = 0 if

$b \cap \gamma = \emptyset$. Fix $m_0 > k, m$. Let $\zeta = \alpha_{m_0}$.

Claim 4. $D_{\xi(\zeta)} \backslash S \subset \bigcup_{\eta \in c} Q^\eta_{\xi(\zeta)}$. Pf: Let $x \in D_{\xi(\zeta)} \backslash S$.

The set $D_{\xi(\zeta)}$ is contained in A_γ, so $x \in A_\gamma \backslash S$. Therefore,

$x \in K_\eta$ for some $\eta \in b$ (by Claim 1). I.e., $x \in Q^\eta_{\xi(\zeta)}$. Suppose

$y_\eta \in \gamma$. Then $\eta < \xi(\alpha_j)$ for some j (assume j is the least such).

The ordinal $\xi(\alpha_m)$ is greater than $\max(b \cap \gamma)$, therefore $j \leq m$.

But $x \in D_{\xi(\zeta)}$, or $\xi(\zeta) \leq \eta < \xi(\alpha_j)$. It follows that $m_0 < j$.

A contradiction arises because $m_0 > m$. Claim 4 is thus estab-

lished.

If $x \in D_{\xi(\zeta)} \cap S$, then $x \in \bigcup_{\eta \in a_{\xi(\zeta)}} P^\eta_{\xi(\zeta)}$, by choice of

the $a_{\xi(\alpha)}$'s. Claim 4 gives $D_{\xi(\zeta)} = \bigcup_{\eta \in c} Q^\eta_{\xi(\zeta)} \cup \bigcup_{\eta \in a_{\xi(\zeta)}} P^\eta_{\xi(\zeta)}$.

Let $d = \{\pi_{\xi(\zeta)}(y_\eta): \eta \in c\}$. Then $d \in D_{\xi(\zeta)}$. If $d \in \bigcup_{\eta \in c} Q^\eta_{\xi(\zeta)}$,

then for some $\eta \in c$, $d \in Q^\eta_{\xi(\zeta)}$, so $\pi_{\xi(\zeta)}(y_\eta) \notin d$. The latter

is clearly impossible, so d cannot be an element of $\bigcup_{\eta \in c} Q^\eta_{\xi(\zeta)}$.

If $d \in \bigcup_{\eta \in a_{\xi(\zeta)}} P^\eta_{\xi(\zeta)}$, then for some $\eta \in a_{\xi(\zeta)}$, $d \in P^\eta_{\xi(\zeta)}$.

Therefore, $\pi_{\xi(\zeta)}(y_\eta) \in d$. But $m_0 > k$, so for each $n' \in c$,

$\pi_\mu(y_\eta) \neq \pi_\mu(y_{\eta'})$, by Claim 3. It follows that for each

$\eta' \in c$, $\pi_{\xi(\zeta)}(y_\eta) \neq \pi_{\xi(\zeta)}(y_{\eta'})$. On the other hand,

$\pi_{\xi(\zeta)}(y_\eta) \in d$, so $\pi_{\xi(\zeta)}(y_\eta) = \pi_{\xi(\zeta)}(y_{\eta'})$ for some $\eta' \in c$,

a contradiction. Therefore, d cannot be an element of

$\eta \underset{\eta \in a_{\xi(\zeta)}}{\bigcup} P_{\xi(\zeta)}^{\eta}$. We now conclude that the assumption of the

existence of the set S must have been false.

We remark that a cardinal κ has normal projective cover
if and only if either $cf(\kappa) = \omega$ or κ is weakly compact.

The following theorem concludes the section:

*Theorem 1.11. Let α be an ordinal. E(α) is normal if
and only if one of the following conditions holds:*

   (a) α is a successor

   (b)   cf α = ω

   *(c)   cf α = κ > ω, where κ is weakly compact and there
        is a β for which $\alpha = \beta + \kappa$*

Proof.

   (a)   implies E(α) is compact.

   (b)   implies E(α) is normal by Theorem 1.2.

   (c)   implies E(α) is the disjoint union of E($\beta+1$) and
E(κ), both which are normal (the normality of the latter
follows from Theorem 1.5).

   Suppose none of a, b, c holds. Then α is a limit ordi-
nal of uncountable cofinality. Let cf α = κ. Because (c)
fails, if κ is weakly compact, then α contains a closed copy
of $\kappa \cdot \kappa$ and E(α) is not normal, by Theorem 1.3. Otherwise
E(κ) is not normal by Theorems 1.4 and 1.5, and E(α) contains
a closed copy of E(κ).

2. The Projective Cover of Ordinal Subspaces

   In this section we show that the normality of E(S) for
any ordinal subspace S can be decided in terms of the nor-
mality of the ordinals in which S is stationary. We remark

that we need only consider sets of ordinals that are dense
in their suprema, for if A is a set of ordinals, we define
inductively a homeomorphism, $f: A \rightarrow \gamma$, that has the property
that sup $f[A] = \gamma$ and cl $f[A] = \gamma$:

$$f(a_0) = 0$$

$$f(a_{\alpha+1}) = f(a_\alpha) + 1$$

If λ is a limit ordinal,

$$f(a_\lambda) = \begin{cases} \sup_{\alpha<\lambda} f(a_\alpha) & \text{if } a_\lambda \text{ is a limit point of A} \\ \sup_{\alpha<\lambda} f(a_\alpha) + 1 & \text{otherwise} \end{cases}$$

Theorem 2.1. *If γ is an ordinal and A is dense in γ,*
then $E(A) = \phi^+[A]$, where $\phi: E(\gamma) \rightarrow \gamma$ is the canonical map.

Proof. Since A contains the isolated points of γ,
$\phi^+[A]$ contains the isolated points of $E(\gamma)$. Thus, $\phi^+[A]$ is
dense in the extremally disconnected space $E(\gamma)$, and is, it-
self, extremally disconnected. [Theorem 0.2]. Moreover,
$\phi|\phi^+[A]$ is a perfect, irreducible map from $\phi^+[A]$ onto A.
We have seen that $\phi^+[A]$ must be the projective cover of A.

A more general version of this theorem appears in [Wo].

Lemma 2.2. *Assume the following: The ordinal γ is a*
limit ordinal; D is the collection of successor ordinals of
γ; S is a stationary set of γ containing D; and F is a closed
set of $E(\gamma)$. Let Z be the set of ultrafilters on D which
contain no subset of D which is bounded in γ; and $\phi: E(\gamma) \rightarrow \gamma$
is the canonical map. Then $\text{cl}_{\beta D}(F \cap \phi^+[S]) \cap Z = \text{cl}_{\beta D}F \cap Z$.

Proof. The left-hand side is clearly included in the
right-hand side. We take $p \in \text{cl}_{\beta D}F \cap Z$ and a neighborhood

thereof, $V = cl_{\beta D}E$, where $E \subset D$.

Consider $V \cap F$. This set is non-empty, since we chose $p \in cl\ F$. Obviously, for each $\beta < \gamma$, $cl_{\beta D}(D \cap \beta)$ cannot contain $V \cap F$; for if it did, then $(\gamma \backslash \beta) \cap E \in p$, so that $(\gamma \backslash \beta) \cap D \in p$. But the βD-closure of the latter then would be a neighborhood of p which does not meet F.

We then have that $\phi[V \cap F] = A$ is unbounded in γ. But A is also closed in γ, since V and F are closed sets and ϕ is a closed map. Thus, A meets S because S is stationary in γ.

Let $\alpha \in A \cap S$. There is an ultrafilter q converging to α, with $q \in V \cap F \cap \phi^{\leftarrow}[S]$. Thus, V meets $F \cap \phi^{\leftarrow}[S]$. But V was an arbitrary basic neighborhood of p, so $p \in Z \cap cl_{\beta D}(F \cap \phi^{\leftarrow}[S])$. The proof of the lemma is now complete.

Theorem 2.3. *If* S *is a stationary set of* γ *which contains the isolated points of* γ, *and* $E(\gamma)$ *is not normal, then* $E(S)$ *is not normal.*

Proof. Let D be the isolated points of γ. If $E(\gamma)$ is not normal, then there are two closed sets H and K which are disjoint in $E(\gamma)$, but whose closures in βD are not disjoint.

Then since $D \subset E(S) \subset E(\gamma) \subset \beta D$, $H \cap E(S)$ and $K \cap E(S)$ are closed sets in $E(S)$ which are disjoint. But by Theorem 2.1, $E(S) = \phi^{\leftarrow}[S]$, and by Lemma 2.2, their closures in βD are not disjoint.

If G_1 and G_2 are open sets separating $H \cap E(S)$ and $K \cap E(S)$, then $cl_{\beta D}G_1$ and $cl_{\beta D}G_2$ are open sets (see Lemma 0.2) separating $cl_{\beta D}H$ and $cl_{\beta D}K$, a contradiction.

Lemma **2.4.** *Let* S *be contained as a dense, non-stationary subset of a limit ordinal* γ. *If every closed initial segment of* S *has normal projective cover, then* E(S) *is normal.*

Proof. Let $K \subseteq \gamma$ be a closed, unbounded set disjoint from S. If $K = \{\xi_\alpha : \alpha < \lambda\}$ is the canonical well ordering of K, then $E(S) \simeq \bigcup_{\alpha < \lambda} E(S \cap [\xi_\alpha, \xi_{\alpha+1}])$, the right-hand side being the free union of normal sets.

Lemma **2.5.** *Let* S *be contained and dense in the limit ordinal* γ. *Suppose* E(γ) *is normal and that every closed initial segment of* S *has normal projective cover. Then* E(S) *is normal.*

Proof. By Theorem 1.11, the cofinality of γ is either ω or $\gamma = \beta + \kappa$ where κ is weakly compact.

In the former case E(S) is the free union of normal sets and, thus, is normal. In the latter, we may assume without loss of generality that γ is a weakly compact cardinal.

For the sake of contradiction, assume that E(S) is not normal. As in Theorem 1.5 we build a γ-Aronszajn tree. Let D be the successor ordinals of γ.

Remark. In this proof we regard the projective cover of a subset of γ as a subset of E(γ). See Theorem 0.1.

There are closed sets H and K of E(S) that cannot be separated by open sets of E(S). Clearly, H and K cannot be separated by a partition of D. Let $\alpha < \gamma$. By hypothesis, E(S $\cap$ α + 1) is normal, so H $\cap$ E(S $\cap$ α + 1) and K $\cap$ E(S $\cap$ α + 1) can be separated by disjoint open subsets of E(S $\cap$ α + 1). These disjoint open sets have disjoint

open closures in $E(\alpha + 1)$ because $E(S \cap \alpha + 1)$ is contained
in $E(\alpha + 1)$ as a dense subset and the latter is extremally
disconnected. Thus, $H \cap E(\alpha + 1)$ and $K \cap E(\alpha + 1)$ may be
separated by a partition of $D \cap (\alpha + 1)$.

Let $T = \{X \subset D\colon X$ separates H and K at some level α
and $\sup X = \alpha\}$. The remainder of the proof is as in the
necessity $(\rightarrow)$ of Theorem 1.5. The tree T is the contradictory
Aronszajn tree.

Theorem 2.6. *Let S be dense in the limit ordinal* γ.
$E(S)$ *is normal if and only if for all limit ordinals* $\alpha \leqq \gamma$,
the conditions S is stationary in α *and* $\alpha \notin S$ *imply* $E(\alpha)$ *is
normal.*

Proof. $(\rightarrow)$ The condition $\alpha \notin S$ implies $E(S \cap \alpha)$ is
a closed subset of $E(S)$ and hence is normal. But, S is
stationary in α, so Theorem 2.3 implies that $E(\alpha)$ is normal.

$(\leftarrow)$ By induction on the closed initial segments of S.

We show that for $\alpha \leqq \gamma$, $S \cap \alpha$ closed in S implies
$E(S \cap \alpha)$ is normal.

Case I. $\alpha = \beta + 1$. If β is a successor or if $\beta \notin S$,
$S \cap \beta$ is closed in S and $S \cap \alpha$ is normal. Otherwise, β is
a limit ordinal and $\beta \in S$. Let U be the collection of ultra-
filters on the successor ordinals of γ that converge to β.
Then $U \subset E(S \cap \beta + 1)$. Let H and K be disjoint closed sets
of $E(S \cap \alpha)$. If $x \in U$, there is a clopen neighborhood A_x
intersecting at most one of H, K. Since U is compact,
there are $A_{x_1}, \ldots, A_{x_n}$ which cover U. Let $B =$
$\beta \setminus (A_{x_1} \cup \ldots \cup A_{x_n})$. The set B is bounded in β by some δ.

But H and K can be separated on $A_{x_1} \cup \ldots \cup A_{x_n}$ and also on
$E(S \cap \delta + 1)$, the latter being normal by the induction
hypothesis.

   Case II. $\alpha = \lambda$ (a limit ordinal). Again, assume all
closed initial segments of S in λ have normal projective
cover. We need only consider the case $\lambda \notin S$, for if $\lambda \in S$,
$S \cap \lambda$ is not closed in S. If S is stationary in λ, then
$E(\lambda)$ is normal by hypothesis and $E(S \cap \lambda)$ is normal by Lemma
2.5. If S is not stationary in λ, then $E(S \cap \lambda)$ is normal
by Lemma 2.4.

   Engelking and Lutzer have shown [EL] that a set of
ordinals is paracompact if and only if it is non-stationary
in every cofinally uncountable limit which it does not con-
tain. In view of this we have:

   *Theorem 2.7. Assume that for each inaccessible cardinal
κ, there is a κ-Aronszajn tree (i.e., no weakly compact
cardinal exists). Then for a set S of ordinals, E(S) is
normal if and only if S is paracompact.*

   Proof. (←) Paracompactness is always preserved from
a space to its projective cover.

   (→) Assume S is not paracompact. Then S is stationary
in some cofinally uncountable limit κ which is not in S.
Without loss of generality, we may assume that S contains
each isolated point of κ, (as above, build a suitable homeo-
morphism which collapses S).

   If there is no weakly compact cardinal, we have seen
(Theorems 1.4 and 1.5) that $E(\kappa)$ is not normal. We infer

from Theorem 2.3 that E(S) is not normal.

3. Closing Remarks

The projective cover of a weakly compact cardinal (if such exist) has been shown to be normal, extremally disconnected, not paracompact. Kunen [K2] has constructed a space with the same properties using only the axioms of ZFC. The former space is locally compact; the latter is not. Both are countably paracompact. Kunen's example is presented below. We begin with a

Definition. A family of sets $\{A_\alpha: \alpha \in I\}$ is said to be independent if given disjoint finite subsets, J and K of I,

$$\bigcap_{\alpha \in J} A_\alpha \cap \bigcap_{\alpha \in K} A_\alpha^c \neq \emptyset.$$

Hausdorff [H] has shown that there exists a family of 2^{ω_1} independent subsets of ω_1, $\{A_\alpha: \alpha < 2^{\omega_1}\}$. A finitely additive measure can be defined so that the A_α's are independent events of probability 1/2, and that the measure of each point is zero.

Let $\{H_\alpha: \alpha < 2^{\omega_1}\}$ enumerate $\mathcal{P}(\omega_1)$; and for $\xi < \omega_1$, let $\mathcal{J}_\xi$ be the filter generated by $\{A_\alpha: \xi \in H_\alpha\} \cup \{A_\alpha^c: \xi \not\in H_\alpha\}$. Pick an ultrafilter p_ξ containing $\mathcal{J}_\xi$, which contains no set of measure zero.

Example 3.1. The example X is $D_{\omega_1} \cup \{p_\xi: \xi < \omega_1\}$.

The space X is extremally disconnected, as it is dense in βD_{ω_1}. But X is not collectionwise Hausdorff, for if the points $\{p_\xi: \xi < \omega_1\}$ can be separated by mutually disjoint

basic open sets, say $\{cl\ B_\xi : \xi < \omega_1\}$, then the B_ξ's are
mutually disjoint sets of positive measure. It follows that
X cannot be paracompact.

But X is normal: Let $H \subset \omega_1$. Then $H = H_\alpha$ for some
$\alpha < \omega_1$. The open set $cl\ A_\alpha$ contains $\{p_\xi : \xi \in H_\alpha\}$, but misses
$\{p_\xi : \xi \notin H_\alpha\}$.

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