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n-SHAPE FIBRATIONS

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1. Introduction

In [11] and [12] the authors introduced and studied a class of maps between compact metric spaces called shape fibrations. A shape fibration is a map $p: E \to B$ which is induced by a level map of ANR-sequences $\underline{p}: \underline{E} \to \underline{B}$, $p = \lim \underline{p}$, such that $\underline{p}$ has a certain homotopy lifting property with respect to the class of all topological spaces. It is shown in [11] that the approximate fibrations of Coram and Duvall [1] are precisely shape fibrations between ANR's. Also, it is shown in [11] that cell-like maps between finite-dimensional metric compacta are shape fibrations. However, the Taylor map [16] is an example of a cell-like map between infinite-dimensional continua which fails to be a shape fibration.

The purpose of this paper is to consider the homotopy lifting property with respect to some other classes $\mathcal{X}$ of spaces. In particular, if $\mathcal{X}$ consists of all topological spaces of dimension $\leq n$, we obtain the notion of an n-shape fibration. Maps which are n-fibrations for all n are also interesting. We call such maps weak shape fibrations. Most of the results for shape fibrations obtained in [11] and [12]

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remain true in appropriate formulations for n-shape fibrations and for weak shape fibrations. For maps between ANR's, weak shape fibrations coincide with approximate fibrations and therefore also with shape fibrations. Furthermore, every cell-like map between metric compacta (regardless of dimension) is a weak shape fibration. Consequently, the Taylor map provides an example of a weak shape fibration which fails to be a shape fibration.

The authors would like to point out that several of the results of this paper are modeled on results of Coram and Duvall in [2]. Also, the authors would like to thank Jerzy Dydak for bringing to their attention Lemma 8.3 of [4] which is referred to in Section 7 of this paper.

2. Shape Fibrations with Respect to a Class of Spaces

We shall consider inverse sequences of metric compacta $\underline{E} = (E_i, q_{ii'})$, $\underline{B} = (B_i, r_{ii'})$. If all E_i and B_i are ANR's, we speak of ANR-sequences. Polyhedral sequences and Q-manifold sequences are especially useful since polyhedra and Q-manifolds are convenient ANR's. A level preserving map of sequences $\underline{p}$: $\underline{E} \to \underline{B}$ (abbreviated as level map) is a sequence of maps p_i: $E_i \to B_i$ such that

(1) $p_i q_{ii'} = r_{ii'} p_{i'}$ for $i \leq i'$.

Definition 1. Let $\mathcal{X}$ be a class of spaces. A level map $\underline{p}$: $\underline{E} \to \underline{B}$ is said to have the *homotopy lifting property with respect to* $\mathcal{X}$ ($\mathcal{X}$-HLP) provided each i admits a $j \geq i$ such that for any $X \in \mathcal{X}$ and any maps h: $X \to E_j$, H: $X \times I \to B_j$ with

(2) $p_j h = H_0$.

There is a homotopy $\tilde{H}: X \times I \to E_i$ satisfying

(3) $\tilde{H}_0 = q_{ij}h$, and

(4) $p_i\tilde{H} = r_{ij}H$.

Every such j is called a *lifting index* for i.

Definition 2. We say that $\underline{p}: \underline{E} \to \underline{B}$ has the *approximate homotopy lifting property with respect to* $\mathcal{X}$ ($\mathcal{X}$-AHLP) provided each i and each $\varepsilon > 0$ admit a $j \geq i$ and a $\delta > 0$ such that for any $X \in \mathcal{X}$ and any $h: X \to E_j$, $H: X \times I \to B_j$ with the distance

(5) $d(p_jh, H_0) \leq \delta$,

there is a homotopy $\tilde{H}: X \times I \to E_i$ satisfying

(6) $d(\tilde{H}_0, q_{ij}h) < \varepsilon$

(7) $d(p_i\tilde{H}, r_{ij}H) < \varepsilon$.

We call δ a *lifting mesh* for (i,ε).

Remark 1. If each E_i is an ANR and if $\underline{p}: \underline{E} \to \underline{B}$ has the $\mathcal{X}$-AHLP where all members of $\mathcal{X}$ are paracompact Hausdorff spaces, then $\underline{p}$ also has the formally stronger property obtained by replacing (6) by (3). This follows from the proof of Proposition 1 of [11]. (Paracompactness is needed in order to construct the function $\phi: X \to (0,1]$ in that proof.)

Remark 2. If each B_i is an ANR and all the members of $\mathcal{X}$ are paracompact Hausdorff spaces, then $\underline{p}: \underline{E} \to \underline{B}$ has the $\mathcal{X}$-AHLP provided it has the formally weaker property obtained by replacing (5) by (2). This follows from the proof of Proposition 2 of [11].

Definition 3. Let $\underline{p}: \underline{E} \to \underline{B}$ be a level map of inverse sequences and let $(E,q_i) = \lim \underline{E}$, $(B,r_i) = \lim \underline{B}$. The unique map $p: E \to B$ such that $p_iq_i = r_ip$ for all i, is said to be induced by $\underline{p}$ or to be the limit of $\underline{p}$, denoted by $p = \lim \underline{p}$.

Definition 4. Let $\mathcal{X}$ be a class of spaces. A map $p: E \to B$ between metric compacta is called a *shape fibration with respect to* $\mathcal{X}$ (*$\mathcal{X}$-shape fibration* for short) provided there exists a level map of ANR-sequences $\underline{p}: \underline{E} \to \underline{B}$ such that $p = \lim \underline{p}$ and $\underline{p}$ has the $\mathcal{X}$-AHLP.

Remark 3. Notice that whenever p is an $\mathcal{X}$-shape fibration and $\mathcal{X}' \subseteq \mathcal{X}$, then p is a fortiori an $\mathcal{X}'$-shape fibration.

The proof of Theorem 1 of [11] applies in the present situation and yields the following result.

Theorem 1. *If* $\underline{p}: \underline{E} \to \underline{B}$ *and* $\underline{p}': \underline{E}' \to \underline{B}'$ *are two level maps of* ANR-*sequences with the same limit* $p = \lim \underline{p} = \lim \underline{p}'$ *and if for some class* $\mathcal{X}$ $\underline{p}$ *has the* $\mathcal{X}$-AHLP, *then so does* $\underline{p}'$.

The next result follows from the proof of Theorem 2 of [11] together with Remark 1.

Theorem 2. *Let all members of* $\mathcal{X}$ *be paracompact Hausdorff spaces. If* $\underline{p}: \underline{E} \to \underline{B}$ *is a level map of* ANR-*sequences such that* $\lim \underline{p} = p: E \to B$ *and* $\underline{p}$ *has the* $\mathcal{X}$-AHLP, *then there is a level map* $\underline{p}': \underline{E}' \to B$ *of* ANR-*sequences such that* $\lim \underline{p}' = p$ *and* $\underline{p}'$ *has the* $\mathcal{X}$-HLP.

Remark 4. If all of the members of $\mathcal{X}$ are paracompact Hausdorff spaces and if $p: E \to B$ is an $\mathcal{X}$-shape fibration, then for every closed subset $B' \subseteq B$ the restriction $p' = p|E': E' \to B'$, $E' = p^{-1}(B')$, is also an $\mathcal{X}$-shape fibration. This is obtained by following the proof of the analogous statement for shape fibrations ([11], Proposition 4) and applying Theorem 2 at the outset.

3. Shape Fibrations

Definition 5. A map $p: E \to B$ between metric compacta is called a *shape fibration* if it is an $\mathcal{X}$-shape fibration where $\mathcal{X}$ is the class of all topological spaces.

Remark 5. The following theorem (Theorem 3) shows that shape fibrations as defined above (Definition 5) coincide with the shape fibrations as defined in [11]. Therefore, for ANR's, shape fibrations (in the sense of Definition 5) agree with approximate fibrations (see [11], Corollary 1).

Theorem 3. *Shape fibrations coincide with $\mathcal{X}$-shape fibrations where $\mathcal{X}$ is the class of all separable metric spaces.*

Proof. Let $\underline{p}: \underline{E} \to \underline{B}$ be a level map of ANR-sequences such that $p = \lim \underline{p}$ and $\underline{p}$ has the AHLP with respect to separable metric spaces. For given i and ε, let $j \geq i$ and δ be the lifting index and mesh respectively. Let X be an arbitrary topological space and let $h: X \to E_j$ and $H: X \times I \to B_j$ be maps satisfying $d(p_j h, H_0) \leq \delta$. Consider the set

(1) $\Delta = \{(e,\omega) \in E_j \times B_j^I: d(p_j(e), \omega(0)) \leq \delta\}$, $\left(B_j^I\right.$ is given the compact-open topology$\left.\right)$. Notice that Δ is a separable metric space because B_j^I is one ([3], Theorem 8.2(3), p. 270 and Theorem 5.2, p. 265).

We shall now define maps $f: X \to \Delta$, $g: \Delta \to E_j$ and $G: \Delta \times I \to B_j$ such that

(2) $gf = h$,

(3) $G(f \times 1) = H$.

(4) The diagram

$$\begin{array}{ccc} \Delta & \xleftarrow{\ f\ } & X \\ \Big\downarrow{\scriptstyle i_0} & & \Big\downarrow{\scriptstyle i_0} \\ \Delta\times I & \xleftarrow[f\times I]{} & X\times I \end{array}$$

commutes strictly (where $i_0(y) = (y,0)$), and

(5) the diagram

$$\begin{array}{ccc} E_j & \xleftarrow{\ g\ } & \Delta \\ {\scriptstyle p_j}\Big\downarrow & & \Big\downarrow{\scriptstyle i_0} \\ B_j & \xleftarrow[G]{} & \Delta\times I \end{array}$$

commutes up to δ, i.e.,

(6) $d(p_j g(y),\ G(y,0)) \leq \delta$ for $y \in \Delta$.

We define f by

(7) $f(x) = (h(x),\ \hat{H}(x))$, where $[\hat{H}(x)](t) = H(x,t)$.

The continuity of f follows from ([3], Theorem 3.1(1), p. 261). We define g and G by

(8) $g(e,\omega) = e$, and

(9) $G((e,\omega),t) = \omega(t)$.

The map G is continuous because the evaluation map $(\omega,t) \longmapsto \omega(t)$ is continuous ([3], Theorem 2.4(2), p. 260).

By the choice of j and δ there is a map $\tilde{G}\colon \Delta \times I \to E_i$ such that

(10) $d(\tilde{G}_0, q_{ij} g) < \varepsilon$, and

(11) $d(p_i \tilde{G}, r_{ij} G) < \varepsilon$.

Now the map $\tilde{H}\colon X \times I \to E_i$ given by

(12) $\tilde{H} = \tilde{G} \circ (f \times 1)$

has the desired properties, i.e., it satisfies 2.(6) and 2.(7).

Theorem 4. *Shape fibrations coincide with* $\mathcal{X}$*-shape fibrations where* $\mathcal{X}$ *is the class of all separable, locally compact polyhedra.*

We shall now prove the theorem by using the following lemma the proof of which we postpone until the next section.

Lemma 1. *Let* E *and* B *be* ANR's, *let* X *be a separable metric space and let the following diagram commute*

(13)
$$\begin{array}{ccc} E & \xleftarrow{\ h\ } & X \\ {\scriptstyle P}\big\downarrow & & \big\downarrow{\scriptstyle i_0} \\ B & \xleftarrow[\ H\ ]{} & X \times I \end{array} .$$

Then for any $\eta > 0$ *and* $\delta > 0$ *there exists a separable, locally compact polyhedron* P *and there exist maps* $f: X \to P$, $g: P \to E$, $G: P \times I \to B$ *such that image of* f *is a dense subset of* P *and*

(14) $d(gf,h) < \eta$

(15) $d(G(f \times 1),H) < \delta$.

If $\dim X \le n$, *then one can achieve that* $\dim P \le n$ *also.*

Proof of Theorem 4. Let j and δ be the lifting index and the lifting mesh for i and $\varepsilon/2$ with respect to separable, locally compact polyhedra. We also assume that δ is so small that δ-close points in E_j and B_j map by q_{ij} and r_{ij} respectively to $\varepsilon/2$-close points in E_i and B_i. Let X be a separable metric space and let $h: X \to E_j$, $H: X \times I \to B_j$ be maps such that $p_j h = H_0$. Apply Lemma 1 to h, H, p_j, $\delta/2$ and η, where $\eta < \delta$ is so small that η-close points map under p_j onto $\delta/2$-close points. We obtain a separable, locally compact polyhedron P and maps $f: X \to P$, $g: P \to E_j$, $G: P \times I \to B_j$ such that $f(X)$ is a dense subset of P and (14) and (15) hold. (Moreover, if $\dim X \leq n$, then $\dim P \leq n$. We shall use this fact in the proof of Theorem 5.)

Since $p_j h = H_0$, (14) and (15) imply

(16) $d(p_j gf, Gi_0 f) < \delta$.

Since $f(X)$ is dense in P, (16) implies

(17) $d(p_j g, Gi_0) \leq \delta$.

In other words, the diagram

(18)
$$\begin{array}{ccc} E_j & \xleftarrow{\;g\;} & P \\ {\scriptstyle p_j}\downarrow & & \downarrow{\scriptstyle i_0} \\ B_j & \xleftarrow[\;G\;]{} & P \times I \end{array}$$

commutes up to δ. By the choice of δ and j there is a homotopy $\tilde{G}: P \times I \to E_i$ such that

(19) $d(\tilde{G}_0, q_{ij} g) < \varepsilon/2$ and

(20) $d(p_i \tilde{G}, r_{ij} G) < \varepsilon/2$.

We now put

(21) $\tilde{H} = \tilde{G} \circ (f \times 1): X \times I \to E_i$.

Clearly,

(22) $d(\tilde{H}_0, q_{ij} h) < \varepsilon$ and

(23) $d(p_i \tilde{H}, r_{ij} H) < \varepsilon$.

4. The Proof of Lemma 1

Let us first recall that a map f of a space X into the nerve $|N(\mathcal{V})|$ of a locally finite open covering $\mathcal{V}$ of X is called a *canonical map* for $\mathcal{V}$ provided

(1) $f^{-1}(St(V,N(\mathcal{V}))) \subseteq V$

for all $V \in \mathcal{V}$. Equivalently, if $x \in V_0 \cap \cdots \cap V_n$, then f(x) is contained in the closed simplex spanned by the vertices $V_0,\cdots,V_n$.

Remark 6. For every map $f: X \to |K|$ of X onto a dense subset of the carrier of a locally finite simplicial complex K there is a star-finite open covering $\mathcal{V}$ of X such that $N(\mathcal{V}) = K$ and f is a canonical map for $\mathcal{V}$. It suffices to consider the star-finite covering ω formed by all St(q,K) where $q \in K^0$. With every q associate the open set V = $f^{-1}(St(q,K))$. These sets form a star-finite open covering of X. Clearly, $V_0 \cap \cdots \cap V_n \neq \phi$ if and only if $St(q_0,K) \cap \cdots \cap St(q_n,K) \neq \phi$, i.e., if and only if the vertices $q_0,\cdots,q_n$ span a simplex in K. In other words $N(\mathcal{V}) = K$. Furthermore, for each vertex q and for the corresponding $V \in \mathcal{V}$ one has

(2) $f^{-1}(St(q,K)) = V$

which shows that f is indeed a canonical map $f: X \to |N(\mathcal{V})| = |K|$ for $\mathcal{V}$.

The next lemma will be used in the proof of Lemma 1.

Lemma 2. *Let* X *be a separable metric space. Then every open covering of* X *admits a countable star-finite refinement* $\mathcal{V}$ *and a canonical map* $f: X \to |N(\mathcal{V})|$ *such that* f(X) *is dense in* $|N(\mathcal{V})|$. *If* dim X $\leq$ n *one can also achieve that* dim $N(\mathcal{V}) \leq n$.

Proof. Separable metric spaces are regular and Lindelöf and therefore strongly paracompact ([15], V. 4.B, p. 172), i.e., every open covering $\mathcal{U}'$ admits a star-finite refinement $\mathcal{U}$. If dim $X \leq n$, then every open covering admits a refinement $\mathcal{U}$ of order $\leq n + 1$ (which is a fortiori star-finite). Notice that every star-finite covering $\mathcal{U}$ of X must be countable. This is true since $\mathcal{U}$ admits a countable refinement $\mathcal{W} = (W_1, W_2, \cdots)$ and each W_i is contained in at most finitely many members of $\mathcal{U}$.

Let $g: X \to |N(\mathcal{U})|$ be a canonical map and let $\mathcal{U} = (U_1, U_2, \cdots)$. For a highest dimensional simplex S in $St(U_1, N(\mathcal{U}))$, whose interior is not entirely contained in $Cl(g(X))$, one composes g with a projection of $S \backslash y$ into ∂S, where $y \in \text{Int } S \backslash Cl(g(X))$. By repeating this procedure finitely many times, we obtain a modified map f_1 such that every simplex S in $St(U_1, N(\mathcal{U}))$ is either entirely contained in $Cl(f_1(X))$ or its interior is disjoint with $f_1(X)$. We now modify f_1 over $St(U_2, N(\mathcal{U}))$ to obtain f_2, etc. Because of local finiteness, $f = \lim f_n$ exists and $Cl(f(X)) = |K|$ for some subcomplex K of $N(\mathcal{U})$.

Notice that if for some $x \in X$ the point $f(x)$ belongs to $St(U_i, N(\mathcal{U}))$, then also $g(x) \in St(U_i, N(\mathcal{U}))$. By Remark 6, $f: X \to |K|$ is a canonical map for the star-finite covering $\mathcal{V}$ which consists of sets $V_i = f^{-1}(St(U_i, K))$ where U_i is a vertex of K. If $x \in V_i$, then $f(x) \in St(U_i, K) \subseteq St(U_i, N(\mathcal{U}))$ and therefore $g(x) \in St(U_i, N(\mathcal{U}))$, i.e., $x \in g^{-1}(St(U_i, N(\mathcal{U}))$. Since g is a canonical map for $\mathcal{U}$, the set $g^{-1}(St(U_i, N(\mathcal{U}))$ is contained in the member U_i of $\mathcal{U}$. Hence, $V_i \subseteq U_i$ and we see that $\mathcal{V}$ refines $\mathcal{U}$ and thus also $\mathcal{U}'$.

If dim $X \leq n$, then $N(\mathcal{V}) = K \subseteq N(\mathcal{U})$ and dim $N(\mathcal{U}) \leq n$ imply that dim $N(\mathcal{V}) \leq n$.

Proof of Lemma 1. It follows from ([6], Theorem 8.1, p. 146) that there exists an open covering $\mathcal{U}'$ of $X \times I$ such that for any locally finite refinement $\mathcal{U}$ of $\mathcal{U}'$ there exist maps $G: |N(\mathcal{U})| \to B$ and $g: |N(\mathcal{U}|X \times 0)| \to E$ such that for any canonical map $\phi: X \times I \to |N(\mathcal{U})|$ one has

(3) $d(G\phi, H) < \delta$ and

(4) $d(g(\phi|X \times 0)i_0, h) < \eta$.

Notice that the restriction $\mathcal{U}|X \times 0$ of $\mathcal{U}$ to $X \times 0$ consists of the sets $U \cap (X \times 0)$ where $U \in \mathcal{U}$. The nerve $N(\mathcal{U}|X \times 0)$ can be viewed as a subcomplex of $N(\mathcal{U})$ if one identifies the vertex $U \cap (X \times 0)$ of $N(\mathcal{U}|X \times 0)$ with the vertex U of $N(\mathcal{U})$, $U \in \mathcal{U}$. Consequently, if $(x,0) \in U_0 \cap \cdots \cap U_r$, then $\phi(x,0)$ is contained in the closed simplex spanned by the vertices $U_0, \cdots, U_r$ and therefore $\phi(x,0) \subseteq |N(\mathcal{U}|X \times 0)|$. The composition $g(\phi|X \times 0)$ is thus well-defined.

It is well-known (e.g. [5], IX Theorem 5.6, p. 241) that one can refine $\mathcal{U}'$ by a stacked covering of $X \times I$ over a covering $\mathcal{V}$ of X. By Lemma 2 one can assume that $\mathcal{V}$ is a countable star-finite covering which admits a canonical map $f: X \to |N(\mathcal{V})| = P$ such that $f(X)$ is dense in P. (Moreover, if dim $X \leq n$, one can achieve that dim $P \leq n$.) By the definition of stacked coverings, for each $V \in \mathcal{V}$ there is a finite collection of open intervals $J_1^V, \cdots, J_{n(V)}^V$ which covers I and whose nerve is a triangulation of I. Furthermore, each $V \times J_i^V$ is contained in a member of $\mathcal{U}'$.

Now consider all of the sets $St(V, N(\mathcal{V})) \times J_i^V$. They form an open covering ω of $P \times I$. Let (L,L') be a countable locally finite triangulation of $(P \times I, P \times 0)$ such that the sets $St(q,L)$, $q \in L^0$, refine ω. Then we put

(5) $\mathcal{U} = \{(f \times 1)^{-1}(St(q,L)) : q \in L^0\}$.

This is a countable star-finite covering of $X \times I$. For each vertex $q \in L^0$ there is a $V \in \mathcal{V}$ and an $i \leq n(V)$ such that

(6) $St(q,L) \subseteq St(V,K) \times J_i^V$.

Therefore,

(7) $$(f \times 1)^{-1}(St(q,L)) \subseteq (f \times 1)^{-1}\left(St(V,N(\mathcal{V})) \times J_i^V\right)$$
$$= f^{-1}\left(St(V,N(\mathcal{V})) \times J_i^V\right).$$

Since $f: X \to |N(\mathcal{V})|$ is a canonical mapping for $\mathcal{V}$, one has $f^{-1}(St(V,N(\mathcal{V})) \subseteq V$ and therefore

(8) $(f \times 1)^{-1}(St(q,L)) \subseteq V \times J_i^V$,

which proves that $\mathcal{U}$ refines $\mathcal{U}'$.

We now identify $q \in L^0$ with $U = (f \times 1)^{-1}(St(q,L)) \in \mathcal{U}$ and conclude by Remark 6 that $N(\mathcal{U}) = L$ and that $f \times 1: X \times I \to P \times I = |N(\mathcal{U})|$ is a canonical map for the covering $\mathcal{U}$.

The proof will be complete if we show that

(9) $|N(\mathcal{U}|X \times 0)| = P \times 0$.

Indeed, by the choice of $\mathcal{U}'$, we then obtain maps $G: P \times I \to B$ and $g: P \times 0 \to E$ such that conditions (14) and (15) of Lemma 1 hold.

Notice that (9) is equivalent to the following assertions:

(i) $U \cap (X \times 0) \neq \phi$ implies that $q \in L'$,

(ii) $q \in L'$ implies that $U \cap (X \times 0) \neq \phi$,

(iii) $U_0 \cap \cdots \cap U_n \cap (X \times 0) \neq \phi$ implies that $\langle q_0,\cdots,q_n\rangle \in L'$, and

(iv) $\langle q_0,\cdots,q_n\rangle \in L'$ implies that $U_0 \cap \cdots \cap U_n \cap (X \times 0) \neq \phi$.

(i) If $q \notin L'$, then $St(q,L) \cap |L'| = \phi$. Thus $(f \times 1)^{-1}(St(q,L)) \cap (f \times 1)^{-1}(|L'|) = \phi$ or $U \cap (X \times 0) = \phi$.

(ii) If $q \in L'$, then $St(q,L) \cap |L'|$ is a nonempty, open subset of $|L'|$. Since $f(X)$ is dense in $|L'|$, there exists an $x \in X$ such that $f(x) \in St(q,L) \cap |L'|$. Thus $(x,0) \in (f \times 1)^{-1}(St(q,L)) \cap (f \times 1)^{-1}(|L'|)$ or $(x,0) \in U \cap (X \times 0)$.

(iii) If $(x,0) \in U_0 \cap \cdots \cap U_n$, then $(f \times 1)(x,0) \in (f \times 1)(U_0) \cap \cdots \cap (f \times 1)(U_n) \cap |L'| \subseteq St(q_0,L) \cap \cdots \cap St(q_n,L) \cap |L'| = St(q_0,L') \cap \cdots \cap St(q_n,L')$. Therefore, $q_0,\cdots,q_n$ span a simplex in L'.

(iv) If $q_0,\cdots,q_n$ span a simplex in L', then $St(q_0,L') \cap \cdots \cap St(q_n,L')$ is a nonempty open subset of $|L'|$. By denseness of $f(X)$ in $|L'|$, there exists an $x \in X$ such that $(f \times 1)(x,0) = (f(x),0) \in St(q_0,L') \cap \cdots \cap St(q_n,L')$. Thus $(x,0) \in (f \times 1)^{-1}(St(q_0,L')) \cap \cdots \cap (f \times 1)^{-1}(St(q_n,L')) \subseteq (f \times 1)^{-1}(St(Q_0,L)) \cap \cdots \cap (f \times 1)^{-1}(St(q_n,L)) \cap (X \times 0) = U_0 \cap \cdots \cap U_n \cap \cdots \cap U_n \cap (X \times 0)$.

5. n-Shape Fibrations

Definition 6. A map $p: E \to B$ between metric compacta is called an n-*shape fibration* if it is an $\mathcal{X}$-shape fibration where $\mathcal{X}$ is the class of all topological spaces X of dimension $\leq n$.

Here we are using covering dimension based on numerable coverings.

Theorem 5. *A map* p: E → B *is an* n*-shape fibration if and only if it is an* $\mathcal{X}$*-shape fibration where* $\mathcal{X}$ *is either of the following two classes:*

(a) $\mathcal{X}$ *is the class of separable metric space* X *with* $\dim X \leq n$,

(b) $\mathcal{X}$ *is the class of separable, locally compact polyhedra* P *with* $\dim P \leq n$.

Proof. We shall first show that p is an n-shape fibration whenever (a) holds. Let j and δ be a lifting index and a lifting mesh for i and ε with respect to separable metric spaces of dimension $\leq n$. Let X be a topological space, $\dim X \leq n$, and let $h: X \to E_j$, $H: X \times I \to B_j$ satisfy

(1) $d(p_j h, H\, i_0) \leq \delta$.

We introduce the separable metric space Δ and the maps $f: X \to \Delta$, $g: \Delta \to E_j$, $G: \Delta \times I \to B_j$ as in the proof of Theorem 3. Then the diagram 3.(5) commutes up to δ. By a well-known factorization theorem (e.g., [14], Lemma 2.2, p. 34) the map f factors through a separable metric space Y where $\dim Y \leq \dim X \leq n$, i.e., there are maps $f': X \to Y$, $f'': Y \to \Delta$ such that

(2) $f''f' = f$.

One can also achieve that f' be surjective.

Now consider the diagram

(3)
$$\begin{array}{ccc} E_j & \xleftarrow{\;gf''\;} & Y \\ {\scriptstyle p_j}\downarrow & & \downarrow{\scriptstyle i_0} \\ B_j & \xleftarrow[\;G(f''\times 1)\;]{} & Y\times I \end{array}$$

which commutes up to δ since $f'\colon X \to Y$ is onto. By the choice of j and δ there exists a map $\tilde{G}\colon Y \times I \to E_i$ such that

(4) $d(\tilde{G}_0, q_{ij}gf'') < \varepsilon$, and

(5) $d(p_i\tilde{G}, r_{ij}G(f'' \times 1)) < \varepsilon$.

We now define $\tilde{H}\colon X \times I \to E_i$ by

(6) $\tilde{H} = \tilde{G}(f' \times 1)$.

It is readily seen that

(7) $d(\tilde{H}_0, q_{ij}h) < \varepsilon$, and

(8) $d(p_i\tilde{H}, r_{ij}H) < \varepsilon$.

The proof that (b) implies (a) follows the proof of Theorem 4 with the single modification of using the case $\dim X \leq n$ in Lemma 1.

We now further reduce the defining class $\mathcal{X}$ for n-shape fibrations.

Theorem 6. *A map* $p\colon E \to B$ *is an* n*-shape fibration if and only if it is an* $\mathcal{X}$*-shape fibration where* $\mathcal{X}$ *is either of the following two classes:*

(a) $\mathcal{X}$ *is the collection of all* k*-cells with* $k \leq n$, *i.e.,* $\mathcal{X} = \{I^0, I^1, \cdots, I^n\}$,

(b) $\mathcal{X}$ *is the class of compact polyhedra of dimension* $\leq n$.

We precede the proof of Theorem 6 by a lemma.

Lemma 3. *Let* $\underline{p}\colon \underline{E} \to \underline{B}$ *be a level map which has the* $\mathcal{X}$-AHLP *where* $\mathcal{X} = \{I^0, I^1, \cdots, I^n\}$. *Then for every* i *and* $\varepsilon > 0$ *there is a* $j \geq 1$ *and a* $\delta > 0$ *such that the following approximate partial homotopy lifting property holds: Let* X *be a compact polyhedron with* $\dim X \leq n$ *and let* $A \subseteq X$ *be a subpolyhedron. Let* $h\colon (X \times 0) \cup (A \times I) \to E_j$, *and* $H\colon X \times I \to B_j$

be maps such that

(9) $d(p_j h, H|(X \times 0) \cup (A \times I)) < \delta$.

Then, there is a homotopy $\tilde{H}: X \times I \to E_i$ *such that*

(10) $\tilde{H}|(X \times 0) \cup (A \times I) = q_{ij}h$, *and*

(11) $d(p_i\tilde{H}, r_{ij}H) < \varepsilon$.

Proof. The proof proceeds by induction on $\dim(X - A)$. If $\dim(X - A) = 0$, let j and δ be a lifting index and a lifting mesh for i and ε. Then we extend $q_{ij}h$ to a map $\tilde{H}: (X \times 0) \cup ((A \cup X^0) \times I) \to E_i$ where for any vertex $x \in X^0 \setminus A$ the restriction $\tilde{H}_x = \tilde{H}|x \times I$ is a homotopy satisfying

(12) $\tilde{H}_x(x,0) = q_{ij}h(x)$ (see Remark 1), and

(13) $d(p_i\tilde{H}_x, r_{ij}H|(x \times I)) < \varepsilon$.

We now assume the assertion true for $\dim(X - A) < k \leq n$. For a given i and ε choose a lifting index i' and a lifting mesh ε' for $\{I^0, I^1, \cdots, I^n\}$. Now choose j and δ in accordance with the inductive hypothesis for the integer $k - 1$ and for i' and ε'. We also require that δ-close points map by $r_{i'j}$ into ε'-close points.

Now assume that $\dim(X - A) = k$ and that we are given h and H satisfying (9). By the inductive hypothesis there is a map $\tilde{H}': (X \times 0) \cup ((X^{k-1} \cup A) \times I) \to E_{i'}$ such that

(14) $\tilde{H}'|(X \times 0) \cup (A \times I) = q_{i'j}h|(X \times 0) \cup (A \times I)$

and

(15) $d(p_{i'}\tilde{H}', r_{i'j}H|(X \times 0) \cup ((X^{k-1} \cup A) \times I) < \varepsilon'$.

For each k-simplex C of $X \setminus A$ we consider the maps $\tilde{H}'|(C \times 0) \cup (\partial C \times I)$ and $r_{i'j}H|C \times I$. By the choice of i' and ε', by (15) and by Remark 1, there is a map $\tilde{H}_C: C \times I \to E_i$ such that

(16) $\tilde{H}_C|(C \times 0) \cup (\partial C \times I) = q_{ii'}\tilde{H}'|(C \times 0) \cup (\partial C \times I)$

and

(17) $d(p_i\tilde{H}_C, r_{ij}H|C \times I) < \varepsilon$.

Now the desired homotopy $\tilde{H}: X \times I \to E_i$ is given by

(18) $\tilde{H}|(X \times 0) \cup ((X^{k-1} \cup A) \times I) = q_{ii'}\tilde{H}'$ and

(19) $\tilde{H}|C \times I = \tilde{H}_C$.

This completes the proof of Lemma 3.

Proof of Theorem 6. It suffices (by Theorem 4) to show that (a) implies that p is an $\mathcal{X}$-shape fibration for the class of all locally compact separable polyhedra X with dim $X \leq n$.

For a given i and ε choose $j \geq i$ and $\delta > 0$ in accordance with Lemma 3. Once again apply Lemma 3, this time to j and δ, to obtain j' and δ'. Let $X = |K|$, where K is a countable locally finite simplicial complex, dim $X \leq n$, and let $h: X \to E_{j'}$ and $H: X \times I \to B_{j'}$ be such that $d(p_{j'}h, H_0) < \delta'$. By a standard construction, one can find two sequences of finite subpolyhedra $X_k, Y_k \subseteq X$, $k = 1,2,\cdots$, which cover X and are such that $X_k \cap X_{k'} = \phi$, $Y_k \cap Y_{k'} = \phi$ for $k \neq k'$ and $Y_k \cap (\cup X_i) = Y_k \cap (X_{k-1} \cup X_k)$. By the choice of j' and δ' there is a homotopy $\tilde{H}': (\cup X_k) \times I \to E_j$ such that

(20) $\tilde{H}'|(\cup X_k) \times 0 = q_{jj'}h|(\cup X_k)$ and

(21) $d(p_j\tilde{H}', r_{jj'}H|(\cup X_k) \times I) < \delta$.

Again we apply Lemma 3, this time to each pair $(Y_k, Y_k \cap (X_{k-1} \cup X_k))$ to obtain a homotopy $\tilde{H}_k: Y_k \times I \to E_i$ which extends $q_{ij}\tilde{H}'|(Y_k \cap (X_{k-1} \cup X_k)) \times I$ and $q_{ij}h|Y_k$ and is such that

(22) $d(p_i\tilde{H}_k, r_{ij'}H|Y_k \times I) < \varepsilon$.

Clearly, the map $\tilde{H}: X \times I \to E_i$ given by $\tilde{H}|(\cup X_k) \times I = q_{ij}\tilde{H}'$

and $\tilde{H}|Y_k \times I = \tilde{H}_k$ has all of the desired properties.

Example 1. For every n there is a map p: E → B which is an n-shape fibration, but which fails to be an (n + 1)-shape fibration. Indeed, let $E = S^{n+1} \subseteq E^{n+2}$ be the unit sphere and let B be the segment $0 \times \cdots \times 0 \times [-1,1] \subseteq E^{n+2}$ whose end-points are the south-pole e_0 and the north-pole e_1 of S^{n+1}. Let p: E → B be the projection defined by $p(x_1,\cdots, x_{n+1},x_{n+2}) = (0,\cdots,0,x_{n+2})$. Then p fails to have the AHLP for $X = S^{n+1}$ and is not an (n + 1)-shape fibration. One can see this by considering the maps h = identity and H defined by $H(x_1,\cdots,x_{n+1},x_{n+2},t) = (0,\cdots,0,(1 - t)x_{n+2})$. However, the trivial level map $\underline{p}$ (i.e., $\underline{p} = (p_i)$ and $p_i = p$) does have the AHLP with respect to compact polyhedra X where dim X ≤ n. This is an easy consequence of the following facts: For an arbitrary ε > 0 each map h: X → E is ε-homotopic to a map $X \longrightarrow E\setminus\{e_0,e_1\}$, $0 \times \cdots \times 0 \times [\varepsilon - 1,1 - \varepsilon]$ is a strong deformation retract of B, and $p|E\setminus\{e_0,e_1\}: E\setminus\{e_0,e_1\} \to B\setminus\{e_0,e_1\}$ is a product map and therefore has the homotopy lifting property.

Theorem 7. *Let* p: E → B *be an* n-*shape fibration, let* e ∈ E, b = p(e), *and* $F = p^{-1}(b)$. *Then,* p *induces an isomorphism of the homotopy pro-groups*

(23) $\underline{p}_*$: pro − $\pi_k(E,F,e)$ → pro − $\pi_k(B,b)$ *for all* k ≤ n.

Theorem 2 of [12] is the analogue of Theorem 7 and its proof applies without change to establish Theorem 7. Notice that one can use Theorem 2 of the present paper to achieve that p be induced by a level map having the n-HLP.

Again by arguing as in [12], one obtains the following theorem.

Theorem 8. *If* p: E → B *is an* n-*shape fibration*, e ∈ E, b = p(e), F = p^{-1}(b), *then the following (finite) sequence is exact:*

$$(24)\quad \text{pro} - \pi_n(F,e) \xrightarrow{\underline{i}_*} \text{pro} = \pi_n(E,e) \xrightarrow{\underline{p}_*} \text{pro} - \pi_n(B,b) \xrightarrow{\underline{\delta}} \text{pro} - \pi_{n-1}(F,e) \to \cdots.$$

The above sequence is obtained from the exact pro-homotopy sequence of the pointed pair (E,F,e) (see [10]) by replacing pro - π_k(E,F,b) by pro - π_k(B,b), k $\leq$ n (using Theorem 6). This also explains the morphisms $\underline{i}_*$, $\underline{p}_*$ and $\underline{\delta}$.

6. Weak Shape Fibrations

Definition 7. A map p: E → B is called a *weak shape fibration* if it is an n-shape fibration for every n.

Notice that for the corresponding level maps $\underline{p}$ and for given i and ε the lifting index and lifting mesh depend on n.

The next corollary follows from Theorems 7 and 8.

Corollary 2. *For a weak shape fibration* p: E → B *the morphism*

$$\underline{p}_*: \text{pro} - \pi_k(E,F,e) \longrightarrow \text{pro} - \pi_k(B,b)$$

is an isomorphism of pro-groups for all k *and one has an (infinite) exact sequence corresponding to* (24).

Theorem 9. *Let* p: E → B *be a weak shape fibration and let the points* x,y ∈ B *belong to the same path component. If* dim E < ∞ *and* dim B < ∞, *then the fibers* X = p^{-1}(x) *and* Y = p^{-1}(y) *have the same shape.*

Proof. The proof of Theorem 9 will follow from the scheme of proof of Theorem 3 of [11] once we establish the following fact: There exist an integer r and a level map $\underline{p}'\colon \underline{E}' \to \underline{B}$ of polyhedral sequences such that

(1) $\dim E_i' \leq r$ for each i,

(2) $p = \lim \underline{p}'$, and

(3) $\underline{p}'$ has the HLP with respect to metric compacta of dimension $\leq r + 1$.

In order to establish this fact, embed E and B in Euclidean spaces R^n and R^m, respectively. One now obtains $\underline{p}\colon \underline{E} \to \underline{B}$ by extending p to a map $\tilde{p}\colon R^n \to R^m$ and by considering suitable decreasing sequences of polyhedral neighborhoods of E and B respectively. It follows from Theorem 1 that for each k, $\underline{p}$ has the $\mathcal{X}$-AHLP where $\mathcal{X}$ is the class of metric compacta of dimension $\leq k$. The proof of Theorem 2 now yields $\underline{p}'\colon \underline{E}' \to \underline{B}$ such that for each k (in particular, for $k = r + 1$ defined below), $\underline{p}'$ has the $\mathcal{X}$-HLP where $\mathcal{X}$ is the class of metric compacta of dimension $\leq k$. Furthermore, the proof of Theorem 2 allows one to assume that the members E_i' of $\underline{E}'$ are subpolyhedra of $E_i \times B_i$ which implies that $\dim E_i' \leq n + m = r$.

Theorem 10. *If* E *and* B *are compact* ANR*'s, then* $p\colon E \to B$ *is a weak shape fibration if and only if it is an approximate fibration, or equivalently a shape fibration.*

Proof. An approximate fibration between compact ANR's is a shape fibration and thus also a weak shape fibration.

Conversely, if p is a weak shape fibration, then for each n the trivial level map of ANR-sequences $\underline{p}$ has the AHLP with respect to $\{I^0, I^1, \cdots, I^n\}$ by Theorems 1 and 6. This implies that p has the AHLP for all I^k, $k = 0,1,\cdots$. Hence, by Theorem 2.6 of [2], p is an approximate fibration.

7. Cell-like Maps are Weak Shape Fibrations

Theorem 11. *Every cell-like map* $p: E \to B$ *between metric compacta is a weak shape fibration.*

By specializing Lemma 8.3 of [4] one obtains the following lemma. (This Lemma can also be proved by using the techniques of Lemma 2.3 of [7].)

Lemma 4. *Let* $p: Q \to Q$ *be a map, let* $B \subseteq Q$ *be a compact subset of* Q, *let* $E = p^{-1}(B)$ *and let* $\mathrm{Sh}(p^{-1}(b)) = 0$ *for each* $b \in B$. *Then for any two neighborhoods* $U \supseteq E$, $V \supseteq B$ *with* $p^{-1}(V) \subseteq U$ *and for any* $n \in N$ *and* $\varepsilon > 0$ *there exist neighborhoods* $U' \supseteq E$, $V' \supseteq B$ *with* $U' \subseteq U$, $V' \subseteq V$, $p^{-1}(V') \subseteq U'$ *having the following property: If* X *is a compact polyhedron,* $\dim X \leq n$, *and* $A \subseteq X$ *a compact subpolyhedron and if* $h: A \to U'$, $g: X \to V'$ *are maps such that* $ph = g|A$, *then there exists a map* $\tilde{g}: X \to U$ *such that* $\tilde{g}|A = h$ *and* $d(p\tilde{g},g) < \varepsilon$.

Proof of Theorem 11. Let $p: E \to B$ be a cell-like map between metric compacta. We embed E in Q and consider the quotient space $\tilde{Q} = Q/\{p^{-1}(b): b \in B\}$ and the quotient map π. One can identify B with $\pi(E)$ so that $\pi|E = p$. Since $\tilde{Q}$ is a compact metric space, it can be embedded in Q. Notice that for $\pi: Q \to \tilde{Q} \subseteq Q$ one has $\pi^{-1}(B) = E$. Applying Lemma 4 to π one can construct by induction compact ANR-neighborhoods U_i and V_i of E and B respectively such that $\pi^{-1}(V_i) \subseteq U_i$, the neighborhoods U_i and V_i, $i \in N$, form inclusion ANR-sequences with limits E and B respectively and $U' = U_i$, $V' = V_i$ satisfy

Lemma 4 for $U = U_{i-1}$, $V = V_{i-1}$, $n = i$ and $\varepsilon = \frac{1}{i}$.

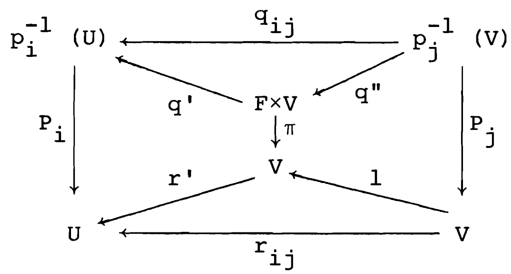
The maps $p_i = \pi|U_i\colon U_i \to V_i$ form a level map $\underline{p}$, which induces p. Furthermore, for given i, $n \in N$ and $\varepsilon > 0$ one can choose $j \in N$ such that $j > \max\{i,n,\frac{1}{\varepsilon}\}$. If X is a compact polyhedron, $\dim X \le n$, and if $h\colon X \to U_j$ and $H\colon X \times I \to V_j$ satisfy $H_0 = p_j h = \pi h$, then there is a homotopy $\tilde{H}\colon X \times I \to U_{j-1} \subseteq U_i$ such that $\tilde{H}_0 = h$ and $d(\pi\tilde{H},H) < \frac{1}{j} < \varepsilon$. In view of Theorem 6 and Remark 2 this completes the proof.

Example 2. The Taylor map is a weak shape fibration (Theorem 11), which fails to be a shape fibration (Example 6 of [11]).

8. Shape Cell-bundles

In this section we exhibit for every cell-like map $p\colon E \to B$ a level map $\underline{p}$ between ANR-sequences which induces p and which has a certain local factorization property. This yields an alternate proof of Theorem 11 and is of some independent interest.

Definition 8. A map $p\colon E \to B$ between metric compacta is called a *shape disk-bundle (shape Q-bundle)* if there is a level map of ANR-sequences $\underline{p}\colon \underline{E} \to \underline{B}$ which induces p and which has the following property: For each i there is a $j \ge i$ such that for each $b \in B_j$ there exists an open neighborhood V of b in B_j and an open set $U \subseteq B_i$ with $r_{ij}(V) \subseteq U$ such that for F the n-cell (the Hilbert cube Q) we have the following factorization diagram where $\pi\colon F \times V \to V$ is the second projection. We call shape disk-bundles and shape Q-bundles jointly *shape cell-bundles*.



Theorem 12. *Every shape cell-bundle is a weak shape fibration.*

Proof. Let $\underline{p}$: $\underline{E} \to \underline{B}$ be a level map as in Definition 8. Notice that it will more than suffice to establish the following: given $i,n \in N$ there exists $j \geq i$ such that for any compact polyhedron X, dim $X \leq n$, and subpolyhedron $A \subseteq X$, and for any maps $H: X \to B_j$, $h: A \to E_j$ with $p_j h = H|A$, there is a map $\tilde{H}: X \to E_i$ such that

(1) $\tilde{H}|A = q_{ij}h$, and

(2) $p_i\tilde{H} = r_{ij}H$.

In order to prove this fact choose integers $i = i_0 \leq i_1 \leq \cdots \leq i_{n+1} = j$ such that i_k satisfies Definition 8 with respect to i_{k-1}. Notice that we can find open covers $\mathcal{V}^{i_k}$ of B_{i_k}, $k = 0,1,\cdots,n+1$, such that any $V \in \mathcal{V}^{i_{k+1}}$ admits a $U \in \mathcal{V}^{i_k}$ satisfying Definition 8.

Suppose we are given the maps $H: X \to B_j$, $h: A \to E_j$ with $p_j h = H|A$. Let K be a triangulation of X of fine enough mesh that simplices of K are mapped into elements of $\mathcal{V}^{i_{n+1}}$ under H. Let L^r denote the set of all simplices from K of dimension $\leq r$ which are not contained in A. One can easily find a map $\tilde{H}^0: A \cup |L^0| \to E_{i_n}$ such that

(3) $\tilde{H}^0|A = q_{i_n j}h$, and

(4) $p_{i_n}\tilde{H}^0 = r_{i_n j}H|A \cup |L^0|$.

Inductively, suppose that $\tilde{H}^r\colon A \cup |L^r| \to E_{i_{n-r}}$ is such that

(5) $\tilde{H}^r|A = q_{i_{n-r}j}h$, and

(6) $p_{i_{n-r}}\tilde{H}^r = r_{i_{n-r}j}H|A \cup |L^r|$.

In order to define $\tilde{H}^{r+1}\colon A \cup |L^{r+1}| \to E_{i_{n-r-1}}$ we extend $q_{i_{n-r-1}i_{n-r}}\tilde{H}^r$ to all (r + 1)-simplices Δ of L^{r+1} as follows. Since $\tilde{H}^r\colon \partial\Delta \to E_{i_{n-r}}$ is already defined, one can consider the map

(7) $\pi' q'' \tilde{H}^r\colon \partial\Delta \to F$,

where $\pi'\colon F \times V \to F$ is the first projection. Since F is contractible, the map (7) can be extended to a map $g\colon \Delta \to F$. Then we define $\tilde{H}^{r+1}|\Delta\colon \Delta \to E_{i_{n-r-1}}$ by

(8) $\tilde{H}^{r+1}|\Delta = q'(g \times r_{i_{n-k}j}H|\Delta)$.

It is easy to check that $\tilde{H}^{r+1}$ so defined has all the properties needed in the inductive step. Finally, $\tilde{H} = \tilde{H}^n\colon X \to E_i$ is the required map satisfying (1) and (2).

Theorem 13. *Every cell-like map* p: E → B *between finite-dimensional metric compacta is a shape disk-bundle.*

We shall prove two preliminary lemmas.

Lemma 5. *Let* p: E → B *be a cell-like map between finite dimensional metric compacta. Then, there exist* m,n ∈ N *and embeddings* $j\colon B \to R^n$, $i\colon E \to R^m \times R^n$ *such that the projection* $\pi\colon R^m \times R^n \to R^n$ *satisfies the following conditions:*

(9) $\pi i = jp$, *and*

(10) *for each* b ∈ B *the set* $ip^{-1}(b)$ *is cellular in the*

hyperplane $\pi^{-1}j(b) = R^m \times j(b)$.

Proof. For $n \geq 2 \dim B + 1$ there is an embedding $j: B \to R^n$. For $r \geq 2 \dim E + 1$ there is an embedding $k: E \to R^r$. An easy general position argument enables one to verify that for each $b \in B$, $kp^{-1}(b) \subseteq r^r \times 0$ satisfies the cellularity criterion [13] in $R^{r+3} = R^r \times R^3$ provided $r \geq 2$ (e.g., p. 600 of [8]). Let $m = r + 3 \geq 5$ and let $i: E \to R^m \times R^n$ be given by

(11) $i(x) = (k(x), jp(x))$.

Then, i is an embedding because k is an embedding. Furthermore, $\pi i = jp$ and for each $b \in B$ the set $ip^{-1}(b) = kp^{-1}(b) \times j(b)$ is cellular in $R^m \times j(b)$ by Theorem 3 of [13]. This completes the proof of Lemma 5.

Remark 7. From now on we shall identify $x \in E$ with $i(x)$ and $b \in B$ with $j(b)$. Therefore, $p: E \to B$ can be viewed as the restriction $\pi|E$ of the projection $\pi: R^m \times R^n \to R^n$. Notice that $p^{-1}(b)$ is now cellular in $R^m \times b$ for each $b \in B$.

Definition 9. A finite collection $\mathcal{C}$ of pairs (B_k, D_k), $k = 1, 2, \cdots, r$, will be called an *admissible collection* for p if it satisfies the following conditions:

(12) $B_1, \cdots, B_r$ are PL n-cells in R^n and $B \subseteq \cup_{k=1}^r \text{Int } B_k$

(13) $B_k \cap B \neq \phi$ for $k = 1, \cdots, r$

(14) $D_1, \cdots, D_r$ are PL m-cells in R^m such that for $k = 1, \cdots, r$,

$p^{-1}(B \cap B_k) \subseteq (\text{Int } D_k) \times B_k$.

With each admissible collection $\mathcal{C}$ we associate two polyhedra

(15) $B_{\mathcal{C}} = \cup_{i=1}^r B_k \subseteq R^n$, and

(16) $E_{\mathcal{C}} = \cup_{i=1}^{r}(D_k \times B_k) \subseteq R^m \times R^n$.

Notice that $\pi(E_{\mathcal{C}}) = B_{\mathcal{C}}$, where $\pi: R^m \times R^n \to R^n$ is the second projection. Also notice that $B_{\mathcal{C}}$ is a neighborhood of B by (12), and that $E_{\mathcal{C}}$ is a neighborhood of E since for $e \in E$ there exists by (12) a B_k such that $p(e) \in \text{Int } B_k$ and so by (14) $e \in \text{Int } D_k \times \text{Int } B_k \subseteq E_{\mathcal{C}}$. However by the invariance of domain $\text{Int } D_k \times \text{Int } B_k$ is open in $R^m \times R^n$ and therefore $e \in \text{Int } E_{\mathcal{C}}$.

Remark 8. An admissible collection always exists. In particular, if B_1 is a PL n-cell in R^n with $B \subseteq \text{Int } B_1$ and D_1 is a PL m-cell in R^m with $\pi'(E) \subseteq \text{Int } D_1$ (where π' is the first projection), then the single pair (B_1, D_1) forms an admissible collection.

Lemma 6. *Let* $\mathcal{C} = \{(B_k, D_k)\}_{k=1}^{r}$ *be an admissible collection for* $p: E \to B$, *let* $X \subseteq E_{\mathcal{C}}$ *be an open neighborhood of* E *in* $R^m \times R^n$ *and let* $Y \subseteq \cup \text{ Int } B_k$ *be a compact neighborhood of* B *in* R^n. *Let* δ *be the Lebesgue number for the open cover* $\{\text{Int } B_k \cap Y\}_{k=1}^{r}$ *of* Y. *Then, there exists an admissible collection* $\mathcal{C}' = \{(B'_{k'}, D'_{k'})\}_{k'=1}^{r'}$ *for* p *such that*

(17) $B_{\mathcal{C}'} \subseteq Y$,

(18) $\text{diam } B'_{k'} < \delta/3$, $k' = 1, \cdots, r'$,

(19) $D'_{k'} \times B'_{k'} \subseteq X$, $k' = 1, \cdots, r'$, *hence* $E_{\mathcal{C}'} \subseteq X$, *and*

(20) $D'_{k'} \times B'_{k'} \subseteq (\text{Int } D_k) \times B'_{k'}$ *for all* k *such that* $B'_{k'} \subseteq \text{Int } B_k$.

Proof. Let $b \in B$ and let $\Delta(b)$ denote the set of all indices $k \in \{1, \cdots, r\}$ such that $b \in \text{Int } B_k$. Then, by (14)

(21) $p^{-1}(b) \subseteq \cap_{k \in \Delta(b)}((\text{Int } D_k) \times B_k)$.

Therefore,

(22) $\pi' p^{-1}(b) \subseteq \cap_{k\in\Delta(b)} \text{Int } D_k$.

By Lemma 5 we may choose a PL m-cell $D_b' \subseteq R^m$ such that

(23) $\pi' p^{-1}(b) \subseteq \text{Int } D_b' \subseteq D_b' \subseteq \cap_{k\in\Delta(b)} \text{Int } D_k$.

Since $p^{-1}(b) \subseteq X$, one can also achieve that

(24) $D_b' \times b \subseteq X$.

Now choose a PL n-cell neighborhood B_b' of b in Y so small that

(25) diam $B_b' < \delta/3$,

(26) $p^{-1}(B \cap B_b') \subseteq (\text{Int } D_b') \times B_b' \subseteq D_b' \times B_b' \subseteq (\cap_{k\in\Delta(b)} \text{Int } D_k) \times B_b'$, and

(27) $D_b' \times B_b' \subseteq X$.

Since B is compact, the covering $\{\text{Int } B_b' | b \in B\}$ of B admits a finite subcover. Thus, we obtain a finite collection $B_1', \cdots, B_{r'}'$ of PL n-cells and a corresponding collection $D_1', \cdots, D_{r'}'$ of PL m-cells. Then $\mathcal{C}' = \{(B_{k'}', D_{k'}')\}_{k'=1}^{r'}$ is an admissible collection which satisfies (17)-(20). Notice in particular that (14) and (20) follow from (26).

Proof of Theorem 13. Let p: E → B be a cell-like map between finite-dimensional compacta. Then we identify p with $\pi|E$ as described in Remark 7.

We shall now define by induction a sequence of admissible collections $\mathcal{C}_i$, i = 1,2,$\cdots$. We choose for $\mathcal{C}_1$ an arbitrary admissible collection (Remark 8). Given $\mathcal{C}_i = \{(B_k, D_k)\}_{k=1}^{r}$ we define $\mathcal{C}_{i+1} = \{(B_{k'}', D_{k'}')\}_{k'=1}^{r'}$ by applying Lemma 6 to $\mathcal{C} = \mathcal{C}_i$, $X = X_i$ contained in the 1/i-neighborhood N(E,1/i), $Y = Y_i$ contained in N(B,1/i) and to $\delta = \delta_i$ the Lebesgue number for $\{\text{Int } B_k \cap Y_i\}_{k=1}^{r}$. We obtain in this way a level map $\underline{p} = (p_i)$ of polyhedral inclusion sequences with $p_i = \pi|E_{\mathcal{C}_i}$

where $\lim \underline{p} = p = \pi|E$:

$$\begin{array}{ccccc} {}^E\mathcal{C}_1 & \longleftarrow & {}^E\mathcal{C}_2 & \longleftarrow & \cdots \\ \downarrow p_1 & & \downarrow p_2 & & \\ {}^B\mathcal{C}_1 & \longleftarrow & {}^B\mathcal{C}_2 & \longleftarrow & \cdots \end{array}$$

We claim that $\underline{p}$ satisfies all the conditions of Definition 8 with $j = i + 1$ and F an m-cell D_k (depending on $b \in B$).

Indeed, if $b \in B$, one can choose by (12) a k_0' such that $b \in \text{Int } B'_{k_0'} = V$. Then by (16) we have

(28) $(p_{i+1})^{-1}(V) = \cup_{k'=1}^{r'} D'_{k'} \times (V \cap B'_{k'})$.

Let $J = \{k' \colon V \cap B'_{k'} \neq \phi\}$. Then by (18) there exists a k such that

(29) $\cup_{k' \in J} B'_{k'} \subseteq \text{Int } B_k = U$.

Notice that $k_0' \in J$ and therefore

(30) $V \subseteq \mathcal{U}$.

By (20) and (29)

(31) $\cup_{k' \in J}(D'_{k'} \times B'_{k'}) \subseteq (\text{Int } D_k) \times U$

and so by definition of J and V

(32) $\cup_{k'=1}^{r'} D'_{k'} \times (V \cap B'_{k'}) \subseteq (\text{Int } D_k) \times V$.

By (28) and (32) we have

(33) $(p_{i+1})^{-1}(V) \subseteq D_k \times V$,

and by (29), (30) and (16) we have

(34) $D_k \times V \subseteq \cup_{k=1}^{r} D_k \times (U \cap B_k) = (p_i)^{-1}(U)$.

Consequently, $V, U, F = D_k$ and the inclusions r', q", q' given by (30), (33) and (34) respectively satisfy all the conditions of Definition 8.

Theorem 14. *Every cell-like map* $p\colon E \to B$ *between metric compacta is a shape* Q-*bundle.*

The proof of Theorem 14 is the same as the proof of Theorem 13 except that Lemma 7 plays the role of Lemma 5.

The roles of the PL-cells in R^m is taken by PL-*Hilbert cubes* in $Q = \Pi_{j=1}^{\infty}[-1,1]$ defined as compacta of the form

(35) $D \times \Pi_{j>i}[-1,1] \subseteq Q$,

where $D \subseteq \Pi_{j=1}^{i}[-1,1] \subseteq R^i$ is a PL i-cell.

Notice that the union of a finite collection of PL-Hilbert cubes is an ANR. (In fact, it is a Q-manifold.)

The role of cellular sets in R^m is taken by *cellular sets* $X \subseteq Q$ defined as intersections of sequences

(36) $Q_1 \supseteq \text{Int } Q_1 \supseteq Q_2 \supseteq \cdots$,

where each Q_i is a PL-Hilbert cube, i.e., is of the form (35).

Remark 9. The proof of Theorem 1 of [9] shows that every continuum X of trivial shape which is contained in $Q^+ = \Pi_{i=1}^{\infty}[0,1]$ is a cellular set in Q.

Lemma 7. *Let* $p: E \to B$ *be a cell-like map between metric compacta. Then, there exist embeddings* $j: B \to Q$ *and* $i: E \to Q \times Q$ *such that the second projection* $\pi: Q \times Q \to Q$ *satisfies the following conditions:*

(37) $\pi i = jp$,

(38) *For each* $b \in B$, *the set* $ip^{-1}(b)$ *is cellular in* $\pi^{-1}j(b) = Q \times j(b)$.

Proof. Let $j: B \to Q$ and $k: E \to Q^+ \subseteq Q$ be arbitrary embeddings. Consequently, by Remark 9 $kp^{-1}(b) \subseteq Q^+$ is cellular in Q. Let $i: E \to Q \times Q$ be given by

(39) $i(x) = (k(x), jp(x))$.

Clearly, (37) and (38) hold because $ip^{-1}(b) = kp^{-1}(b) \times j(b)$ is cellular in $Q \times j(b)$.

Example 3. By Theorem 14, the Taylor map p, [16], is a shape Q-bundle and therefore a weak shape fibration. However, the Taylor map is not a shape fibration (Example 6 of [11]). Consequently, if p = lim $\underline{p}$, the maps p_i cannot be bundle maps.

In spite of Example 3, we may pose the following questions for the finite-dimensional case.

Question 1. Is every cell-like map between finite-dimensional metric compacta induced by a level map of ANR-sequences $\underline{p}$: $\underline{E} \to \underline{B}$ such that each p_i: $E_i \to B_i$ is a disk-bundle?

An affirmative answer to this question was answered by T. B. Rushing in "A characterization of inverse limits of n-disk bundle maps," Notices, A.M.S. 193, V. 26, no. 3, abstract 765-627.

Question 2. Is there a notion of a "shape bundle" which satisfactorily generalizes the notion of a shape cell-bundle?

This question has been answered affirmatively by Šime Ungar in "Shape Bundles" (to appear).

References

[1] D. Coram and P. Duvall, *Approximate fibrations*, Rocky Mountain J. Math. 7 (1977), 275-288.

[2] ________, *Approximate fibrations and a movability condition for maps*, Pacific J. Math. 72 (1977), 41-56.

[3] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1966.

[4] J. Dydak, *The Whitehead and Smale theorems in shape theory*, Dissertationes Math., Warszawa (to appear).

[5] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, Princeton, 1952.

[6] S. T. Hu, *Theory of retracts*, Wayne State Univ. Press, Detroit, 1965.

[7] R. C. Lacher, *Cell-like mappings, I*, Pacific J. Math. 30 (1969), 717-731.

[8] ________, *Cell-like spaces*, Proc. Amer. Math. Soc. 20 (1969), 598-602.

[9] S. Mardešić, *Decreasing sequences of cubes and compacta of trivial shape*, General Topology and Appl. 2 (1972), 17-23.

[10] ________, *On the Whitehead theorem in shape theory, I*, Fund. Math. 91 (1976), 51-64.

[11] ________ and T. B. Rushing, *Shape fibrations, I*, General Topology and Appl. 9 (1978), 193-215.

[12] ________, *Shape fibrations, II*, Rocky Mountain J. Math. (to appear).

[13] D. R. McMillan, *A criterion of cellularity in a manifold*, Ann. of Math. 79 (1964), 327-337.

[14] K. Morita, *Čech cohomology and covering dimension for topological spaces*, Fund. Math. 87 (1975), 31-52.

[15] J. Nagata, *Modern general topology*, North Holland, Amsterdam, 1968.

[16] J. L. Taylor, *A counter-example in shape theory*, Bull. Amer. Math. Soc. 81 (1975), 629-632.

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