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\mathcal{C} -CALMLY REGULAR CONVERGENCE

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ζ -CALMLY REGULAR CONVERGENCE

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1. Introduction

Let X be a metric space and let d denote the distance-function defined on X . By 2^X we denote the hyperspace of all non-empty compacta lying in X . In [4] Borsuk defined the fundamental metric d_F on 2^X such that two compacta in X which are close with respect to d_F have similar shape properties. In particular, it was demonstrated in [4] and [8] that there is a large number of hereditary shape properties α (i.e., properties preserved by shape dominations) such that the following holds.

(1.1) If A_0, A_1, \dots is a sequence in 2^X with $\lim_{d_F} (A_k, A_0) = 0$ and $A_k \in \alpha$ for $k = 1, 2, \dots$, then $A_0 \in \alpha$.

On the other hand, Boxer and Sher [5] observed that in general (1.1) is not true for every hereditary shape property α . But, by assuming that A_0 is an FANR, they established (1.1) for every α . Finally, in [8] the same was proved under a weaker assumption that A_0 is a calm compactum. The class of calm compacta was introduced by the author in [6]. It intersects the class of all movable compacta in the class of all FANR's (see Theorem (4.5) in [8]) but calm compacta need not be movable (e.g. solenoids).

In view of the above results, it is interesting to find conditions under which $\lim_{d_F} (A_k, A_0) = 0$ for a sequence A_0, A_1, \dots in 2^X implies that A_0 is calm.

This paper introduces the notion of a calmly regular

(or ca-regular) convergence of compacta lying in X such that if the sequence A_1, A_2, \dots converges ca-regularly to A_0 , then A_0 is calm and $\lim d_F(A_k, A_0) = 0$.

The definition of the ca-regular convergence, besides providing generalizations of theorems about calm compacta from [6] (see §§2 and 3), is justified by the new information that it gives about the collection $ca(X)$ of all calm compacta in a metric space X . The main result (4.6) of this paper shows that one can define a metric d_{ca} on $ca(X)$ such that the convergence with respect to d_{ca} is equivalent to the ca-regular convergence.

The paper is organized as follows. In §2 we recall definitions of \mathcal{C} -calmness from [6], where \mathcal{C} is an arbitrary (non-empty) class of topological spaces, and the fundamental metric d_F from [4] and introduce the notion of a \mathcal{C} -calmly regular (or $ca_{\mathcal{C}}$ -regular) convergence of compacta in a metric space X which lies in an ANR space M . The ca-regular convergence is by definition $ca_{\mathcal{C}}$ -regular convergence for \mathcal{C} the class of all finite polyhedra. We first prove (see (2.3)) that in this definition the choice of the embedding of X and of M is immaterial. In (2.4) we prove that if a sequence $\{A_k\}$ in 2^X converges $ca_{\mathcal{C}}$ -regularly to a compactum A_0 (in symbols, $A_k \xrightarrow{ca_{\mathcal{C}}} A_0$), then A_0 is \mathcal{C} -calm regardless of the nature of A_k 's. Then we discuss the role of the class \mathcal{C} ((2.7), (2.8), and (2.9)) and give several examples of situations in which $ca_{\mathcal{C}}$ -regular convergence naturally appears ((2.10), (2.12), and (2.14)).

The short §3 shows that taking finite products and suspensions preserves $ca_{\mathcal{C}}$ -regular convergence ((3.1) and (3.2),

respectively). Also, $A_k \text{---} \text{ca} \zeta \rightarrow A_0$ iff for every component C_0 of A_0 there is a component C_k of A_k such that $C_k \text{---} \text{ca} \zeta \rightarrow C_0$ ((3.4)).

The final §4 proves, relying heavily on Begle's technique in [1], that the hyperspace $\text{ca} \zeta(X)$ of all ζ -calm compacta in X with the topology induced by $\text{ca} \zeta$ -regular convergence is a metric space. We close by raising two questions about the topological properties of $\text{ca} \zeta(X)$.

This paper is the second in a series in which we study various types of globally regular convergence. In the first [7] we considered ζ -movably regular convergence.

We assume that the reader is familiar with the theory of shape [2].

Throughout the paper ζ and \mathcal{D} will be arbitrary (non-empty) classes of topological spaces. By \mathcal{P} we denote the class of all finite polyhedra.

If not stated otherwise, we reserve X for an arbitrary metric space with a fixed metric d ; A_0, A_1, A_2, \dots are compact subsets of X ; M is an absolute neighborhood retract for the class of all metric spaces (in notation, an ANR) which contains X ; a neighborhood means an open neighborhood; and $N_\varepsilon(A_0)$ denotes the ε -neighborhood of A_0 in M .

2. ζ -Calmly Regular Convergence

Let B be a subset of an ANR M , and let V be an open subset of M containing B . We denote by $\zeta_h(V; B)$ the following statement.

$\boxed{\zeta_h(V; B)}$ For every neighborhood W of B in M there is a neighborhood W_0 , $W_0 \subset V \cap W$, of B in M such that if

$f, g: K \rightarrow W_0$ are maps of a member K of \mathcal{C} into W_0 which are homotopic in V , then f and g are homotopic in W .

A compactum A is \mathcal{C} -*calm* if for some (and hence for every) embedding of A into an ANR M the following holds. There is a neighborhood V of A in M such that $\mathcal{C}_h(V; A)$ is true. It is easy to see that this definition is equivalent to the one given in [6] (see Theorem (4.2) in [6]). \mathcal{P} -calm compacta are called *calm*.

Recall [4] (see also [8]) the definition of the fundamental metric d_F on 2^X . Let M be an AR-space containing X metrically.

(2.1) If $A, B \in 2^X$, then $d_F(A, B)$ is the greatest lower bound of those $\varepsilon > 0$ for which there exist ε -fundamental sequences $\underline{f} = \{f_k, A, B\}_{M, M}$ and $\underline{g} = \{g_k, B, A\}_{M, M}$. (By an ε -fundamental sequence we mean a fundamental sequence $\underline{f} = \{f_k, A, B\}_{M, M}$ for which there exists a neighborhood U of A in M such that $f_k|_U$ is an ε -map (i.e., $d(f_k(x), x) < \varepsilon$ for every $x \in U$) for almost all indices k). It is known [4, Theorem 3.1] that the choice of M is irrelevant when computing $d_F(A, B)$. The hyperspace 2^X with the metric d_F is denoted by 2_F^X .

(2.2) *Definition.* A sequence $\{A_1, A_2, \dots\}$ of compacta in a metric space X which lies in an ANR M is said to *converge \mathcal{C} -calmly regularly* (or *ca \mathcal{C} -regularly*) in M to a compactum $A \subset X$ provided

- (i) $\lim_{d_F} (A_n, A) = 0$, and
- (ii) there is a neighborhood V of A in M such that $\mathcal{C}_h(V; A_n)$ holds for almost all indices n .

We first prove that the definition (2.2) is shape theoretic in the sense that \mathcal{C} -calmly regular convergence is independent of the choice of M and the embedding of X into M .

(2.3) *Proposition.* Let X be embedded in ANR's M and M' and let a sequence $\{A_n\}$ of compacta in X converge \mathcal{C} -calmly regularly in M to a compactum $A_0 \subset X$. Then the convergence $A_n \rightarrow A_0$ is also \mathcal{C} -calmly regular in M' .

Proof. Since $A = \bigcup_{i=0}^{\infty} A_i$ is compact, there are neighborhoods Z and Z' of A in M and M' , respectively, and maps $h: Z \rightarrow M'$ and $h': Z' \rightarrow M$ such that $h|_A = h'|_A = \text{id}$.

Let V be a neighborhood of A_0 in M and let k_V be an index such that $\mathcal{C}_h(V; A_n)$ holds for each $n \geq k_V$. Put $V' = (h')^{-1}(V)$ and let $k_{V'} \geq k_V$ be such that $n \geq k_{V'}$ implies $A_n \subset V'$.

We claim that $\mathcal{C}_h(V'; A_n)$ is true for each $n \geq k_{V'}$. Indeed, consider an arbitrary neighborhood W' of A_n in M' , where $n \geq k_{V'}$. Let $W = h^{-1}(W')$. Since $n \geq k_V$ we conclude that there is a neighborhood W_0 of A_n in M , $W_0 \subset V \cap W$, such that if two maps of a member of \mathcal{C} into W_0 are homotopic in V then they are already homotopic in W . Let $W_0^* = (h')^{-1}(W_0)$ and let $W'_0, W'_0 \subset W_0^* \cap W'$, be a neighborhood of A_n in M' with the property that $h \circ h'|_{W'_0} \approx i_{W'_0, W'}$ in W' ($i_{W'_0, W'}$ denotes the inclusion of W'_0 into W' and " \approx " stands for "is homotopic to").

Now, if $f, g: K \rightarrow W'_0$ are maps of $K \in \mathcal{C}$ into W'_0 homotopic in V' , then $h' \circ f$ and $h' \circ g$ are maps of K into W_0 homotopic in V . It follows that $h' \circ f \approx h' \circ g$ in W . But then $h \circ h' \circ f \approx h \circ h' \circ g$ in W' . The choice of W'_0 implies that $h \circ h' \circ f \approx f$ and

$h \circ h' \circ g \approx g$ in W' . Hence $f \approx g$ in W' .

If a sequence $\{A_n\}$ of compacta in X converges $\text{ca}\mathcal{C}$ -regularly to a compactum A_0 in X in some (and hence, by (2.3), in every) ANR containing X we shall write $A_n \text{---}\text{ca}\mathcal{C} \rightarrow A_0$.

(2.4) *Proposition.* Let $A_n \text{---}\text{ca}\mathcal{C} \rightarrow A_0$. Then A_0 is \mathcal{C} -calm regardless of the nature of A_n 's.

Proof. We can assume that all compacta under consideration lie in the Hilbert cube Q . Select a neighborhood V of A_0 in Q and an index $k_1 = k_V$ such that $\mathcal{C}_h(V; A_n)$ holds for each $n \geq k_1$. We shall prove that $\mathcal{C}_h(V; A_0)$ is also true.

Let W be an arbitrary neighborhood of A_0 in Q . Let W' , $W' \subset W$, be a compact ANR neighborhood of A_0 . Pick $k_2 \geq k_1$ such that $n \geq k_2$ implies $A_n \subset \text{int}W'$. Let $\varepsilon > 0$ has the property that ε -close maps into W' are homotopic in W' [10]. Take $k_3 \geq k_2$ such that $d_F(A_{k_3}, A_0) < \varepsilon$. By assumption, there is a neighborhood W'_0 of A_{k_3} , $W'_0 \subset V \cap W'$, such that any two maps of a member of \mathcal{C} into W'_0 homotopic in V are already homotopic in W' .

Now, take an ε -fundamental sequence $\underline{f} = \{f_k, A_0, A_{k_3}\}_{Q, Q}$. We can find an index n_0 and a neighborhood W_0 of A_0 in Q , $W_0 \subset V \cap W'$, such that $d(f_{n_0}(x), x) < \varepsilon$ for each $x \in W_0$ and $f_{n_0}(W_0) \subset W'_0$.

Consider maps $f, g: K \rightarrow W_0$ of $K \in \mathcal{C}$ into W_0 that are homotopic in V . Then $f_{n_0} \circ f \approx f$ in W' (because these are ε -close maps into W') and, similarly, $f_{n_0} \circ g \approx g$ in W' . It follows that $f_{n_0} \circ f$ and $f_{n_0} \circ g$ are maps of K into W'_0 homotopic in V . By the choice of W'_0 and k_3 , we see that they are homotopic in W' . But then f and g are homotopic in W' and therefore also in W .

Let $/C_A/$ denote the number of components of a compactum A .

(2.5) *Proposition.* Let $A_n \xrightarrow{\text{ca}} C A_0$. Then A_0 has finitely many components and $/C_{A_0}/ = /C_{A_n}/$ for almost all indices n .

Proof. The first claim follows from (2.4) and Theorem (4.6) in [6]. The second is proved by the method used in the above proof.

(2.6) *Example.* (a) A constant sequence $\{A_n\} = \{A\}$ converges ca_C -regularly to A iff A is C -calm.

(b) In the interval $X = [-1, 1]$ consider the sequence $A_n = \{-(1/n)\} \cup \{1/n\}$ ($n = 1, 2, \dots$) of compacta. Then $\lim_F (A_n, A_0) = 0$, where $A_0 = \{0\}$ is C -calm for every class C , but $\{A_n\}$ does not converge ca_C -regularly to A_0 . Hence, the converse of (2.4) is not true.

Now we shall discuss the role of a class C in our definition. With obvious changes the proof of Theorem (4.8) in [6] gives the following.

(2.7) *Theorem.* Let $A_n \xrightarrow{\text{ca}} C A_0$ and let a class C shape dominate a class D . Then $A_n \xrightarrow{\text{ca}} D A_0$.

Here a class of topological spaces C shape dominates another such class D provided for every $X \in D$ there is $Y \in C$ such that Y shape dominates X . We use the shape theory of arbitrary topological spaces in the form described by Kozłowski [12] (see also §3 in [13]).

Similarly, when C and D are classes of compacta, by replacing in the above definition shape domination with

Borsuk's notion of quasi-domination [3], we say that \mathcal{C} quasi-dominates \mathcal{D} . With this concept we can improve (2.7) (and by (2.6) (a) also (4.8) in [6]) for classes of compacta.

(2.8) *Theorem.* Let $A_n \text{---} \text{ca} \mathcal{C} \rightarrow A_0$ and let a class of compacta \mathcal{C} quasi-dominate another such class \mathcal{D} . Then $A_n \text{---} \text{ca} \mathcal{D} \rightarrow A_0$.

Proof. Let V be a neighborhood of A_0 in M and let k_V be an index such that $\mathcal{C}_h(V; A_n)$ holds for each $n \geq k_V$. We claim that for such indices $\mathcal{D}_h(V; A_n)$ also holds.

Indeed, let $n \geq k_V$ and let W be an arbitrary neighborhood of A_n in M . Select a neighborhood W_0 of A_n in M , $W_0 \subset V \cap W$, using $\mathcal{C}_h(V; A_n)$. Consider $K \in \mathcal{D}$ and maps $f, g: K \rightarrow W_0$ homotopic via a homotopy $H_t: K \rightarrow V$ ($0 \leq t \leq 1$). Since the class \mathcal{C} quasi-dominates the class \mathcal{D} , there is a compactum $L \in \mathcal{C}$ which quasi-dominates K . We can assume that both K and L lie in the Hilbert cube Q . Since V and W_0 are ANR's, there is a neighborhood Z of K in Q and extensions $\hat{f}, \hat{g}: Z \rightarrow W_0$ of f and g , respectively, and $\hat{H}_t: Z \rightarrow V$ of H_t such that $\hat{H}_0 = \hat{f}$ and $\hat{H}_1 = \hat{g}$ [10]. Now, we select a neighborhood Z_0 of K in Q , $Z_0 \subset Z$, fundamental sequences $\underline{f} = \{f_k, L, K\}_{Q, Q}$ and $\underline{g} = \{g_k, K, L\}_{Q, Q}$, and an index k_1 such that $k \geq k_1$ implies $f_k \circ g_k|_{Z_0} \approx i_{Z_0, Z}$ in Z . Next, we pick $k_2 \geq k_1$ and a neighborhood T of L in Q such that $f_k|_T \approx f_k|_T$ in Z whenever $k, k' \geq k_2$.

Note that $\hat{H}_t \circ f_{k_2}|_L$ is a homotopy of V connecting maps $\hat{f} \circ f_{k_2}|_L$ and $\hat{g} \circ f_{k_2}|_L$ of L into W_0 . By assumption, these maps are homotopic in W . As before, we conclude that there is a neighborhood T_1 of L in Q , $T_1 \subset T$, and a homotopy $\hat{G}_t: T_1 \rightarrow W$

($0 \leq t \leq 1$) such that $\hat{G}_0 = \hat{f} \circ f_{k_2}|_{T_1}$ and $\hat{G}_1 = \hat{g} \circ f_{k_2}|_{T_1}$.

Select $k_3 \geq k_2$ and a neighborhood T_0 of K in Q , $T_0 \subset Z_0$, such that $g_{k_3}(T_0) \subset T_1$. Then $G_t = \hat{G}_t \circ g_{k_3}|_K$ is a homotopy in W connecting $G_0 = \hat{f} \circ f_{k_2} \circ g_{k_3}|_K$ and $G_1 = \hat{g} \circ f_{k_2} \circ g_{k_3}|_K$. We leave to the reader to check that G_0 is in W homotopic to f and that G_1 is in W homotopic to g . Hence f and g are homotopic in W .

(2.9) *Proposition.* Let $A_n \text{---} \text{ca} \mathcal{C} \rightarrow A_0$. If each compactum A_n ($n = 1, 2, \dots$) is $(\mathcal{C}, \bar{\mathcal{D}})$ -smooth [6], then $A_n \text{---} \text{ca} \mathcal{C} \rightarrow A_0$.

Proof. Let a neighborhood V of A_0 in M and an index k_V be chosen using the fact that $A_n \text{---} \text{ca} \mathcal{C} \rightarrow A_0$. We claim that $\mathcal{C}_h(V; A_n)$ holds for every $n \geq k_V$.

Let W be an arbitrary neighborhood of A_n in M ($n \geq k_V$). Since A_n is $(\mathcal{C}, \bar{\mathcal{D}})$ -smooth, there is a neighborhood W' of A_n in M , $W' \subset W$, such that every two $\bar{\mathcal{D}}$ -homotopic [6] maps of a member of \mathcal{C} into W' are homotopic in W . The required neighborhood W_0 is picked with respect to W' using $\bar{\mathcal{D}}_h(V; A_n)$.

Consider maps $f, g: K \rightarrow W_0$ of $K \in \mathcal{C}$ and assume that they are homotopic in V . Then for every $L \in \bar{\mathcal{D}}$ and every map $h: L \rightarrow K$, compositions $f \circ h$ and $g \circ h$ are homotopic in V . The choice of W_0 implies $f \circ h \simeq g \circ h$ in W' . In other words, f and g are $\bar{\mathcal{D}}$ -homotopic in W' . Hence, $f \simeq g$ in W .

We shall give now three examples of situations in which $\text{ca} \mathcal{C}$ -regular convergence appears naturally.

(2.10) *Example.* Let $\lim_F(A_n, A_0) = 0$ and assume that each compactum A_n ($n = 1, 2, \dots$) is \mathcal{C} -trivial [8] and connected. Then $A_n \text{---} \text{ca} \mathcal{C} \rightarrow A_0$.

We observed in the example (2.6) (b) that the convergence with respect to the fundamental metric to a \mathcal{C} -calm compactum is not sufficient for the $\text{ca}\mathcal{C}$ -regular convergence. We shall now introduce a stronger metric d_{SF} on a subset $X[A]$ of 2^X consisting of all compacta in X with the same shape as a compactum A and prove that if A is \mathcal{C} -calm then the convergence with respect d_{SF} implies $\text{ca}\mathcal{C}$ -regular convergence.

(2.11) *Definition.* If $B, C \in X[A]$, then $d_{\text{SF}}(B, C)$ is the greatest lower bound of those $\varepsilon > 0$ for which there exist ε -fundamental sequences $\underline{f} = \{f_k, B, C\}_{M, M}$ and $\underline{g} = \{g_k, C, B\}_{M, M}$ such that $\underline{g} \circ \underline{f} \approx \underline{\text{id}}_B$ and $\underline{f} \circ \underline{g} \approx \underline{\text{id}}_C$.

One easily proves that this is indeed a metric and that the computation of $d_{\text{SF}}(B, C)$ is independent of the choice of the AR-space M containing X .

(2.12) *Example.* Let A_0, A_1, A_2, \dots be elements of $X[A]$ and assume that $\lim_{\text{SF}} (A_n, A_0) = 0$. If A is a \mathcal{C} -calm compactum, then $A_n \text{---}\text{ca}\mathcal{C} \rightarrow A_0$.

Proof. Without loss of generality we can assume that X lies in the Hilbert cube Q . Let V be a compact ANR neighborhood of A_0 in Q such that $\mathcal{C}_h(V; A_0)$ holds. Select $\varepsilon > 0$ with the property that ε -close maps into V are homotopic in V . Pick an index k_V such that $n \geq k_V$ implies $A_n \subset \text{int} V$ and $d_{\text{SF}}(A_n, A_0) < \varepsilon$. Hence for each $n \geq k_V$ there are ε -fundamental sequences $\underline{f}^n = \{f_k^n, A_0, A_n\}_{Q, Q}$ and $\underline{g}^n = \{g_k^n, A_n, A_0\}_{Q, Q}$, neighborhoods U_0^n and V_0^n of A_0 and A_n , respectively, and an index k_n such that $k \geq k_n$ implies that $f_k^n|_{U_0^n}$ and $g_k^n|_{V_0^n}$ are ε -maps and $\underline{f}^m \circ \underline{g}^n \approx \underline{\text{id}}_{A_m}$.

Consider an arbitrary neighborhood W of A_n ($n \geq k_V$).

Pick $k_W \geq k_n$ and a neighborhood W' of A_0 such that $f_k^n(W') \subset W$ for all $k \geq k_W$. Now, select a neighborhood W'_0 of A_0 with respect to W' using $\mathcal{C}_h(V; A_0)$. Then we take a required neighborhood W_0 of A_n inside $W \cap V_0^n \cap V$ and an index $k_0 \geq k_W$ for which $g_k^n(W_0) \subset W'_0$ and $f_k^n \circ g_k^n|_{W_0} \approx i_{W_0, W}$ in W whenever $k \geq k_0$.

Let $f, g: K \rightarrow W_0$ be maps of $K \in \mathcal{C}$ into W_0 and assume that they are homotopic in V . Observe that $g_{k_0}^n \circ f \approx f$ in V (these are ε -close maps into V) and, similarly, $g_{k_0}^n \circ g \approx g$ in V . Hence, $g_{k_0}^n \circ f, g_{k_0}^n \circ g: K \rightarrow W'_0$ are homotopic in V . The choice of W'_0 implies that $g_{k_0}^n \circ f \approx g_{k_0}^n \circ g$ in W' . But then $f_{k_0}^n \circ g_{k_0}^n \circ f \approx f_{k_0}^n \circ g_{k_0}^n \circ g$ in W . Finally, since $f_{k_0}^n \circ g_{k_0}^n \circ g \approx f$ in W and $f_{k_0}^n \circ g_{k_0}^n \circ g \approx g$ also in W , it follows that $f \approx g$ in W .

Examples of sequences converging with respect to d_{SF} are provided by sequences of n -dimensional ANR's converging homotopy n -regularly [9]. In fact, one easily checks that the proof of Theorem (4.2) in [9] contains the proof of the following statement.

(2.13) Let A_0, A_1, A_2, \dots be n -dimensional compact ANR's in a metric space X . If $\{A_n\}$ converges homotopy n -regularly to A_0 , then there is a subsequence $\{A_k\}$ of $\{A_n\}$ such that $\lim_{SF} (A_k, A_0) = 0$.

(2.14) *Example.* Under the assumptions of (2.13) we see from (2.12) that $A_n - \text{ca}\mathcal{C} \rightarrow A_0$, for every class \mathcal{C} .

3. Operations Preserving $\text{ca}\mathcal{C}$ -Regular Convergence

In this section we shall prove that by taking finite products and suspensions of $\text{ca}\mathcal{C}$ -regularly converging sequences of compacta we get $\text{ca}\mathcal{C}$ -regularly converging sequences. We also investigate in what way $\text{ca}\mathcal{C}$ -regular convergence of

components of the members of the sequence $\{A_n\}$ to components of A_0 imply that $A_n \text{---} \text{ca}\zeta \rightarrow A_0$.

(3.1) *Theorem.* If for each $i = 1, \dots, m$, $\{A_n^i\}$ is a sequence of compacta in a metric space X_i converging $\text{ca}\zeta$ -regularly to a compactum A_0^i in X_i , then $A_n = \prod_{i=1}^m A_n^i \text{---} \text{ca}\zeta \rightarrow A_0 = \prod_{i=1}^m A_0^i$.

Proof. We can assume that each X_i lies in an ANR space M_i . Then $X = \prod_{i=1}^m X_i$ lies in the ANR space $M = \prod_{i=1}^m M_i$. It suffices to prove that $\{A_n\}$ converges $\text{ca}\zeta$ -regularly to A_0 in M .

For each $i = 1, \dots, m$ pick a neighborhood V_i of A_0^i in M_i and an index k_i such that $\zeta_h(V_i; A_k^i)$ holds for all $k \geq k_i$. Let $V = \prod_{i=1}^m V_i$ and let $k_V = \max\{k_1, \dots, k_m\}$. We claim that $\zeta_h(V; A_k)$ is true for each $k \geq k_V$.

Indeed, let $k \geq k_V$ and consider an arbitrary neighborhood W of A_k in M . We can find neighborhoods W_1, \dots, W_m of A_k^1, \dots, A_k^m in M_1, \dots, M_m , respectively, such that $W' = \prod_{i=1}^m W_i \subset W$. For each $i = 1, \dots, m$ inside $V_i \cap W_i$ select a neighborhood W_{i0} of A_k^i using the property of V_i and k_i . Put $W_0 = \prod_{i=1}^m W_{i0}$.

If $f, g: K \rightarrow W_0$ are maps of $K \in \zeta$ into W_0 that are homotopic in V , then the compositions $\pi_i \circ f, \pi_i \circ g: K \rightarrow W_{i0}$ (where π_i is the projection of M onto M_i) are homotopic in V_i ($i = 1, \dots, m$). It follows that they are homotopic in W_i . Hence, f and g are homotopic in W' and therefore also in W .

(3.2) *Theorem.* If $A_n \text{---} \text{ca}\zeta \rightarrow A_0$, then the sequence $\{SA_n\}$ of the (unreduced) suspensions of A_n ($n = 1, 2, \dots$) converges $\text{ca}\zeta$ -regularly to SA_0 .

Proof. We can assume that $A = \bigcup_{i=0}^{\infty} A_i$ lies in a compact convex infinite-dimensional subset M of a Banach space N_0 . Setting $N = N_0 \times \mathbb{R}$ (\mathbb{R} denotes the real line), let us identify every point $y \in N_0$ with the point $(y, 0) \in N$. We select a point $c \in A$ and define the suspension SA_i as the union of all segments $|ay|$ and $|a'y|$, where $a = (c, 1)$, $a' = (c, -1)$, and $y \in A_i$ ($i = 0, 1, 2, \dots$).

We shall show that $\{SA_n\} \rightarrow SA_0$ caC -regularly in SM . Observe that both M and SM are homeomorphic to Q [11].

Since $\{A_n\} \rightarrow A_0$ caC -regularly in M , there is a compact ANR neighborhood \tilde{V} of A_0 in M and an index $k_{\tilde{V}}$ such that for every $n \geq k_{\tilde{V}}$ the statement $\mathcal{C}_h(\tilde{V}; A_n)$ holds. Let $V = SM[-1, -(1/2)] \cup S\tilde{V} \cup SM[1/2, 1]$, where $SM[\alpha, \beta]$ denotes all points of SM with the second coordinate in the interval $[\alpha, \beta]$, $-1 \leq \alpha \leq \beta \leq 1$, and let $k_V = k_{\tilde{V}}$. We claim that $\mathcal{C}_h(V; SA_n)$ is true for all $n \geq k_V$.

Indeed, let $n \geq k_V$ and let W be an arbitrary neighborhood of SA_n in SM . Select a neighborhood \tilde{W} of A_n in M and an ϵ , $0 < \epsilon < 1/2$, such that $SM[-1, -1+\epsilon] \cup S\tilde{W} \cup SM[1-\epsilon, 1] \subset W$. Now take a compact ANR neighborhood \tilde{W}_0 , $\tilde{W}_0 \subset \tilde{W} \cap \tilde{V}$, of A_n in M using the choice of \tilde{V} and $k_{\tilde{V}}$ and put $W_0 = SM[-1, -1+\epsilon] \cup S\tilde{W}_0 \cup SM[1-\epsilon, 1]$.

Let $C_-(D_-)$ be the set obtained as the union of all segments $|b'y|$ where $b' = (c, -1+(\epsilon/2))$ and $y \in \tilde{S}\tilde{W}_0[-1+\epsilon]$ ($y \in SV[-1/2]$), and let $C_+(D_+)$ be the set obtained as the union of all segments $|by|$ where $b = (c, 1-(\epsilon/2))$ and $y \in \tilde{S}\tilde{W}_0[1-\epsilon]$ ($y \in SV[1/2]$).

Observe that $(SM[-1, -1+\epsilon], C_-)$, $(SM[1-\epsilon, 1], C_+)$, $(SM[-1, -1/2], D_-)$ and $(SM[1/2, 1], D_+)$ are pairs of AR's.

Hence in each of these pairs there is a strong deformation retraction of the first set onto the second set [10].

Consider maps $f, g: K \rightarrow W_0$ of $K \in \mathcal{C}$ into W_0 and assume that they are homotopic in V . Applying strong deformation retractions determined by the first two pairs, we see that f and g are homotopic in W_0 to maps f' and g' , respectively, of K into $S\tilde{W}_0[-1+\epsilon, 1-\epsilon] \cup C_- \cup C_+$. By applying strong deformation retractions determined by the last two pairs, we see that f' and g' are homotopic in $S\tilde{V}[-1/2, 1/2] \cup D_- \cup D_+$.

Since both $S\tilde{V}[-1/2, 1/2] \cup D_- \cup D_+$ and $S\tilde{W}_0[-1+\epsilon, 1-\epsilon] \cup C_- \cup C_+$ are contained in $S\tilde{V}[-1+(\epsilon/2), 1-(\epsilon/2)]$ it follows (after projecting onto \tilde{V}) that f' and g' are homotopic in $S\tilde{W}[-1+(\epsilon/2), 1-(\epsilon/2)]$. Hence, f and g are homotopic in W .

(3.3) *Corollary.* The (unreduced) suspension of a \mathcal{C} -calm compactum is also \mathcal{C} -calm.

The proof of the following proposition is left to the reader.

(3.4) *Proposition.* If $A_n \text{---} \text{ca}\mathcal{C} \rightarrow A_0$, then for every component C_0 of A_0 there is a component C_n of A_n such that $C_n \text{---} \text{ca}\mathcal{C} \rightarrow C_0$. Conversely, let \mathcal{C} be a component hereditary class of topological spaces and let every compactum A_n ($n = 0, 1, 2, \dots$) has precisely k ($k < \infty$) components C_n^1, \dots, C_n^k such that $C_n^i \text{---} \text{ca}\mathcal{C} \rightarrow C_0^i$, $1 \leq i \leq k$. Then $A_n \text{---} \text{ca}\mathcal{C} \rightarrow A_0$.

4. The Metric of $\text{ca}\mathcal{C}$ -Regular Convergence

The collection of all \mathcal{C} -calm compacta in a metric space X can be made into a hyperspace $\text{ca}\mathcal{C}(X)$ by defining the notion of convergence by means of \mathcal{C} -calmly regular convergence. In

this section, using Begle's method in [1], we shall define the metric d_{ca} on the space $caC(X)$ in such a way that $\lim_{ca} (A_n, A_0) = 0$ iff $A_n \xrightarrow{caC} A_0$.

In fact, it is clear from the explanation on the page 444 in [1] that such a metric can be introduced provided we can prove the analogues of Lemmas 1, 3, 4, and 5 in [1] for the function $\gamma_C(\epsilon, A)$ (defined in (4.1)) corresponding to Begle's function $\delta_n(\epsilon, P)$. The analogue of Lemma 4 was established in (2.4) while Lemmas (4.2), (4.3), and (4.5) below correspond to Lemmas 1, 3, and 5, respectively.

Throughout this section we assume that X lies in an AR space M of diameter 1.

(4.1) *Definition.* For each compact subset A of M and for each $\epsilon > 0$, let $\gamma_C(\epsilon, A)$ be the least upper bound of all numbers γ , $0 \leq \gamma \leq \epsilon$, such that $C_h(N_\gamma(A); A)$ holds.

It is clear that for each compactum A in M , $\gamma_C(\epsilon, A)$ always exists and is a non-negative monotone non-decreasing, and hence measurable, function on the half-open interval $I^* = (0, 1]$. If A is C -calm, then $\gamma_C(\epsilon, A) > 0$ everywhere in I^* and conversely.

The relation between definitions (2.2) and (4.1) is provided by the following.

(4.2) *Lemma.* If $\lim_F (A_n, A_0) = 0$, then the sequence $\{A_n\}$ converges caC -regularly to A_0 iff $\liminf \gamma_C(\epsilon, A) > 0$ for each ϵ in I^* .

Proof. The proof is similar to the proof of Lemma (4.2) in [7].

The following two lemmas resemble Lemmas (4.3) and (4.4) in [7]. The proofs are similar in spirit but technically more complicated.

(4.3) *Lemma.* Let $\liminf_F(A_n, A_0) = 0$. Then $\limsup \gamma_C(\varepsilon, A_n) \geq \gamma_C(\varepsilon, A_0)$ for all but countably many points ε in I^* .

Proof. We shall prove that if $\limsup \gamma_C(\varepsilon_0, A_n) > \gamma_C(\varepsilon_0, A_0)$ at the point $\varepsilon_0 \in I^*$, then the function $\gamma_C(\varepsilon, A_0)$ has a jump at the point ε_0 (in particular, it is not continuous at ε_0). Since $\gamma_C(\varepsilon, A_0)$ is a monotone function, there are at most countably many points of I^* at which this can happen.

Take an $e > 0$ and a subsequence $\{A_{n_i}\}$ of $\{A_n\}$ such that $\gamma_C(\varepsilon_0, A_{n_i}) \geq \gamma_C(\varepsilon_0, A_0) + e$ for all $i > 0$. Let b , $0 < b < e/2$, be an arbitrary number and choose an index i_0 so that $i \geq i_0$ implies $d_F(A_{n_i}, A_0) < b$. Observe that $N_{\gamma_C(\varepsilon_0, A_0) + (e/2)}(A_0) \subset N_{\gamma_C(\varepsilon_0, A_{n_i})}(A_{n_i})$. We claim that $C_h(N_{\gamma_C(\varepsilon_0, A_0) + (e/8)}(A_0); A_0)$ holds. Since $N_{\gamma_C(\varepsilon_0, A_{n_i})}(A_{n_i}) \subset N_{\varepsilon_0 + b}(A_0)$, this would imply that for every b , $0 < b < e/8$, $\gamma_C(\varepsilon_0 + b, A_0) \geq \gamma_C(\varepsilon_0, A_0) + (e/8)$, i.e., that the function $\gamma_C(\varepsilon, A_0)$ has a jump at least $e/8$ at the point ε_0 .

Let W be an arbitrary compact ANR neighborhood of A_0 in M (which we can assume is the Hilbert cube Q). Pick ξ , $0 < \xi < b/4$, such that 2ξ -close maps into W are homotopic in W and ξ -close maps into $N_{\gamma_C(\varepsilon_0, A_0) + (e/4)}(A_0)$ are homotopic in $N_{\gamma_C(\varepsilon_0, A_0) + (e/2)}(A_0)$. Select $j \geq i_0$ such that $d_F(A_{n_j}, A_0) < \xi$. Let $\underline{f} = \{f_k, A_{n_j}, A_0\}_{M, M}$ and $\underline{g} = \{g_k, A_0, A_{n_j}\}_{M, M}$ be ξ -fundamental sequences. Let Z be a neighborhood of A_0 and let $k_{\underline{g}}$ be an index such that $k \geq k_{\underline{g}}$ and $x \in Z$ implies $d(g_k(x), x) < \xi$.

Similarly, let Z' be a neighborhood of A_{n_j} and let $k_{\underline{f}}$ be an index such that $k \geq k_{\underline{f}}$ and $x \in Z'$ implies $d(f_k(x), x) < \xi$.

Now we pick a neighborhood W' of A_{n_j} and $k_{W'} \geq k_{\underline{f}}, k_{\underline{g}}$ such that $f_k(W') \subset W$ for all $k \geq k_{W'}$. Inside W' let us select a neighborhood W'_0 of A_{n_j} with the property that maps of $K \in \mathcal{C}$ into W'_0 homotopic in $N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_{n_j})}(A_{n_j})$ are already homotopic in W' . Finally, the required neighborhood W_0 of A_0 , $W_0 \subset Z \cap W \cap N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_0) + (e/8)}(A_0)$ and the index $k_0 \geq k_{W'}$ are picked so that $g_{k_0}(W_0) \subset Z' \cap W'_0$.

Consider maps $f, g: K \rightarrow W_0$ defined on a member K of \mathcal{C} and suppose that they are homotopic in $N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_0) + (e/8)}(A_0)$. Since $(g_{k_0} \circ f, f)$ and $(g_{k_0} \circ g, g)$ are two pairs of ξ -close maps into $N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_0) + (e/4)}(A_0)$, it follows that $g_{k_0} \circ f$ and $g_{k_0} \circ g$ are homotopic in $N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_{n_j})}(A_{n_j})$. The choice of W'_0 implies that these maps are already homotopic in W' . But, then $f_{k_0} \circ g_{k_0} \circ f$ and $f_{k_0} \circ g_{k_0} \circ g$ are homotopic in W . We conclude that f and g are homotopic in W because $f_{k_0} \circ g_{k_0} \circ f \approx f$ in W and $f_{k_0} \circ g_{k_0} \circ g \approx g$ in W .

(4.4) *Lemma.* Let $A_n \xrightarrow{\text{ca}} A_0$. Then $\gamma_{\mathcal{C}}(\varepsilon_0, A_0) \leq \liminf \gamma_{\mathcal{C}}(\varepsilon_0, A_n)$ at every point $\varepsilon_0 \in I^*$ in which the function $\gamma_{\mathcal{C}}(\varepsilon, A_0)$ is continuous.

Proof. Let us consider a point $\varepsilon_0 \in I^*$ at which the function $\gamma_{\mathcal{C}}(\varepsilon, A_0)$ is continuous. Suppose that $\gamma_{\mathcal{C}}(\varepsilon_0, A_0) > \liminf \gamma_{\mathcal{C}}(\varepsilon_0, A_n)$. Then there is an e , $0 < e < \varepsilon_0$, and a subsequence $\{A_{n_i}\}$ of $\{A_n\}$ such that $\gamma_{\mathcal{C}}(\varepsilon_0, A_{n_i}) + 2e < \gamma_{\mathcal{C}}(\varepsilon_0, A_0) - 2e$, for all $i > 0$. Since the function $\gamma_{\mathcal{C}}(\varepsilon, A_0)$ is continuous at ε_0 , there is a number d , $0 < d < e$, such that $\gamma_{\mathcal{C}}(\varepsilon, A_0) \in$

$(\gamma_{\mathcal{C}}(\varepsilon_0, A_0) - 2e, \gamma_{\mathcal{C}}(\varepsilon_0, A_0) + 2e)$ for all $\varepsilon \in (\varepsilon_0 - 2d, \varepsilon_0 + 2d)$. In particular, $\gamma_{\mathcal{C}}(\varepsilon_0 - d, A_0) > \gamma_{\mathcal{C}}(\varepsilon_0, A_0) - 2e > \gamma_{\mathcal{C}}(\varepsilon_0, A_{n_1}) + 2e$.

We claim that there is an index k such that

$N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_{n_k}) + e(A_{n_k})} \subset N_{\varepsilon_0}(A_{n_k})$ and such that

$\mathcal{C}_h(N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_{n_k}) + e(A_{n_k})}; A_{n_k})$ holds. This would imply that

$\gamma_{\mathcal{C}}(\varepsilon_0, A_{n_k}) \geq \gamma_{\mathcal{C}}(\varepsilon_0, A_{n_k}) + e$ which is an obvious contradiction.

By using the fact that $A_n \xrightarrow{\text{ca}} A_0$, inside

$N_{\gamma_{\mathcal{C}}(\varepsilon_0 - d, A_0)}(A_0)$ we pick a compact ANR neighborhood V of A_0

and an index k_V so that $\mathcal{C}_h(V; A_i)$ is true for all $i > k_V$.

Let ξ , $0 < \xi < d$, has the property that ξ -close maps into V

are homotopic in V and that $N_{\xi}(A_0) \subset V$. Pick an integer k

so large that $n_k \geq k_V$ and $d_F(A_{n_k}, A_0) < \xi$. Let $\underline{f} =$

$\{f_k, A_{n_k}, A_0\}_{M, M}$ be an ξ -fundamental sequence, let Z' be a

neighborhood of A_{n_k} , and let n_Z , be such that $d(f_n(x), x) < \xi$

for all $x \in Z'$ and $n \geq n_Z$.

Now we pick a neighborhood W_0^* , $W_0^* \subset V$, of A_0 in M with the property that maps of $K \in \mathcal{C}$ into W_0^* which are homotopic in $N_{\gamma_{\mathcal{C}}(\varepsilon_0 - d, A_0)}(A_0)$ are already homotopic in V .

Consider now an arbitrary neighborhood W' of A_{n_k} . Let a neighborhood \tilde{W}_0' of A_{n_k} be selected with respect to W' and V using $\mathcal{C}_h(V; A_{n_k})$. Finally, let a neighborhood W_0' , $W_0' \subset \tilde{W}_0' \cap Z'$, of A_{n_k} and $n_0 \geq n_Z$, be such that $f_{n_0}(W_0') \subset W_0$.

Suppose $f, g: K \rightarrow W_0'$ are maps of a member K of \mathcal{C} homotopic in $N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_{n_k}) + e(A_{n_k})}$. Observe that $N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_{n_k}) + e(A_{n_k})} \subset N_{\gamma_{\mathcal{C}}(\varepsilon_0 - d, A_0)}(A_0)$ and that $f' = f_{n_0} \circ f$ and f and $g' = f_{n_0} \circ g$ are two pairs of ξ -close maps into V . It follows that $f': K \rightarrow W_0^*$ and $g': K \rightarrow W_0^*$ are homotopic in $N_{\gamma_{\mathcal{C}}(\varepsilon_0 - d, A_0)}(A_0)$.

The choice of W_0^* gives that f' and g' are homotopic in V . But then f and g are homotopic in V so that the choice of V and k_V implies that f and g are homotopic in W' and this is what we wanted to prove.

Combining the last two lemmas we have the following theorem.

(4.5) *Theorem.* If $A_n \xrightarrow{\text{ca}} A_0$, then $\lim \gamma_{\mathcal{C}}(\epsilon, A_n)$ exists and equals $\gamma_{\mathcal{C}}(\epsilon, A_0)$ almost everywhere in I^* .

We are now ready to introduce the metric d_{ca} on the hyperspace $\text{ca}_{\mathcal{C}}(X)$ of all \mathcal{C} -calm compacta in a metric space X . Let E be the Banach space of all bounded measurable functions on the interval I^* , the norm of an element f in E being defined as:

$$\|f\| = \int_0^1 |f| d\epsilon.$$

We define a correspondence between $\text{ca}_{\mathcal{C}}(X)$ and a subset of $2_F^X \times E$ by assigning to each element A of $\text{ca}_{\mathcal{C}}(X)$ the element $(A, \gamma_{\mathcal{C}}(\epsilon, A))$ of $2_F^X \times E$. This correspondence is one-to-one, so a metric is defined in $\text{ca}_{\mathcal{C}}(X)$ by letting the distance between two points in $\text{ca}_{\mathcal{C}}(X)$ be the distance between the corresponding points in $2_F^X \times E$. Specifically,

$$d_{\text{ca}}(A, B) = [d_F^2(A, B) + (\int_0^1 |\gamma_{\mathcal{C}}(\epsilon, A) - \gamma_{\mathcal{C}}(\epsilon, B)| d\epsilon)^2]^{1/2}.$$

With obvious modifications the argument on the page 444 in [1] shows that this metric induces the same topology on $\text{ca}_{\mathcal{C}}(X)$ as that naturally defined in terms of \mathcal{C} -calmly regular convergence. Hence, one can prove the following.

(4.6) *Theorem.* There is a metric d_{ca} on the hyperspace $\text{ca}_{\mathcal{C}}(X)$ of all \mathcal{C} -calm compacta in a metric space X such that

for a sequence A_0, A_1, A_2, \dots in $\text{ca}\zeta(X)$ $\lim_{\text{ca}} d_{\text{ca}}(A_n, A_0) = 0$ iff $A_n \xrightarrow{\text{ca}\zeta} A_0$.

At present we can state only the following three corollaries that describe topological properties of the metric space $(\text{ca}\zeta(X), d_{\text{ca}})$.

(4.7) *Corollary.* If X is homeomorphic to Y , then $\text{ca}\zeta(X)$ is homeomorphic to $\text{ca}\zeta(Y)$.

Proof. The proof is similar to the proof of (4.7) in [7].

(4.8) *Corollary.* The identity map $\text{id}: (\text{ca}\zeta(X), d_{\text{ca}}) \rightarrow (\text{ca}\zeta(X), d_F)$ is continuous.

Proof. See (2.1).

(4.9) *Corollary.* Let α be a hereditary shape property. Then the collection of all elements of $\text{ca}_\alpha(X)$ which have property α constitute a closed subset of $\text{ca}_\alpha(X)$.

Proof. See (4.4) in [8].

We leave many questions concerning the topological structure of the space $\text{ca}\zeta(X)$ open. The most natural problem would be to see what properties of X are carried over onto $\text{ca}\zeta(X)$. In particular, is $\text{ca}\zeta(X)$ separable (topologically complete) if X is separable (topologically complete)? The last two questions are in view of (4.8) and the method of the proofs for Theorems 2 and 3 in [1] equivalent to the following questions.

(4.10) If X is a separable (topologically complete) metric space, is $(\text{ca}\zeta(X), d_F)$ also separable (topologically

complete) metric space?

References

1. E. G. Begle, *Regular convergence*, Duke Math. J. 11 (1944), 441-450.
2. K. Borsuk, *Theory of shape*, Monografie Matematyczne 59, Warsaw, 1975.
3. _____, *Some quantitative properties of shapes*, Fund. Math. 93(1976), 197-212.
4. _____, *On a metrization of the hyperspace of a metric space*, Fund. Math. 94 (1977), 191-207.
5. L. Boxer and R. B. Sher, *Borsuk's fundamental metric and shape domination*, Bull. Acad. Polon. Sci. 26 (1978), 849-853.
6. Z. Čerin, *Homotopy properties of locally compact spaces at infinity--calmness and smoothness*, Pacific J. Math. 79 (1978), 69-91.
7. _____, *(\mathcal{C} -movably regular convergence* (preprint).
8. _____ and A. P. Šostak, *Some remarks on Borsuk's fundamental metric* (preprint).
9. M. L. Curtis, *Deformation-free continua*, Ann. of Math. 57 (1953), 231-247.
10. S. T. Hu, *Theory of retracts*, Wayne State University Press, Detroit, 1965.
11. O. H. Keller, *Die Homeomorphie der kompakten konvexen Mengen in Hilbertschen Raum*, Math. Ann. 105 (1931), 748-758.
12. G. Kozłowski, *Images of ANR's* (preprint).
13. _____ and J. Segal, *Locally well-behaved paracompacta in shape theory*, Fund. Math. 95 (1977), 55-71.

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