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by

William G. Fleissner

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
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A COLLECTIONWISE HAUSDORFF NONNORMAL MOORE SPACE WITH A σ-LOCALLY COUNTABLE BASE

William G. Fleissner[1]

The theory of locally finite collections of open sets, which is intimately involved with metrizability and paracompactness, is fairly well developed. In contrast, the theory of locally countable collections of open sets is fragmentary. All that is known, except for some recent extensions of Theorem 2.1 of [FR], due to Burke [B], is contained in [FR].

The space described in this paper is the result of a search for a regular paralindelöf nonnormal space. While this search has failed, it has produced a collectionwise Hausdorff nonnormal Moore space with a σ-locally countable base. This space is the second published example of a collectionwise Hausdorff nonnormal Moore space constructed without extra axioms of set theory. (The first is Wage's [W]). It is the third published example of a regular nonmetrizable space with a σ-locally countable base. (The first two are in [FR] and [DGN]. Actually, the space of this paper was constructed before those two, but was not included in [FR] because of its length. Gruenhage has also constructed a space which is essentially the space described in this paper.)

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1. What Would a Regular Paralindelöf Nonnormal Space X Look Like?

To start with X must contain two disjoint closed sets H,K which cannot be separated. Because X is regular, there is a cover $\mathcal{U}_0$ of H by open sets whose closures miss K, and similarly a cover $\mathcal{V}_0$ of K by open sets whose closures miss H. Now $\mathcal{U}_0 \cup \mathcal{V}_0 \cup \{X - (H \cup K)\}$ is an open cover of X, so because X is paralindelöf there is a locally countable refinement $\mathcal{U}_1 \cup \mathcal{V}_1 \cup \{X - (H \cup K)\}$, (where, of course, $\mathcal{U}_1$ covers H and $\mathcal{V}_1$ covers K).

While $(\mathcal{U}_1 \cup \mathcal{V}_1)$ is locally countable, it cannot be star-countable--that is, its own witness to its local countability. For if every $U \in \mathcal{U}_1$ met only countably many V's $\in \mathcal{V}_1$, and simultaneously every V met only countably many U's $\in \mathcal{U}_1$, then by the techniques used below to prove the space collectionwise Hausdorff, H and K can be separated by disjoint open sets. (See, in this context, the proof of Theorem 1.7 in [FR]).

Because $(\mathcal{U}_1 \cup \mathcal{V}_1)$ is locally countable, there is a cover of X by open sets each of which meet only countably many elements of $(\mathcal{U}_1 \cup \mathcal{V}_1)$. This cover can be refined so that it refines $\mathcal{U}_1 \cup \mathcal{V}_1 \cup \{X - (H \cup K)\}$ and is itself locally countable. Repeating this process, there are, for all $n \in \omega$, locally countable open covers $\mathcal{U}_{n+1}$ of H and $\mathcal{V}_{n+1}$ of K which refine $\mathcal{U}_n$ and $\mathcal{V}_n$, respectively, and witness that $\mathcal{U}_n \cup \mathcal{V}_n$ is locally countable.

The above conditions are very general and will be satisfied by every regular, paralindelöf nonnormal space (if there are any). We now make some assumptions about the space, not true in general, to guide us towards constructing a specific example.

We assume that the points of $X - (H \cup K)$ are isolated. We assume that for each $n \in \omega$, $\mathcal{U}_n$ and $\mathcal{V}_n$ are families of ω_1-many disjoint clopen sets. We assume that each $U \in \mathcal{U}_{n+1}$ meets ω-many V's $\in \mathcal{V}_n$ and ω_1-many V's $\in \mathcal{V}_{n+1}$; similarly for $V \in \mathcal{V}_{n+1}$. Finally, we assume that $\cup_{n \in \omega}(\mathcal{U}_n \cup \mathcal{V}_n) \cup \{\{x\}: x \in X - (H \cup K)\}$ is a base for X.

With these assumptions, we can be more specific in describing X. First, we can index $\mathcal{U}_n$ as $\{U^n_\alpha: \alpha < \omega_1\}$. Each $h \in H$ is contained in one and only one element of each $\mathcal{U}_n$ so h determines a function $\hat{h}: \omega \to \omega_1$ by the requirement that $h \in U^n_{\hat{h}(n)}$. Since each $U \in \mathcal{U}_2$ meets only one $U \in \mathcal{U}_1$, it seems better to index $\mathcal{U}_2$ as $\{U^2_{\alpha,\beta}: \alpha,\beta < \omega_1\}$ in such a way that $U^2_{\alpha,\beta} \subset U^1_\alpha$. In deciding which elements of $\mathcal{U}_2$ meet which elements of $\mathcal{V}_2$ we must remember that $U^2_{\alpha,\beta} \cap V^2_{\gamma,\delta} \neq \phi$ implies $U^1_\alpha \cap V^1_\gamma \neq \phi$.

Further analysis of this sort leads to more reindexing, and the introduction of some device to make the U's and V's closed. It leads fairly naturally to defining a space in essentially the same way as in the next section.

2. The Space[2] and its Base

We introduce our set theoretic notation. An ordinal is the set of its predessors. Set ${}^a b = \{f: f \text{ is a function}, \text{dom } f = a, \text{range } f \subset b\}$; ${}^{<\omega}b = \cup_{n<\omega}{}^n b$; $[a]^{<\omega} = \{c \subset a: c \text{ is finite}\}$. An ordinal α can be uniquely written $\lambda(\alpha) + i(\alpha)$ where $\lambda(\alpha)$ is a limit ordinal and $i(\alpha) < \omega$. We say that α

[2] I have resisted the temptation of calling the space Michelline.

is odd or even according to whether $i(\alpha)$ is odd or even.

Now we define the space X. Set $H = \{h \in {}^{\omega}\omega_1 : h(0)$ is even$\}$ and $K = \{k \in {}^{\omega}\omega_1 : k(0)$ is odd$\}$. (What is happening is that we have identified h with the function $\hat{h}$ it determines. H and K are similar but disjoint. The use of odd and even is simply a device to paint H red and K blue.) Let I_n be the set of pairs, (σ,τ) where $\sigma,\tau \in {}^{\omega}\omega_1$, such that $\sigma(0)$ is even, $\tau(0)$ is odd, and for all $j < n, \sigma(j) < \tau(j+1)$ and $\tau(j) < \sigma(j+1)$. Set $I = \cup_{n\in\omega} I_n$, $X = H \cup K \cup I$.

A point $(\sigma,\tau) \in I$ is isolated. For $\sigma \in {}^{n}\omega_1$, $\sigma(0)$ even, and $s \in [\sigma(1)]^{<\omega}$, set $B(\sigma,s) = \{h \in H: h|n = \sigma\} \cup \{(\sigma',\tau) \in I_m : m \geq n, \sigma'|n = \sigma$, and $\tau(0) \notin s\}$. For $\tau \in {}^{n}\omega_1$, $\tau(0)$ odd, $t \in [\tau(1)]^{<\omega}$, set $B(\tau,t) = \{k \in K: k|n = \tau\} \cup \{(\sigma,\tau') \in I_m : m \geq n, \tau'|n = \tau, \sigma(0) \notin t\}$. These are the basic open sets.

It is routine to verify that points are closed and that the above basic open sets are closed. Hence the space is regular. (In order to get a regular space, some device such as introducing the s's and t's and the requirement $\tau(0) \notin s$ seems to be needed. The complications of the next section are due to the s's and t's).

Since $\alpha < \omega_1$ is countable, we can enumerate it, $\alpha = \{\eta(\alpha,j): j \in \omega\}$. Set $r(\alpha,n) = \{\eta(\alpha,j): j \leq n\}$. For each $n < \omega$, set

$$\mathcal{G}_n^1 = \{B(\sigma,r(\sigma(1),n)): \sigma \in {}^{n}\omega_1, \sigma(0) \text{ even}\},$$

$$\mathcal{G}_n^2 = \{B(\tau,r(\tau(1),n)): \tau \in {}^{n}\omega_1, \tau(0) \text{ odd}\},$$

$$\mathcal{G}_n^3 = \{\{x\}: x \in I - \cup(\mathcal{G}_n^1 \cup \mathcal{G}_n^2)\},$$

$$\mathcal{G}_n = \mathcal{G}_n^1 \cup \mathcal{G}_n^2 \cup \mathcal{G}_n^3.$$

It is easy to verify that for each $e \in \{1,2,3\}$, $n \in \omega$, $\mathcal{G}_n^e$ is a disjoint family of open sets, $\mathcal{G}_n$ covers X, $\cup_{n<\omega}\mathcal{G}_n$ is a base for X, and $(\mathcal{G}_n)_{n<\omega}$ is a development for X. So X is a Moore space with a σ-disjoint base.

It remains to show that every element of $\mathcal{G}_{n+1}$ meets only countably many elements of $\mathcal{G}_n$. We do this by cases. If $\{(\sigma,\tau)\} \in \mathcal{G}_{n+1}$, then the only elements of $\mathcal{G}_n$ it could possibly meet are $B(\sigma|n, r(\sigma(1),n))$, $B(\tau|n, r(\tau(1),n))$, and $\{(\sigma,\tau)\}$. If $B(\sigma,s) \in \mathcal{G}_{n+1}^1$, then it meets exactly one element of $\mathcal{G}_n^1$ and none of $\mathcal{G}_n^3$. If $B(\tau,t) \in \mathcal{G}_n^2$ and $B(\sigma,s) \cap B(\tau,t) \neq \phi$, then since $B(\sigma,s) \cap K = \phi$ and $B(\tau,t) \cap H = \phi$, there must be some $(\sigma',\tau') \in B(\sigma,s) \cap B(\tau,t)$. From the definitions of $B(\sigma,s)$ and $B(\tau,t)$, we have $\sigma'|n+1 = \sigma$ and $\tau'|n = \tau$. From the definition of I_m, we have $\tau(j) < \sigma(j+1)$ for all $j \leq n$. Since $\sigma(j+1)$ is countable for all $j \leq n$, there are only countably many τ's $\in {}^n\omega_1$ such that $B(\tau,t)$ could possibly meet $B(\sigma,s)$. Thus $B(\sigma,s) \in \mathcal{G}_{n+1}^1$ meets at most countably many elements of $\mathcal{G}_n^2$. The same argument, mutis mutandis, shows that every element of $\mathcal{G}_{n+1}^2$ meets at most countably many elements of $\mathcal{G}_n$.

3. Separation Properties

In this section we will show that X is collectionwise Hausdorff, but not normal, paralindelöf, or strongly collectionwise Hausdorff. By other results, to show that X is not normal and not paralindelöf, it would be sufficient to show that X is not strongly collectionwise Hausdorff, but we prefer to give direct proofs. Recall that a space is paralindelöf if every open cover has a locally countable open refinement and is strongly collectionwise Hausdorff if every closed discrete set of points can be separated by a discrete (not merely disjoint) family of open sets.

We begin by showing that X is collectionwise Hausdorff. The referee has suggested a general proof rather than the specific case X.

Lemma. Let M *be a regular space with a base* $\mathcal{G} = \cup_{n\in\omega}\mathcal{G}_n$ *satisfying, for each* $n \in \omega$

i) $\cup\mathcal{G}_n = X$

ii) each element of $\mathcal{G}_{n+1}$ *meets only countably many elements of* $\mathcal{G}_n$

iii) for all $U \in \mathcal{G}_n$, $U = \cup\{V \in \mathcal{G}_{n+2}: V \subset U\}$.

Then M *is hereditarily collectionwise Hausdorff.*

Proof. Let $D \subset M$ be a (relatively) discrete collection of points. By regularity, for each $d \in D$ we may choose $U(d)$ and $n(d)$ so that $d \in U(d) \in \mathcal{G}_{n(d)}$ and $\overline{U(d)} \cap D = \{d\}$. For each $d \in D$, choose $V(d) \in \mathcal{G}_{n(d)+2}$ such that $d \in V(d) \subset U(d)$. Set $D_n = \{d \in D: n(d) = n\}$ and $\mathcal{V}_n = \{V(d): d \in D_n\}$.

For each $n \in \omega$ define an equivalence relation $\sim_n$ on D_n by $d \sim_n e$ iff there is a finite sequence $(W_i)_{i\leq m}$ such that

iv) $\forall\, i < m$, $W_i \cap W_{i+1} \neq \phi$

v) $W_o = V(d)$, $W_m = V(e)$

vi) if i is odd, $W_i \in \mathcal{G}_{n+1}$

vii) if i is even, $W_i \in \mathcal{V}_n$.

We can verify by cases that for a given W_i there are only countably many W_{i+1}'s satisfying iv), vi) and vii). Therefore, each equivalence class is countable. Choose D(m,n) so that $D_n = \cup_{m\in\omega} D(m,n)$ and $D(m,n)$ contains at most one element of any $\sim_n$ equivalence class. For all $m,n \in \omega$, the open cover $\mathcal{G}_{n+1}$ witnesses that $\{V(d): d \in D(m,n)\}$ is discrete.

Reindex $\{D(m,n): m,n \in \omega\}$ as $\{D(k): k \in \omega\}$. For each $k \in \omega$, $d \in D(k)$, there is an open set S_d, $d \in S_d$, such that for each $j < k$, S_d meets at most one element of $\{V(e): e \in D(j)\}$. Thus, $E_d = \{e \in D_j: j < k, S_d \cap V_e \neq \phi\}$ is finite. Since for all $e \in D$, $\overline{U_e} \cap D = \{e\}$, $\{S_d - \cup\{\overline{U_e}: e \in E_d\}: d \in D\}$ is a disjoint family open sets separating D.

Because we used only that D was relatively discrete, rather than closed discrete, we may conclude M is hereditarily collectionwise Hausdorff.

The proofs that X is not normal, not paralindelöf, and not strongly collectionwise Hausdorff require some combinatorics, which we now introduce.

Definitions. Let $\rho \in {}^m\omega_1$. Say that $A \subset {}^{m+1}\omega_1$ is 1-*full over* ρ if for all $\sigma \in A$, $\sigma|m = \rho$ and A is uncountable. Say that $A \subset {}^{m+n+1}\omega_1$ is n+1 *full over* ρ if $A = \cup\{C_\sigma: \sigma \in D\}$ where D is n-full over ρ and each C_σ is 1-full over σ. $A \subset {}^n\omega_1$ is n-*full* if A is n-full over the empty function (which is the only element of ${}^0\omega_1$).

For $A \subset {}^n\omega_1$ set $prA = \{\rho|m: m \leq n, \rho \in A\}$.

Say that $(r_\rho: \rho \in A)$ is an n-*full* Δ-*system* if A is n-full and there is a family $(s_\sigma: \sigma \in prA)$ such that for all $\rho,\rho' \in A$ $\rho \cap \rho' = \sigma$ implies $r_\rho \cap r_{\rho'} = s_\sigma$. Note that since $\rho \cap \rho = \rho$, it follows that $s_\rho = r_\rho$. Also note that a 1-full Δ-system is a Δ-system in the usual sense. The root is s_ϕ, where ϕ is the empty function.

The proofs of Lemmas 0 and 2 are easy and left to the reader. The proofs of Lemmas 1, 3, 4 and 5 are deferred to section 4.

Lemma 0. *a) If* $\rho \in {}^{m}\omega_1$ *and* $A \subset {}^{m+i+j}\omega_1$, *then* A *is* (i+j)*-full over* ρ *iff* $A = \bigcup\{C_\sigma : \sigma \in D\}$ *where* D *is* i*-full over* ρ *and each* C_σ *is* j*-full over* σ.

b) If A *is* n*-full then* $\{\sigma \in A: (\forall i < n-1)(\sigma(i) < \sigma(i+1))\}$, *the subset of increasing functions, is also* n*-full.*

c) If τ, σ, σ' *are increasing functions* dom $\tau \leq$ dom σ, *and* $\sigma \subset \sigma'$, *then* $B(\tau,t) \cap B(\sigma,s) \neq \phi$ *iff* $B(\tau,t) \cap B(\sigma',s) \neq \phi$.

Lemma 1. *If* $\bigcup\{B(\sigma_\alpha, s_\alpha): \alpha < \omega_1\}$ *is a union of basic open sets covering* H *(or* K*), then for some* $n < \omega$, $A = \{\sigma_\alpha : \alpha < \omega_1\}$ *contains an* n*-full set.*

Lemma 2. *If* $f: A \to \omega$, *and* A *is* n*-full then there is an* n*-full* A', *a subset of* A, *such that* f *is constant on* A'.

Lemma 3. *Let* $(r_\rho : \rho \in A)$ *be a family of finite sets indexed by an* n*-full set* A. *Then there is a* n*-full subset* A' *of* A *such that* $(r_\rho : \rho \in A')$ *is an* n*-full* Δ*-system.*

Lemma 4. *Let* $(s_\sigma : \sigma \in S)$ *and* $(t_\tau : \tau \in T)$ *be families of finite sets, where* S *is* n*-full and* T *is* m*-full. Then there are* S', *an* n*-full subset of* S, *and* T', *an* m*-full subset of* T, *such that for all* $\sigma \in S'$ *and* $\tau \in T'$, $\sigma(0) \notin t_\tau$ *and* $\tau(0) \notin s_\sigma$.

Lemma 5. *Let* $(s_\sigma : \sigma \in S)$ *and* $(t_\tau : \tau \in T)$ *be families of finite sets, where* S *is* n*-full and* T *is* m*-full, and* $\sigma(0)$

is even for all $\sigma \in S$ *and* $\tau(0)$ *is odd for all* $\tau \in T$. *Then there is* $\tau \in T$ *such that* $B(\tau,t_\tau) \cap B(\sigma,s_\sigma) \neq \phi$ *for some* $\sigma \in S$. *Moreover, if* $m \leq n$ *then there are uncountably many such* σ*'s in* S.

With these lemmas it is straightforward to show that X is not normal, not paralindelöf and not strongly collectionwise Hausdorff. First, let us show that X is not normal. Let U and V be open sets, $H \subset U$, $K \subset V$. By Lemma 1, there are families $\mathcal{B} = \{B(\sigma,s_\sigma): \sigma \in S\}$, $\mathcal{B}' = \{B(\tau,t_\tau): \tau \in T\}$ of basic open sets such that $\cup\mathcal{B} \subset U$, $\cup\mathcal{B}' \subset V$, and for some $n,m \in \omega$, S is n-full, T is m-full. By Lemma 5 there is $\sigma \in S$ and $\tau \in T$ such that $B(\sigma,s_\sigma) \cap B(\tau,t_\tau) \neq \phi$. Thus U and V are not disjoint and X is not normal.

Next, we show that X is not paralindelöf nor strongly collectionwise Hausdorff.

Let $\Sigma = \{\sigma \in {}^{<\omega}\omega_1: \sigma(0)$ is even, dom $\sigma = i(\sigma(0)) + 2$, σ is increasing$\}$. Let $\mathcal{W} = \{B(\sigma,\phi): \sigma \in \Sigma\}$. Then $\mathcal{W} \cup \{X - (H \cap \cup \mathcal{W})\}$ is an open cover of X. For each $\sigma \in \Sigma$, choose an increasing function $y_\sigma \in H \cap B(\sigma,\phi)$. Then $\{y_\sigma: \sigma \in \Sigma\}$ is a closed discrete subset of X. For each $\sigma \in \Sigma$ choose a basic open set U_σ satisfying $y_\sigma \in U_\sigma \subset B(\sigma,\phi)$.

If X were paralindelöf, then we could have chosen the U_σ's so that there is a cover of K by basic open sets each meeting only countably many U_σ's. If X were strongly collectionwise Hausdorff, then we could have chosen the U_σ's so that there is a cover of K by basic open sets each meeting at most one U_σ.

Let $\mathcal{V}$ be a cover of K by basic open sets. By Lemma 1 there is a family $\mathcal{B}' = \{B(\tau, t_\tau): \tau \in T\}$ of basic open sets such that $\cup\mathcal{B}' \subset V$ and T is m-full for some $m < \omega$. Let n be an even natural number greater than m. Then $S = \{\sigma: i(\sigma(0)) = n-2, \sigma$ is increasing, and dom $\sigma = n\}$ is n-full. Each U_σ is of the form $B(\sigma', s_\sigma)$, where $\sigma' \supset \sigma$, and σ' is increasing. By Lemma 0. c), for $\sigma \in S$, $\tau \in T$, $B(\sigma, s_\sigma) \cap B(\tau, t_\tau) \neq \phi$ iff $U_\sigma \cap B(\tau, t_\tau) \neq \phi$. Thus by Lemma 5 some $B(\tau, t_\tau)$ meets uncountably many U_σ's, and X is not paralindelöf nor strongly collectionwise Hausdorff.

Similar arguments show that X is not countably paracompact.

4. Proofs of the Lemmas

The proofs of Lemmas 0 and 2 are straightforward and left to the reader.

Proof of Lemma 1. We will prove the contrapositive. So assume that for all $n < \omega$, $A = \{\sigma_\alpha: \alpha < \omega_1\}$ does not contain an n-full set. We will define by induction on $i < \omega$, functions $\eta_i \in {}^i\omega_1$ so that $h = \cup_{i<\omega}\eta_i \notin \cup\{B(\sigma_\alpha, s_\alpha): \alpha < \omega_1\}$.

Because no subset of A is 1-full, $E_1^1 = A \cap {}^1\omega_1$ is countable. Because no subset of A is n+1 full, the set $E_{n+1}^1 = \{\rho \in {}^1\omega_1$: some subset of A is n-full over $\rho\}$ is countable. Thus we can find $\eta_1 \in {}^1\omega_1$ such that $\eta_1(0)$ is even, $\eta_1 \notin A$, and for all $n < \omega$, no subset of A is n-full over η_1.

Now assume that we have defined $\eta_i \in {}^i\omega_1$ so that $\eta_i \notin A$ and for all $n < \omega$, no subset of A is n-full over η_i. In the same way we found η_1 we can find $\eta_{i+1} \in {}^{i+1}\omega_1$ so that $\eta_i \subset \eta_{i+1}$, $\eta_{i+1} \notin A$, and for all $n < \omega$ no subset of A is n-full over η_i. Set $h = \cup_{i<\omega}\eta_i$. For all $i < \omega$. $h|i = \eta_i$ $\notin A$, so $h \notin \cup\{B(\sigma_\alpha, s_\alpha): \alpha < \omega_1\}$.

Proof of Lemma 3. The proof is by induction on $n < \omega$. For $n = 1$ it is the ordinary Δ-system lemma for finite sets. We give the proof for completeness. By the pigeonhole principle, we can find an 1-full $A^1 \subset A$ such that for all $\rho \in A'$, r_ρ has m elements. If there is an element δ such that $A^1_\delta = \{\rho \in A': \delta \in r_\rho\}$ is uncountable, set $\delta = \delta_1$ and set $A^2 = A^1_\delta$. Again, if there is $\delta \neq \delta_1$ such that $A^2_\delta = \{\rho \in A^2: \delta \in r_\rho\}$ is uncountable, set $\delta = \delta_2$ and $A^3 = A^2_\delta$. Continue. Because each r_ρ has m elements, we can find at most m-1 such δ's. Set $d = \{\delta_1, \cdots, \delta_j\}$ and $A = A^{j+1}$. (It may be that $d = \phi$ and $A = A^1$).

Note that if $\delta \notin d$, then $A_\delta = \{\rho \in A: \delta \in r_\rho\}$ is countable. So for any countable set C, $\{A_\delta: \delta \in r_\rho - d, \rho \in C\}$ is countable. Thus we may define by induction on $\beta < \omega_1$, $\rho(\beta) \in A$, $\rho(\beta) \notin \cup\{A_\delta: \delta \in r_{\rho(\gamma)} - d, \gamma < \beta\}$. Then, setting $R = \{\rho(\beta): \beta < \omega_1\}$, $(r_\rho: \rho \in R)$ is a 1-full Δ-system.

Now assume that Lemma 3 is true for n = m. We must show that it is true for n = m + 1. So let $(r_\rho: \rho \in A)$ be a family of finite sets indexed by an m+1-full set A. Then $A = \cup\{C_\sigma: \sigma \in D\}$ where D is m-full and each C_σ is 1-full over σ. First we apply the ordinary Δ-system lemma to each C_σ to get a Δ system C'_σ with root t_σ. Next we apply our induction hypothesis to $(t_\sigma: \sigma \in D)$ and obtain a m-full Δ-system $(t_\sigma: \sigma \in D')$. $A' = \cup\{C'_\sigma: \sigma \in D'\}$ is almost the set we want, but it needs to be refined. Note that if $\sigma' \in D'$ and $\delta \in r_\rho - t_{\sigma'}$, where $\rho \in C'_\sigma$ and $\sigma \in D'$, then

$\{\rho' \in C'_\sigma : \delta \in r_{\rho'}\}$ has at most one element.

Let $(\sigma(\beta): \beta < \omega_1)$ enumerate D' so that each $\sigma \in D'$ is counted ω_1 times. By induction on $\beta < \omega_1$, we can define $\rho(\beta)$ so that $\rho(\beta) \neq \rho(\gamma)$ for $\gamma < \beta$, $\rho(\beta) \in C'_{\sigma(\beta)}$, and $(r_{\rho(\beta)} - t_{\sigma(\beta)}) \cap (\cup\{r_{\rho(\gamma)} - t_{\sigma(\gamma)}: \gamma < \beta\}) = \phi$. Then setting $R = \{\rho(\beta): \beta < \omega_1\}$, $(r_\rho: \rho \in R)$ is an m+1-full Δ-system.

Proof of Lemma 4. Let $(s_\sigma: \sigma \in S)$ and $(t_\tau: \tau \in T)$ be as in the hypothesis. By Lemma 3, we can reduce to the case where $(s_\sigma: \sigma \in S)$ is an n-full Δ-system and $(t_\tau: \tau \in T)$ is an m-full Δ-system.

We will prove Lemma 4 by induction on the sum n+m. First, we do the case where n = 1 = m. Let s_ϕ be the root of the Δ-system $(s_\sigma: \sigma \in S)$, t_ϕ the root of $(t_\tau: \tau \in T)$. Set $S_1 = \{\sigma \in S: \sigma(0) \notin t_\phi\}$; $T_1 = \{\tau \in T: \tau(0) \notin s_\phi\}$. S_1 and T_1 are 1-full. Note that if $\delta \in s_\sigma - s_\phi$, where $\sigma \in S_1$, there is at most one $\tau \in T_1$ such that $\tau(0) = \delta$; similarly if $\delta \in t_\tau - t_\phi$. Also if $\tau \in T_1$, then $\tau(0) \in s_\sigma - s_\phi$ for at most one $\sigma \in S_1$; similarly for $\sigma \in S_1$. In summary, placing σ from S_1 into S' prohibits only finitely many τ's $\in$ T, from being placed in T', and conversely. So by induction on $\beta < \omega_1$, we can define $\sigma(\beta)$ and $\tau(\beta)$ so that $\sigma(\beta) \neq \sigma(\gamma)$ for $\gamma < \beta$, $\tau(\beta) \neq \tau(\gamma)$ for $\gamma < \beta$, $\sigma(\beta)(0) \notin \cup\{t_{\tau(\gamma)} - t_\phi: \gamma < \beta\}$, $s_{\sigma(\beta)} \cap \{\tau(\gamma)(0): \gamma < \beta\} = \phi$, $\tau(\beta)(0) \notin \cup\{s_{\sigma(\gamma)} - s_\phi: \gamma \leq \beta\}$, and $t_{\tau(\beta)} \cap \{\sigma(\gamma)(0): \gamma \leq \beta\} = \phi$. Set $S' = \{\sigma(\beta): \beta < \omega_1\}$, $T' = \{\tau(\beta): \beta < \omega_1\}$; S', T' have the properties of the conclusion of Lemma 4.

Now assume Lemma 4 is true for m + n = j; we must prove

Lemma 4 with $m + n = j + 1$. At least one of m,n is greater than 1, let us assume it is m, the proof in the case $m = 1$, $n > 1$ being parallel. T can be written as $\cup\{C_\rho: \rho \in D\}$ where D is $m-1$-full and C_ρ is 1-full over ρ for each $\rho \in D$. Let $\{t_\rho: \rho \in \text{pr}T\}$ be as in the definition of n-full Δ-system. By the induction hypothesis, we can apply Lemma 4 to $(s_\sigma: \sigma \in S)$ and $(t_\rho: \rho \in D)$ to get $(s_\sigma: \sigma \in S_1)$ and $(t_\rho: \rho \in T_1)$, where S_1 is an n-full subset of S and T_1 is an $m-1$ full subset of D. For $\sigma \in S_1$, $\rho \in T_1$, there is at most one $\tau \in C_\rho$ such that $\sigma(0) \in t_\tau$. Because S_1 and T_1 are the result of applying Lemma 4, if $\sigma \in S_1$ and $\tau \in \cup\{C_\rho: \rho \in T_1\}$, then $\tau(0) \notin s_\sigma$.

Let $\{\rho(\beta): \beta < \omega_1\}$ enumerate D so that each $\rho \in D$ is counted ω_1 times. By induction on $\beta < \omega_1$ we can define $\nu(\beta) \in \{\sigma(0): \sigma \in S_1\}$ and $\tau(\beta) \in C_{\rho(\beta)}$ so that for all $\gamma < \beta$, $\nu(\beta) \neq \nu(\gamma)$ and $\tau(\beta) \neq \tau(\gamma)$, $\nu(\beta) \notin \cup\{t_{\tau(\gamma)}: \gamma < \beta\}$, and $t_{\tau(\beta)} \cap \{\nu(\gamma): \gamma \leq \beta\} = \phi$. Set $N = \{\nu(\beta): \beta < \omega_1\}$, $T' = \{\tau(\beta): \beta < \omega_1\}$, $S' = \{\sigma \in S_1: \sigma(0) \in N\}$. Then S', T' satisfy the conclusion of Lemma 4.

Proof of Lemma 5. By Lemma 4, we may reduce to the case where for all $\sigma \in S$ and $\tau \in T$, $\sigma(0) \notin t_\tau$ and $\tau(0) \notin s_\sigma$. By Lemma 0. b) we may reduce to the case where each $\sigma \in S$ and $\tau \in T$ is increasing. Let us assume $m \leq n$, the case $n \leq m$ being parallel. By induction on $i \leq m$, we define $\sigma_i \in \text{pr}S \cap {}^i\omega_1$ and $\tau_i \in \text{pr}T \cap {}^i\omega_1$ so that for all $j < i$, $\sigma_j \subset \sigma_i$, $\tau_j \subset \tau_i$, $\sigma_i(j-1) < \tau_i(j)$ and $\tau_i(j-1) < \sigma_i(j)$. Set $\sigma^* = \cup\{\sigma_i: i \leq m\}$, $\tau = \cup\{\tau_i: i \leq m\}$. Let σ be any element of S such that $\sigma^* \subset \sigma$. Because σ and τ are increasing it is easy to verify $B(\sigma,s_\sigma) \cap B(\tau,t_\tau) \neq \phi$. Since there were uncountably many allowable choices for σ_m, we have that there are uncountably many σ's $\in S$ such that $B(\sigma,s_\sigma) \cap B(\tau,t_\tau) = \phi$.

5. A Variation

Many variations of the space X suggest themselves. The one we describe below, Y, has the same topological properties as X. It is interesting because different combinatorial lemmas are needed.

Let Y have the same point set as X. Points (σ,τ) are again isolated. For $\sigma \in {}^{n+1}\omega_1$, $\sigma(0)$ even, and $c \in {}^{n}\omega_1$ satisfying for all $j \leq n$, $c(j) < \sigma(j+1)$, define $B(\sigma,c) = \{h \in H: \sigma \subset h\} \cup \{(\sigma',\tau) \in \tau: \sigma \subseteq \sigma'$ and for all $j \leq n$, $c(j) < \tau(j) < \sigma'(j+1)\}$. Define $B(\tau,d)$ analoguously.

In order to show that Y is not normal and not paralindelöf, we need some combinatorial lemmas. Let $\rho \in {}^{m}\omega_1$. Say that $A \subset {}^{m+1}\omega_1$ is 1-*stafull over* ρ if for all $\sigma \in A$ $\sigma|m = \rho$ and $\{\alpha: \langle m,\alpha\rangle \in A\}$ is stationary. Say that $A \subset {}^{m+n+1}\omega_1$ is n+1-*stafull over* ρ if $A = \cup\{C_\sigma: \sigma \in D\}$ where D is n-stafull over ρ and each C_σ is 1-stafull over σ. $A \subset {}^{n}\omega_1$ is n-*stafull* if A is n-stafull over the empty function.

Lemma 1'. *If* $\cup\{B(\sigma_\alpha,s_\alpha): \alpha < \omega_1\}$ *is a union of basic open sets covering* H *(or* K*), then for some* $n < \omega$, $A = \{\sigma_\alpha: \alpha < \omega_1\}$ *contains an* n-*stafull set.*

Lemma 2'. *If* $f: A \to \omega$ *and* A *is* n-*stafull, then there is an* n-*stafull* $A' \subset A$ *such that* f *is constant on* A'.

Lemma 3'. *Let* $(c_\rho: \rho \in A)$ *be a family of elements of* ${}^{n}\omega_1$ *indexed by an* n+1-*stafull set* A. *Then there is an* n+1 *stafull set* $A' \subset A$ *such that if* $\rho, \rho' \in A'$ *and* $\rho|j = \rho'|j$, *then* $c_\rho(j) = c_{\rho'}(j)$.

Lemma 5'. *Let* $(c_\sigma: \sigma \in S)$ *be a family of elements of* ${}^n\omega_1$ *indexed by an* n+1-*stafull set* S, *where* $\sigma(0)$ *is even for all* $\sigma \in S$. *Let* $(d_\tau: \tau \in T)$ *be a family of elements of* ${}^m\omega_1$ *indexed by an* m+1-*stafull set* T, *where* $\tau(0)$ *is odd for all* $\tau \in T$. *There is* $\tau \in T$ *such that* $B(\tau, d_\tau) \cap B(\sigma, c_\sigma) \neq \phi$ *for some* $\sigma \in S$. *Moreover, if* $m \leq n$ *then there are* ω_1-*many such* σ*'s*.

The proofs of these lemmas and the proof that Y has the desired properties is analogous to the previous proofs and are therefore omitted.

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University of Pittsburgh

Pittsburgh, Pennsylvania 15261