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by

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THE INVERSE LIMIT OF HOMOTOPY EQUIVALENCES BETWEEN TOWERS OF FIBRATIONS IS A HOMOTOPY EQUIVALENCE—A SIMPLE PROOF

Ross Geoghegan[1]

The following theorem is due to Edwards and Hastings [1; 3.4.1], but their proof is buried in a considerable amount of machinery, both their own and that of Quillen [3]. For some time I have wanted to see an elementary proof in the literature, both because the theorem is obviously relevant to shape theory and related parts of geometric topology, and because my paper [2] relies upon it. The original version of the present note contained a short proof along the same lines as that of Edwards and Hastings: essentially it separated out the relevant parts of [1] and [3]. On reading that proof, J. Dydak suggested further simplifications, and it is this extremely simple version which will be given here (with Dydak's permission). Let me repeat that the theorem is due to Edwards and Hastings, and that this exposition is intended to be merely a service.

Theorem. Suppose given a strictly commutative diagram of topological spaces and maps

$$\begin{array}{ccccccc} \cdots & \leftarrow & X_n & \leftarrow & X_{n+1} & \leftarrow & \cdots \\ & & \simeq\downarrow f_n & & \simeq\downarrow f_{n+1} & & \\ \cdots & \leftarrow & Y_n & \leftarrow & Y_{n+1} & \leftarrow & \cdots \end{array}$$

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in which each f_n *is a homotopy equivalence and each bond is a fibration, then the inverse limit map* $f\colon X \to Y$ *is a homotopy equivalence.*

Proof. We will repeatedly use the elementary fact [4; Theorem 3] that if the solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & Y \\ {\scriptstyle i}\downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle p} \\ X & \longrightarrow & B \end{array}$$

commutes, where i is both a closed cofibration and a homotopy equivalence, and p is a fibration, then f exists making the whole diagram commute.

We *claim* that maps h_n can be chosen inductively so that the following diagrams $(*_n)$ commute, and commute with the bonds:

$$\begin{array}{ccc} X_n & \xrightarrow{\ 1\ } & X_n \\ {\scriptstyle i_n}\downarrow & \overset{h_n}{\nearrow} & \uparrow \\ M(f_n) & \xleftarrow[\ j_n\ ]{} & Y_n \end{array} \qquad g_n \equiv h_n \circ j_n \qquad (*_n)$$

Here $M(f_n)$ is the mapping cylinder of f_n, i_n and j_n are the usual inclusions, and the bonding map $q_n\colon M(f_{n+1}) \to M(f_n)$ is induced by the given bonds. To prove this *claim* assume $h_1, \cdots, h_n$ exist and obtain h_{n+1} from the following diagram in the obvious way:

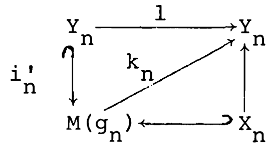
$$\begin{array}{ccc} X_{n+1} & \xrightarrow{\ 1\ } & X_{n+1} \\ \downarrow & \overset{h_{n+1}}{\nearrow} & \downarrow \\ M(f_{n+1}) & \xrightarrow[\ h_n \circ q_n\ ]{} & X_n \end{array}$$

Then let $g_{n+1} = h_{n+1} \circ j_{n+1}$. Claim proved.

The map $H_n\colon X_n \times I \xrightarrow{\text{identif.}} M(f_n) \xrightarrow{h_n} X_n$ commutes with bonds, and $H_n\colon 1_{X_n} \simeq g_n \circ f_n$. Let $g = \varprojlim_n g_n$

and $H = \varprojlim_n H_n$. Then $H: 1_X \simeq g \circ f$.

Now apply the Claim again to get maps k_n so that the following diagrams commute, and commute with the bonds



$f'_n \equiv k_n \circ j'_n$

Define $K_n: Y_n \times I \xrightarrow{\text{identif.}} M(g_n) \xrightarrow{k_n} Y_n$. Let $f' = \varprojlim_n f'_n$ and $K = \varprojlim_n K_n$. Then $K: 1_Y \simeq f' \circ g$. $f' \simeq f' \circ g \circ f \simeq f$, so g is homotopy inverse for f. The theorem is proved.

References

[1] D. A. Edwards and H. M. Hastings, *Čech and Steenrod homotopy theories, with applications to geometric topology*, Lecture Notes in Math., Vol. 542, Springer-Verlag, Berlin and New York, 1976.

[2] R. Geoghegan, *Fibered stable compacta have finite homotopy type*, Proc. Amer. Math. Soc. 71 (1978), 123-129; erratum 74 (1979), 391.

[3] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Math., Vol. 43, Springer-Verlag, Berlin and New York, 1967.

[4] A. Strøm, *Note on cofibrations*, Math. Scand. 19 (1966), 11-14.

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