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It is well-known [B, D] that minimality in the category of Hausdorff groups may fail to be preserved even in finite products. However, it has also been shown [D, G] that the group U of complex roots of units is well-behaved in this respect, all finite powers of U being minimal.

R. M. Stephenson asked [St2, Question 10] whether the countably infinite power of U is a minimal group. We answer this question in the affirmative, as a corollary of a stronger result.

Let us recall from [H] that a topological group G is called a B(A) (resp., $B_r(A)$) group is every continuous, almost open (resp., and one-to-one) homomorphism from G onto a Hausdorff group is open. The author, generalizing earlier results of L. J. Sulley [Su], showed [G, Theorem 1.4] that G is a B(A) group (resp., $B_r(A)$ group) if and only if its completion H with respect to the two-sided uniformity is a B(A) (resp., $B_r(A)$) group and $G \cap N$ is dense in N for every closed normal subgroup N of H (resp., $G \cap N$ is non-trivial for every non-trivial closed normal subgroup of H). Stephenson showed [St1] that a condition identical to the one stated for the $B_r(A)$ property guarantees the inheritance of minimality by precompact groups from their compact completions. It therefore follows that, for precompact groups, the $B_r(A)$ property is equivalent to minimality, and both are consequences of the B(A) property. It is now evident that the following theorem implies an affirmative answer to Stephenson's question.

Theorem. Let A *be any set. Then* U^A *is a* B(A) *topological group.*

Let T denote the circle group, e the function $R \to T$ given by $e(t) = e^{2\pi it}$, (t) the fractional part of t. A subset $\{t_1, \cdots, t_n\}$ of R is said to be rationally independent if $r_1 t_1 + \cdots + r_n t_n = 0$ with each r_i rational implies $r_i = 0$ for each i. We will require the following lemma which is a trivial extension of Theorem 443 of [HW].

Lemma. Let $\{1, \beta_1, \cdots, \beta_n\}$ *be rationally independent,* p *a positive integer,* $r \in \{0, 1, \cdots, p-1\}$. *Then* $\{((k\beta_1), \cdots, (k\beta_n)) : k \equiv r \pmod p\}$ *is dense in the* n*-dimensional unit cube, and its image under* e^n *therefore dense in* T^n.

We now proceed to the proof of the theorem.

Proof. The case where A is finite has already been proved as Example 2 of [G]. We therefore assume A to be infinite. By a result of Soundararajan [So], it is sufficient to show that U^A intersects the closure of every singly-generated subgroup of T^A in a dense subgroup of that closure.

Let $x = (x_\alpha)_{\alpha \in A} \in T^A$, $\langle x \rangle$ the subgroup it generates, and X the closure of this subgroup in T^A. If supp x = $\{\alpha \in A : x_\alpha \neq 1\}$ is finite, then a similar argument to that in [G] will establish that $U^A \cap X$ is dense in X. One may then further assume, without loss of generality, that supp x = A.

Let $A_o = \{a \in A: x_\alpha \in U\}$, $A_1 = A \backslash A_o$. Let $x_\alpha = e(\beta_\alpha)$; then $\beta_\alpha \in Q$ iff $\alpha \in A_o$. Let B be a maximal subset of $\{\beta_\alpha: \alpha \in A_1\}$ such that $B \cup \{1\}$ is rationally independent, $A_2 = \{\alpha \in A_1: \beta_\alpha \in B\}$. Then, for each $\gamma \in A_1 \backslash A_2$, there exist an integer $n(\gamma)$, a finite subset $F_\gamma = \{\beta_{\gamma 1}, \cdots, \beta_{\gamma n(\gamma)}\}$ of B of minimal size, and a finite set of non-zero rational numbers $Q_\gamma = \{c_{\gamma 1}, \cdots, c_{\gamma n(\gamma)}\}$ such that $\beta_\gamma = \sum_{i=1}^{n(\gamma)} c_{\gamma i}\beta_{\gamma i}$.

The elements of $\langle x \rangle$ can then be characterized as follows: $y = (y_\alpha) \in \langle x \rangle$ if and only if, for some integer r, $y_\alpha = e(r\beta_\alpha)$ for each α. In particular, for $\alpha \in A_1 \backslash A_2$, $y_\alpha = \sum_{i=1}^{n(\alpha)} rc_{\alpha i}\beta_{\alpha i}$.

Let $K = \langle (x_\alpha)_{\alpha \in A_o} \rangle \subseteq U^{A_o}$. For $(y_\alpha) \in T^{A_1}$, let $y_\alpha = e(t_\alpha)$. For $\gamma \in A_1 \backslash A_2$, define a subgroup L_γ of U^{A_1} by

$$L_\gamma = \{(y_\alpha): t_\gamma = \sum_{i=1}^{n(\gamma)} c_{\gamma i} t_{\gamma i}\},$$

the coefficients being those from Q_γ. Let L denote the intersection of the subgroups L_γ for all $\gamma \in A_1 \backslash A_2$.

Clearly, $\langle x \rangle$ is a subset of $K \times L$, and we claim that it is in fact a dense subset. Since $(K \times L) \cap U^A$ is dense in $K \times L$ and so in its closure, it would then follow that

$$Cl[U^A \cap Cl\langle x \rangle] = Cl[U^A \cap Cl(K \times L)] = Cl(K \times L) = Cl\langle x \rangle.$$

To establish this density property, we let $y = (y_\alpha) \in K \times L$, $y_\alpha = e(t_\alpha)$ for each α. Let $V = \prod_{\alpha \in A} V_\alpha$ be a neighbourhood of y, $E = \{\alpha \in A: V_\alpha \neq T\}$, $E_i = E \cap A_i$ for $i = 0,1,2$. For $\alpha \in E$, let V'_α be a neighbourhood of t_α such that $e(V'_\alpha) \subseteq V_\alpha$. Now, for $\alpha \in E_1 \backslash E_2$, $t_\alpha = \sum_{i=1}^{n(\alpha)} c_{\alpha i} t_{\alpha i}$, $c_{\alpha i} = m_{\alpha i}/n_{\alpha i}$, g.c.d.$(m_{\alpha i}, n_{\alpha i}) = 1$. For each $\gamma \in E_2$, let $G_\gamma = \{a: \beta_\gamma = \beta_{\alpha, i(\gamma)}$ for some $i(\gamma) \in \{1, \cdots, n(\alpha)\}\}$, and $n_\gamma = $ l.c.m.$\{n_{\alpha, i(\gamma)}: \alpha \in G_\gamma\}$. For each $\gamma \in E_2$ and $\alpha \in G_\gamma$, let $d_{\alpha, i(\gamma)}$ be the integer $c_{\alpha, i(\gamma)} n_\gamma$. Furthermore, let $y_\alpha = x_\alpha^r$ for each $\alpha \in A_o$, and let p denote the least common multiple of the orders of the elements x_α, $\alpha \in E_o$. Then, for any integer j, $y_a = x_\alpha^r = x_\alpha^{r+pj}$, for each $\alpha \in E_o$.

For each $\alpha \in E_1 \backslash E_2$, let $C_\alpha = \{\gamma \in A_2 : \alpha \in G_\gamma\}$, and $C = \cup\{C_\alpha : \alpha \in E_1 \backslash E_2\}$. For each $\alpha \in C$, define $\beta'_\alpha = \beta_\alpha / n_\alpha$, and $t'_\alpha = t_\alpha / n_\alpha$; for $\alpha \in A \backslash C$, $\beta'_\alpha = \beta_\alpha$ and $t'_\alpha = t_\alpha$. It is then trivial to see that $\{\beta'_\alpha : \alpha \in A\} \cup \{1\}$ is rationally independent, and that

$$t_\alpha = \textstyle\sum_{\gamma \in C_\alpha} d_{\alpha, i(\gamma)} t'_\gamma$$

for each $\alpha \in E_1 \backslash E_2$.

For all $\alpha \in C$, select neighbourhoods V''_α of t'_α such that $\sum_{\gamma \in C_\alpha} d_{\alpha, i(\gamma)} V''_\gamma \subseteq V'_\alpha$. Let $Y = \prod_{\alpha \in A} Y_\alpha$ be the neighbourhood of (t'_α) given by

$$Y_\alpha = \begin{cases} V''_\alpha, & \alpha \in C \\ V'_\alpha, & \alpha \in A \backslash C. \end{cases}$$

Now, by the Lemma, there exists $j \equiv r \pmod p$ such that $e(j\beta'_\alpha) \in e(Y_\alpha)$ for each $\alpha \in E_1 \cup C$. Then

$$\begin{aligned} e(j\textstyle\sum_{i=1}^{n(\alpha)} c_{\alpha i}\beta_{\alpha i}) &= e(j\textstyle\sum_{\gamma \in C_\alpha} d_{\alpha, i(\gamma)} \beta'_\gamma) \\ &= e(\textstyle\sum_{\gamma \in C_\alpha} d_{\alpha, i(\gamma)} j\beta'_\gamma) = e(\textstyle\sum_{\gamma \in C_\alpha} d_{\alpha, i(\gamma)} [y_\gamma + k_\gamma]), \\ &\qquad \text{for some integers } k_\gamma \text{ and } y_\gamma \in Y_\gamma, \\ &= e(\textstyle\sum_{\gamma \in C_\alpha} d_{\alpha, i(\gamma)} y_\gamma) \cdot 1 \subseteq e(\textstyle\sum_{\gamma \in C_\alpha} d_{\alpha, i(\gamma)} Y_\gamma) \subseteq \\ & e(V'_\alpha) \subseteq V_\alpha. \end{aligned}$$

It therefore follows that $x^r \in V$, and our claim is established.

Corollary. U^A *is a minimal Hausdorff topological group for any set* A.

Remark. Prodanov [P] has defined a topological group to be *totally minimal* if all its Hausdorff quotient groups are minimal. Since the B(A) property and precompactness are both divisible, and together imply minimality, it follows that U^A has this stronger property, as well.

Question. If every finite power of a group is minimal or a B(A) group, must arbitrary powers of the group have the same property?

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Editor's Note:

When this paper was first received, the referee B. Banaschewski, found a major error in the proof. This fact was communicated to the author and some weeks later, Prof. Banaschewski sent to the editor a correct proof by entirely different reasoning. The existence--but not the essence--of this proof was communicated to the author who responded some months later with his own corrected version which is here printed.