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1. Introduction

It is well-known [2] that the property that all the neighborhoods of the diagonal form a compatible uniform structure imposes strong normality conditions on a space. Here we are interested in the nature of those spaces for which all the neighbornets form a compatible quasi-uniform structure.

A relation V on a topological space X is a *neighbornet* of X provided $V(x) = \{y: (x,y) \in V\}$ is a neighborhood of X for each x in X. A sequence $(V_n, n < \omega)$, of neighbornets of a space X is called a *normal sequence* provided $V_{n+1}^2 \subset V_n$ for every $n < \omega$. A neighbornet V of X is *normal* if V is a member of a normal sequence of neighbornets of X. A topological space X such that each neighbornet of X is normal, is called a z-*space*. Clearly, a topological space X is a z-space provided all the neighbornets of X form a compatible quasi-uniform structure. It is sometimes useful to have the 'covering' definition of a z-space. An *indexed open cover* of a topological space X is an open cover $\{G_x: x \in X\}$ such that $x \in G_x$ for all x in X. Then, a z-space is a topological space for which each indexed open cover $\{G_x: x \in X\}$ has an indexed open refinement (called a z-*refinement*) $\{H_x: x \in X\}$, such that $y \in H_x$ implies $H_y \subset G_x$.

2. Main Results

We first remark that if X is a T_1 topological space and if X is countable, then X is a z-space. While this result is easy to prove directly, it can also be derived from the results of [7]. We will soon have an interesting partial converse to this result stating that all compact metrizable z-spaces are countable. In view of the fact that any disjoint union of z-spaces is a z-space, a more genuine converse is not hoped for.

Let α be a cardinal (initial ordinal) number. A topological space X is said to be *α-resolvable* if there exists a sequence $\{D_\beta : \beta < \alpha\}$ of pairwise disjoint dense subsets of X. Clearly every 2-resolvable space is dense-in-itself.

Theorem 1. *Let* X *be an* ω*-resolvable* z*-space. If some* G_δ *subset of* $X \times X$ *containing the diagonal has empty interior, then* X *is first category.*

Proof. Let (V_n) be a decreasing sequence of open neighborhoods of the diagonal $\{(x,x): x \in X\}$, such that $\cap_1^\infty V_n$ has empty interior. Let (D_n) be a sequence of pairwise disjoint dense subsets of X such that $\cup_1^\infty D_n = X$. We define an indexed open cover $\{G_x : X \in X\}$ of X such that if $x \in D_n$ then $G_x \times G_x \subset V_n$; and we let $\{H_x : x \in X\}$ be a z-refinement of $\{G_x : x \in X\}$. Now we set $P_n = \cup(H_x : x \in D_n)$. Then (P_n) is a sequence of dense open subsets of X. We claim that $\cap P_n = \phi$. If possible, suppose $y \in \cap P_n$. Let m be the largest integer such that $H_y \times H_y \subset V_m$. As $y \in P_{m+1}$, so $y \in H_t$ for some $t \in D_{m+1}$. Consequently $H_y \subset G_t$ and $G_t \times G_t \subset V_{m+1}$. But then, $H_y \times H_y \subset V_{m+1}$, a contradiction.

There are several interesting consequences of the above theorem. We recall ([1], [4]) that every dense-in-itself, first countable T_0 space is ω-resolvable and every dense-in-itself, locally compact T_2 space is (exp ω)-resolvable. We also note that each subspace of a z-space is a z-space.

Corollary 1. *A dense-in-itself, first countable,* T_0 z-*space with a* G_δ-*diagonal is first category. In particular, every dense-in-itself, metrizable* z-*space must be first category.*

Corollary 2. *A locally compact* T_2 z-*space with a* G_δ *diagonal is scattered.*

Corollary 3. *A compact metrizable* z-*space is countable. Consequently, each such space is homeomorphic to a subspace of the rational numbers.*

Corollary 4. *Every Hausdorff* z-*space is totally path-disconnected.*

Corollary 5. *Every separable,* ω-*resolvable,* T_1, z-*space is first category.*

In view of corollary 4, one might ask if all Hausdorff z-space are totally disconnected. That is not so, because there do exist connected Hausdorff topologies on a countably infinite set. However, we do not know whether every completely regular Hausdorff z-space is totally disconnected.

We note that product of two z-spaces need not be a z-space. For example, if X is the real line with the usual topology expanded by isolating all the irrationals and if Y is the convergent sequence $\{0\} \cup \{1/n: n = 1,2,3,\cdots\}$ then it is easy to show that $X \times Y$ is not a z-space.

Theorem 2. *Let* X *be a* T_1, *first countable* z-*space*. *Then* X *is quasi-metrizable*.

Proof. For each $x \in X$, let $\{g_n(x)\}_{n=1}^{\infty}$ be a countable open base at x. For each positive integer n, let $V_n^1 = \cup\{\{x\} \times g_n(x)\colon x \in X\}$; and then for each n, let (V_n^j), $j = 1,2,\cdots$ be a normal sequence. The collection $\{V_n^j\colon n,j$ are positive integers$\}$ is clearly a countable subbase for a quasi uniform structure $\mathcal{U}$ on X; and a comparison of neighborhood filters easily shows that $\mathcal{U}$ is compatible with the given topology.

Finally, we note that (i) z-spaces are precisely those spaces for which the finest compatible quasi uniform structure is the finest compatible local quasi uniform structure [6], and (ii) every first countable z-space is a co-Nagata space.

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