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Introduction

Let d, L, c, s, χ and ψ denote the following standard cardinal functions: density, Lindelöf degree, cellularity, spread (= hereditary cellularity), character, and pseudo-character. (For definitions, see [7] or [14].) The following inequalities are basic in the theory of cardinal invariants: (1) if X is Hausdorff, then $|X| \leq 2^{c(X)\chi(X)}$; (2) if X is T_1, then $|X| \leq 2^{s(X)\psi(X)}$; (3) if X is Hausdorff, then $d(X) \leq 2^{s(X)}$; (4) if X is Hausdorff, then $|X| \leq 2^{2^{s(X)}}$; (5) if X is Hausdorff, then $|X| \leq 2^{L(X)\chi(X)}$. (See [11] and [1].) Partition calculus and ramification arguments are used in the original proofs of these five inequalities. (See [8] and [9].) Specifically, the Erdös-Rado theorem $(2^{\kappa})^+ \rightarrow (\kappa^+)^2_\kappa$ is used in the proof of (1) and (2), the Erdös theorem $\kappa \rightarrow (\kappa,\omega)^2$ is used in the proof of (3), the Erdös-Rado theorem $(2^{2^\kappa})^+ \rightarrow (\kappa^+)^3_\kappa$ is used in the proof of (4), and in proving (5) Arhangel'skiĭ uses a difficult ramification argument to construct a free sequence of length κ^+.

In [16] Šapirovskiĭ proved a fundamental theorem about the cardinal function s, and from this theorem one easily obtains the two inequalities $d(X) \leq 2^{s(X)}$ and $|X| \leq 2^{2^{s(X)}}$. Pol [15] has modified Šapirovskiĭ's technique to give proofs of the two inequalities $|X| \leq 2^{c(X)\chi(X)}$ and $|X| \leq 2^{L(X)\chi(X)}$, and I have used this technique to prove the inequality

$|X| \leq 2^{s(X)\psi(X)}$. In summary, the work of Pol and Šapirovskiĭ gives an alternate, unified approach to the five inequalities stated above.

The point I would like to emphasize in this paper is that the Pol-Šapirovskiĭ technique plays a fundamental, unifying role in the theory of cardinal invariants and can be used to prove a wide variety of cardinal function inequalities. Specifically, I will illustrate their technique by proving that every χ_1-compact space with a G_δ-diagonal has cardinality at most 2^ω. The generalized version of this inequality is due to Ginsburg and Woods [10]; their proof uses the Erdös-Rado theorem $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$. In addition, I will survey several other inequalities in cardinal functions, each of which can be proved using the Pol-Šapirovskiĭ technique.

The Technique Illustrated

In order to take advantage of well known terminology, I will just prove the countable version of the Ginsburg-Woods inequality. (The proof I give can easily be extended to higher cardinality.) The following notation is used: if X is a set, $\mathcal{G}$ is a cover of X, and D is a subset of X, then $st(D,\mathcal{G}) = \cup\{st(x,\mathcal{G}): x \in D\}$. Recall that a space is χ_1-*compact* if every uncountable subset has a limit point.

Lemma. Let X *be a* T_1*-space which is* χ_1*-compact, let* $\mathcal{G}$ *be an open cover of* X*, let* $C \subseteq X$*. Then there is a countable subset* D *of* C *such that* $C \subseteq st(D,\mathcal{G})$.

Proof. Suppose false. Construct a subset $E = \{x_\alpha: 0 \leq \alpha < \omega_1\}$ of C such that for all $\alpha < \omega_1$, $x_\alpha \notin \cup_{\beta<\alpha} st(x_\beta,\mathcal{G})$.

Let p be a limit point of E, and let G be a member of $\mathcal{G}$ such that p belongs to G. Since p is a limit point of E and X is T_1, there exists α and β, $\alpha > \beta$, such that x_α and x_β belong to G. This contradicts $x_\alpha \notin \cup_{\beta<\alpha} st(x_\beta, \mathcal{G})$.

Theorem (Ginsburg and Woods). *Let X be an* χ_1*-compact space with a* G_δ*-diagonal. Then* $|X| \leq 2^\omega$.

Proof. Since X has a G_δ-diagonal, there is a countable sequence $\mathcal{G}_1$, $\mathcal{G}_2$, ... of open covers of X such that if p and q are any two distinct points in X, then for some $n < \omega$, $q \notin st(p, \mathcal{G}_n)$. (See [4].) Construct a sequence $\{E_\alpha: 0 \leq \alpha < \omega_1\}$ of subsets of X such that (1) $|E_\alpha| \leq 2^\omega$, $0 \leq \alpha < \omega_1$; (2) for $1 \leq \alpha < \omega_1$, if $\{D_n: n < \omega\}$ is a countable collection of countable subsets of $\cup_{\beta<\alpha} E_\beta$, and $\cup_{n=1}^\infty st(D_n, \mathcal{G}_n) \neq X$, then $E_\alpha - \cup_{n=1}^\infty st(D_n, \mathcal{G}_n) \neq \emptyset$.

Let $E = \cup_{\alpha<\omega_1} E_\alpha$; since $|E| \leq 2^\omega$, the proof is complete if we can show that E = X. Suppose not, and let $p \in X$, $p \notin E$. For each $n < \omega$ let $C_n = \{x: x \in E, p \notin st(x, \mathcal{G}_n)\}$; clearly $E = \cup_{n=1}^\infty C_n$. For each $n < \omega$, apply the Lemma to $\mathcal{G}_n$ and C_n: there is a countable subset D_n of C_n such that $C_n \subseteq st(D_n, \mathcal{G}_n)$. Note that $E \subseteq \cup_{n=1}^\infty st(D_n, \mathcal{G}_n)$ and $p \notin \cup_{n=1}^\infty st(D_n, \mathcal{G}_n)$. Now choose $\alpha < \omega_1$ such that $\cup_{n=1}^\infty D_n \subseteq \cup_{\beta<\alpha} E_\beta$. By (2), there is some q in E_α such that $q \notin \cup_{n=1}^\infty st(D_n, \mathcal{G}_n)$. This contradicts $E \subseteq \cup_{n=1}^\infty st(D_n, \mathcal{G}_n)$.

Survey of Other Inequalities

First we need some definitions. For a T_1 space X, the *point separating weight* of X, denoted psw(X), is the smallest infinite cardinal κ such that X has a separating open cover $\mathcal{S}$ with the property that every point of X is in at most κ members of $\mathcal{S}$. (The cover $\mathcal{S}$ is *separating* if given any two distinct points p and q in X, there is some S in $\mathcal{S}$ such that $p \in S$, $q \notin S$.) If $psw(X) = \omega$, we say that X has a *point-countable separating open cover.* The *extent* of X, denoted e(X), is the smallest infinite cardinal κ such that every closed, discrete subset of X has cardinality at most κ. (See [7], [13]). Note that for a T_1 space X, $e(X) = \omega$ if and only if X is χ_1-compact. The *weak Lindelöf number* of X, denoted wL(X), is the smallest infinite cardinal κ such that every open cover of X has a subcollection of cardinality $\leq \kappa$ whose union is dense in X. Note that $wL(X) \leq L(X)$ and $wL(X) \leq c(X)$. If $wL(X) = \omega$, we say that X is *weakly Lindelöf.*

Each of the following inequalities can be proved using the Pol-Šapirovskiǐ technique. (1) If X is T_1, then $|X| \leq 2^{e(X)psw(X)}$. (2) If X is T_1, then $|X| \leq psw(X)^{L(X)\psi(X)}$. (3) If X is normal and T_1, then $|X| \leq 2^{wL(X)\chi(X)}$. (See [3], [5], and [2] respectively.)

The countable version of (1) states that an χ_1-compact space with a point-countable separating open cover has cardinality at most 2^{ω}. (In fact, the number of compact subsets has cardinality at most 2^{ω}.) This result should be compared with the Ginsburg-Woods inequality. Two proofs of (1) are given in [3]; the first uses an intersection theorem of Erdös and Rado while the second proof uses the Pol-Šapirovskiǐ technique. (This second proof is also closely related to a construction due to M. E. Rudin [6].)

Arhangel'skiǐ has asked if every Lindelöf Hausdorff space with countable pseudo-character has cardinality at most 2^ω, and (2) gives a partial answer to this question. Specifically, the countable version of (2) states that a Lindelöf space having countable pseudo-character and point separating weight at most 2^ω has cardinality at most 2^ω.

The countable version of (3) states that a weakly Lindelöf first countable Hausdorff space which is also normal has cardinality at most 2^ω. Except for the normality assumption, inequality (3) unifies the two inequalities $|X| \le 2^{c(X)\chi(X)}$ and $|X| \le 2^{L(X)\chi(X)}$.

The reader is referred to [2], [5], [15], and [17] for additional inequalities in cardinal functions which can be proved using the Pol-Šapirovskiǐ technique.

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