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1. Introduction

According to the Nagata-Smirnov Theorem, a topological space is pseudometrizable iff the space is regular and the topology of the space has a σ-locally finite base. This theorem has quite generally been accepted as a "natural" characterization of pseudometrizable spaces in purely topological terms. This characterization has given rise to the definitions of several "generalized metric spaces". One of the most successful and natural generalizations of pseudometrizable spaces is obtained by substituting "network" for "base" in the condition of the Nagata-Smirnov Theorem (recall that a family $\mathcal{L}$ of subsets of a set X is a *network* for a topology τ on X provided that every set in τ is the union of some subfamily of $\mathcal{L}$). If we require that the network consists of closed sets, then we can omit the condition of regularity of the space (note that if $\mathcal{L}$ is a σ-locally finite network for the topology τ of a regular space (X,τ), then the family $\{Cl_\tau(L) | L \in \mathcal{L}\}$ is a σ-locally finite closed network for τ); accordingly, we make the following definition.

Definition 1.1. A topological space (X,τ) is a σ-*space* provided that τ has a σ-locally finite closed network.

Spaces with a σ-locally finite (in fact, σ-discrete) network were first considered by A. V. Arhangel'skij in [1]. These spaces were thoroughly studied by A. Okyama in [8];

this paper contains many of the basic results on σ-spaces. The term "σ-space" was introduced by F. Siwiec and J. Nagata in [10]; this paper also contains the important result that if the topology of a space has a σ-closure-preserving closed network, then the topology has a σ-discrete closed network. As a consequence of the last-mentioned result, the topology of a σ-space always admits a σ-discrete closed network. In [4], R. W. Heath and R. E. Hodel gave a characterization for σ-spaces that yields the result of Siwiec and Nagata as a corollary.

In this paper we characterize σ-spaces in terms of the existence of pseudometrizable topologies tied in certain ways with the original topologies of the spaces; in the end of the paper we show that for first countable spaces, one of the conditions in our characterization may or may not be redundant, depending on some extra set-theoretic assumptions. We also prove some results concerning the existence of closed pseudometrizable subspaces in σ-spaces.

Notation. The set $\{1,2,\ldots\}$ consisting of the natural numbers is denoted by $\mathbf{N}$. A sequence whose n^{th} term is x_n (for $n \in \mathbf{N}$) is denoted by $\langle x_n \rangle_{n=1}^{\infty}$.

If $\{K_n | n \in \mathbf{N}\}$ is a cover of a set X such that for each $n \in \mathbf{N}$, $K_n \subset K_{n+1}$, then we say that $\{K_n | n \in \mathbf{N}\}$ is an *increasing cover* of X, and we write $K_n \uparrow_n X$. If L is a family of subsets of X, then for each $A \subset X$, we denote by $(L)_A$ the family $\{L \in L | L \cap A \neq \phi\}$; if $A = \{x\}$, then we write $(L)_x$ instead of $(L)_A$.

When (X,τ) is a topological space, we denote by $\mathcal{J}_\sigma(\tau)$

the family consisting of all F_σ-subsets of X. Note that the space (X,τ) is perfect iff $\tau \subset F_\sigma(\tau)$.

For the meaning of concepts used without definition in this paper, see [3]; note, however, that in our terminology, regular spaces are not necessarily T_1.

2. σ-Spaces and Pseudometrizable Spaces

Most of the following results deal with the existence of finer pseudometrizable topologies on the ground-set of a σ-space. The discrete topology is, of course, always a finer pseudometrizable topology; to exclude this trivial case, we only consider topologies whose sets are F_σ-sets with respect to the original topology. Hence, for a topological space (X,τ) we consider the existence of a pseudometrizable topology π such that $\tau \subset \pi \subset F_\sigma(\tau)$.

It is a simple matter to show that for a σ-space (X,τ), there always exists a pseudometrizable topology π on X such that $\tau \subset \pi \subset F_\sigma(\tau)$; in the next section we attend to the question whether the existence of such a topology π is sufficient to make (X,τ) a σ-space.

Lemma 2.1. *Let* (X,τ) *be a σ-space, and let* F *be a σ-locally finite closed network for* τ. *Denote by* π *the topology that has* F *as a subbase. Then* π *is a pseudometrizable topology, and* $\tau \subset \pi \subset F_\sigma(\tau)$.

Proof. Denote by F' the family consisting of all finite intersections of sets from the family F. Since F is σ-locally finite and closed in (X,τ), so is F'. The family F' is a base for the topology π and a network for the topology τ; consequently $\tau \subset \pi$. It follows that F' is σ-locally finite and closed with respect to the topology π. Since π has a closed base, π is a regular topology. By the Nagata-Smirnov Theorem, the space (X,π) is pseudometrizable.

We already know that $\tau \subset \pi$. To see that $\pi \subset F_\sigma(\tau)$, we only have to observe that any set belonging to π is the union of some subfamily of the family F' and that any such union belongs to the family $F_\sigma(\tau)$ since F' is σ-locally finite and closed with respect to τ.

Remark 2.1.1. In the above lemma, if $F = \bigcup_{n\in \mathbf{N}} F_n$ with each F_n locally finite, then we obtain a pseudometric d compatible with the topology π by setting $d(x,y) = \inf\{\frac{1}{n} | n \in \mathbf{N}$ and $(\bigcup_{k<n} F_k)_x = (\bigcup_{k<n} F_k)_y\}$ for all x,y in X.

Remark 2.1.2. In Lemma 2.1, if F is a semicompact family (that is, if $\cap K \neq \phi$ whenever K is a subfamily of F with the finite intersection property), then the space (X,π) is completely pseudometrizable.

We now use Lemma 2.1 to obtain a characterization of σ-spaces in terms of the existence of pseudometrizable topologies.

Proposition 2.2. *A topological space* (X,τ) *is a* σ-*space iff there exists a pseudometrizable topology* π *on* X *such that* $\tau \subset \pi \subset F_\sigma(\tau)$ *and every locally finite family of subsets of the space* (X,π) *has a refinement that is* σ-*locally finite in the space* (X,τ).

Proof. Necessity. Assume that (X,τ) is a σ-space. Let F be a σ-locally finite closed network for τ. We may assume that F is a base for a topology π on X. By Lemma

2.1, the topology π is pseudometrizable and $\tau \subset \pi \subset F_\sigma(\tau)$. Let L be a locally finite family of subsets of the space (X,π). Let $K = \{F \in F | \text{the family } (L)_F \text{ is finite}\}$. Since F is a base for π and the family L is locally finite with respect to π, the family K covers X. It is easily seen that the family $N = \{K \cap L | K \in K \text{ and } L \in (L)_K\}$ is a refinement of L and that this family is σ-locally finite with respect to τ.

Sufficiency. Assume that there exists a topology π on X which satisfies the conditions stated in the proposition. To show that (X,τ) is a σ-space, let $B = \bigcup_{n \in \mathbf{N}} B_n$ be a base for π such that for each $n \in \mathbf{N}$, the family B_n is discrete with respect to π. For each $n \in \mathbf{N}$, there exists a refinement N_n of the family B_n such that N_n is σ-locally finite with respect to τ. Since $\pi \subset F_\sigma(\tau)$, there exist closed sets $F_k(B)$, $k \in \mathbf{N}$ and $B \in B$, in the space (X,τ) such that for each $B \in B$, we have $\bigcup_{k \in \mathbf{N}} F_k(B) = B$. For all $n \in \mathbf{N}$ and $k \in \mathbf{N}$, let $F_{n,k} = \{Cl_\tau(N) \cap F_k(B) | N \in N_n \text{ and } N \subset B \in B_n\}$ and note that this family is σ-locally finite and closed with respect to τ. It is easily seen that for every $n \in \mathbf{N}$ and for each $B \in B_n$, we have $B = \bigcup\{F \in \bigcup_{k \in \mathbf{N}} F_{n,k} | F \subset B\}$. It follows that the family $F = \bigcup\{F_{n,k} | n \in \mathbf{N} \text{ and } k \in \mathbf{N}\}$ is a network for π; since $\tau \subset \pi$, the family F is also a network for τ. This completes the proof since F is a σ-locally finite and closed with respect to τ.

Remark 2.2.1. The result of the proposition remains true if "locally finite" is replaced by "discrete" in the proposition (once or twice).

The class of σ-spaces is considerably larger than the class of pseudometrizable spaces; nevertheless, the above result shows that large portions of the theory of pseudometrizable spaces, e.g. much of the Borel-theory for pseudometrizable spaces, carry over to the setting of σ-spaces.

The remaining results of this section deal with the existence of closed pseudometrizable subspaces in σ-spaces.

Proposition 2.3. *Let* (X,τ) *be a* σ*-space. Then there are increasing closed covers* $\{F_{n,k} | k \in \mathbf{N}\}$, $n \in \mathbf{N}$, *of the space* (X,τ) *and there is a pseudometrizable topology* π *on* X *such that for every sequence* $\langle k(n)\rangle_{n=1}^{\infty}$ *of natural numbers, the topologies* τ *and* π *agree on the set* $\bigcap_{n\in\mathbf{N}} F_{n,k(n)}$.

Proof. Let $\mathcal{F} = \bigcup_{n\in\mathbf{N}} \mathcal{F}_n$ be a network for τ such that each $\mathcal{F}_n$ is a discrete family of closed subsets of (X,τ). For every $n \in \mathbf{N}$, let $\langle K_{n,k}\rangle_{k=1}^{\infty}$ be a sequence of closed subsets of (X,τ) such that $K_{n,k} \uparrow_k X \sim \bigcup \mathcal{F}_n$. For all $n \in \mathbf{N}$ and $k \in \mathbf{N}$, let $F_{n,k} = K_{n,k} \cup (\bigcup \mathcal{F}_n)$ and note that every set of the family $\mathcal{F}_n$ is open in the topology inherited by $F_{n,k}$ from τ. For every $n \in \mathbf{N}$, the family $\{F_{n,k} | k \in \mathbf{N}\}$ is an increasing closed cover of the space (X,τ).

Denote by π the topology generated on X by the family $\mathcal{F}$. By Lemma 2.1, the topology π is pseudometrizable and $\tau \subset \pi$. To complete the proof, let $\langle k(n)\rangle_{n=1}^{\infty}$ be a sequence of natural numbers, and let $S = \bigcap_{n\in\mathbf{N}} F_{n,k(n)}$. Denote by τ' and π', respectively, the restrictions of τ and π to S. Since $\tau \subset \pi$, we have $\tau' \subset \pi'$. On the other hand, for each $n \in \mathbf{N}$, and for every $F \in \mathcal{F}_n$, we have $F \cap S \in \tau'$ since $S \subset F_{n,k(n)}$ and F is open in the topology inherited by

$F_{n,k(n)}$ from τ. Consequently, $\{F \cap S | F \in \mathcal{F}\} \subset \tau'$. Since $\mathcal{F}$ is a subbase for π, we see that $\pi' \subset \tau'$.

Corollary 2.3.1. ([2]) *Every σ-space is the union of a family of not more than* $2^{\aleph_0}$ *closed pseudometrizable subspaces.*

Corollary 2.3.2. *If* μ *is a finite measure defined for all Borel-subsets of a σ-space* (X,τ), *then for every* $\varepsilon > 0$, *there is a closed pseudometrizable subspace* S *of* X *such that* $\mu(X \sim S) < \varepsilon$.

Proof. Let the increasing closed covers $\{F_{n,k} | k \in \mathbf{N}\}$ of X satisfy the conclusion of Proposition 2.3. Let ε be a positive real number. For each $n \in \mathbf{N}$, since $F_{n,k} \uparrow_k X$, there exists $k(n) \in \mathbf{N}$ such that $\mu(X \sim F_{n,k(n)}) < 2^{-n} \cdot \varepsilon$. The set $S = \bigcap_{n \in \mathbf{N}} F_{n,k(n)}$ has the properties required in the corollary.

Note that there are σ-spaces (X,τ) which admit finite Borel-measures such that for no pseudometrizable $A \subset X$ does the set $X \sim A$ have zero measure (e.g. any countable T_1-space that is not first countable). For meager sets we do get the following result: every σ-space contains a pseudometrizable subspace whose complement is meager in the space [if (X,τ) is a σ-space, and if $\mathcal{F}$ is a σ-discrete closed network for τ, then the subspace $X \sim \bigcup\{\partial F | F \in \mathcal{F}\}$ has the required properties (note that for any discrete family $\mathcal{L}$, the sets $\bigcup\{\partial L | L \in \mathcal{L}\}$ and $\partial \bigcup \mathcal{L}$ coincide)].

3. Pseudometrizable Topologies and σ-Spaces

In Proposition 2.2 we characterized σ-spaces in terms of the existence of certain pseudometrizable topologies on the ground-set of a space; in the following we try to determine whether the result of that proposition can be improved. More precisely, we try to answer the following question.

Question 3.1. Is it true for every topological space (X,τ) that (X,τ) is a σ-space if there exists a pseudometrizable topology π on X such that $\tau \subset \pi \subset F_\sigma(\tau)$?

Putting a further restriction on the topology π, we get an affirmative answer to the above question.

Proposition 3.2. *The topology* τ *of a space* (X,τ) *has a countable closed network iff there exists a separable pseudometrizable topology* π *on* X *such that* $\tau \subset \pi \subset F_\sigma(\tau)$.

Proof. Necessity follows from Lemma 2.1. Sufficiency is obvious.

Note that for regular spaces, Proposition 3.2 characterizes cosmic spaces (see [6]).

In general, we cannot hope to obtain an affirmative answer to Question 3.1 if we stay within the framework of ordinary set-theory (ZFC). Recall that a Q-set is an uncountable set $A \subset \mathbf{R}$ such that every subset of A is an F_σ-set relative to the Euclidean topology on A. It is known (see e.g. [9]) that if ZFC is consistent, then so is (ZFC + there is a Q-set); assuming the existence of a Q-set, it is easy to find a space (X,τ) which provides a negative answer to Question 3.1.

Example 3.3. *Let* $A \subset \mathbf{R}$ *be a* Q-*set. Then there are* *two topologies* τ *and* π *on* A *such that the following conditions are satisfied:*

(i) (A,τ) *is a first countable regular Lindelöf-space*

(ii) (A,τ) *is not a* σ*-space*

(iii) (A,π) *is a metrizable space*

(iv) $\tau \subset \pi \subset F_\sigma(\tau)$.

Proof. Let π be the discrete topology on A, and let τ be the topology that A inherits from the Sorgenfrey line. Clearly, conditions (i) and (iii) hold, and $\tau \subset \pi$. Since every subset of A is an F_σ-set in the Euclidean topology of A and since τ is finer than the Euclidean topology, we have $F_\sigma(\tau) = P(A)$; consequently, condition (iv) is satisfied. It remains to show that (A,τ) is not a σ-space. Assume on the contrary that τ has a σ-locally finite network, and let F be such a network. For each $a \in A$, the set $A \cap (\leftarrow,a]$ is open in A and hence there exists $F_a \in F$ such that $a \in F_a \subset (\leftarrow,a]$. For any two distinct elements a and b of A, the sets F_a and F_b are distinct. Since the set A is uncountable, it follows that the family F is uncountable; this, however, is a contradiction since any σ-locally finite family of subsets of a Lindelöf-space is countable.

Remark 3.3.1. If $|A| = \omega_1$, and if we take τ to be the supremum of the Euclidean topology on A and the order topology obtained by identifying A with the well-ordered set ω_1, then conditions (ii), (iii) and (iv) of the above example still hold and (i) can be replaced by the following: (A,τ) is a regular first countable space that is not subparacompact.

Our next result shows that under the Product Measure Extension Axiom (PMEA), Question 3.1 does have an affirmative answer in the class of first countable spaces; this result and Example 3.3 make it seem likely that at least for first countable spaces, Question 3.1 cannot be settled using only the axioms of ZFC (for a discussion on the consistency of PMEA relative to that of some other axioms, as well as for other background on PMEA, see [7]).

First, an auxiliary result. To state the result, we use the following terminology: a family L of subsets of a space (X,τ) is F_σ-*additive* in (X,τ) if $\cup L' \in F_\sigma(\tau)$ for every $L' \subset L$.

Lemma 3.4. *Let* (X,τ) *be a topological space such that every* F_σ*-additive partition of* (X,τ) *has a* σ*-discrete refinement. Assume that there exists a pseudometrizable topology* π *on* X *such that* $\tau \subset \pi \subset F_\sigma(\tau)$. *Then* (X,τ) *is a* σ*-space.*

Proof. The proof of sufficiency for Proposition 2.2 shows that (X,τ) is a σ-space provided that every discrete family of open subsets of the space (X,π) has a refinement that is σ-discrete in the space (X,τ). Let U be a discrete family of open sets in (X,π). Since $\pi \subset F_\sigma(\tau)$, we have $\cup U' \in F_\sigma(\tau)$ for each $U' \subset U$. In particular, there are closed subsets F_n, $n \in \mathbf{N}$, of the space (X,τ) such that $\bigcup_{n\in\mathbf{N}} F_n = \cup U$. For each $n \in \mathbf{N}$, let $L_n = \{X \sim F_n\} \cup \{U \cap F_n \mid U \in U\}$ and note that L_n is a partition of X. For each $n \in \mathbf{N}$, the family L_n is F_σ-additive in (X,τ) [note that $X \sim F_n \in F_\sigma(\tau)$ since $\tau \subset F_\sigma(\tau)$]. For each $n \in \mathbf{N}$, let M_n be a

σ-discrete refinement of $\mathcal{L}_n$ in the space (X,τ), and let $\mathcal{M}'_n = \{M \in \mathcal{M}_n | M \subset F_n\}$. Then the family $\bigcup_{n\in\mathbf{N}} \mathcal{M}'_n$ is σ-discrete in the space (X,τ) and this family is a refinement of the family $\mathcal{U}$.

Note that a set $A \subset \mathbf{R}$ is a Q-set iff the partition $\{\{a\}|a \in A\}$ is F_σ-additive with respect to the Euclidean topology on A. Consequently, if there exists a Q-set, then there exists a metrizable space and an F_σ-additive partition of the space such that the partition has no σ-discrete refinement. In our last result we show that if PMEA holds, then F_σ-additive partitions of σ-spaces always have σ-discrete refinements.

Proposition 3.5. (PMEA) *Let* (X,τ) *be a topological space such that there exists a pseudometrizable topology* π *on* X *such that* $\tau \subset \pi \subset F_\sigma(\tau)$.

A. *If* (X,τ) *is weakly first countable, then* (X,τ) *is* σ-*space*

B. *The following conditions on* (X,τ) *are mutually equivalent:*

(i) (X,τ) *is a* σ-*space*

(ii) (X,τ) *is semi-stratifiable*

(iii) Every F_σ-*additive partition of* (X,τ) *has a* σ-*discrete refinement.*

Proof. In [5], it is shown that if PMEA holds and if (X,τ) is either weakly first countable or semi-stratifiable, then condition (iii) holds. By Lemma 3.4, if condition (iii) holds, then (X,τ) is a σ-space. This completes the proof since every σ-space is semi-stratifiable.

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