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## AN EXAMPLE CONCERNING THE CANCELLABILITY OF CYCLES

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## AN EXAMPLE CONCERNING THE CANCELLABILITY OF CYCLES

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### 1. Introduction

Throughout this note, *space* means compact metric space and *map* means continuous function. If  $X$  is a space and  $G$  is an Abelian group, then  $H_n(X, G)$  denotes the  $n$ -dimensional Vietoris-Čech homology group of  $X$  with coefficients in  $G$  (see for instance [2], p. 36).

Let  $X$  and  $Y$  be spaces and let  $\underline{f}$  be a fundamental sequence from  $X$  to  $Y$  (see [2] as a reference to this and other notions of shape theory). The homomorphism from  $H_n(X, G)$  into  $H_n(Y, G)$  induced by  $\underline{f}$  is denoted by  $\underline{f}_{*n}$ .

Let  $A$  be a subset of  $H_n(X, G)$  and let  $a$  be a  $k$ -dimensional cycle. Following K. Borsuk [1], we say that the cycle  $a$  is *cancellable rel. A* provided that there exists a fundamental sequence  $\underline{f}$  from  $X$  to  $X$  such that  $\underline{f}_{*n}(x) = (x)$  for every  $x \in A$  and  $\underline{f}_{*k}(a) = 0$ . Each  $\underline{f}$  with these properties is called a *cancellation of a rel. A*.

The notion of cancellability of cycles was used in [1] to establish some facts about simplicity of certain shapes. The aim of this note is to solve the following problem posed in [1]: Let  $A$  be a subset of  $H_n(X, G)$  and  $k \neq n$ . Is the set of cycles  $a \in H_k(X, G)$  cancellable rel.  $A$  always a subgroup of  $H_k(X, G)$ ? We construct an example solving this problem in the negative.

## 2. Preliminaries

Let  $P$  denote the projective plane and let  $X$  be the infinite countable "bouquet" of projective planes; that is, let  $X = \bigcup_{i=1}^{\infty} P_i$  where each set  $P_i$  is homeomorphic to  $P$ , the sequence of diameters of the  $P_i$ 's converges to zero, and, for some  $x_0 \in X$ ,  $P_i \cap P_j = \{x_0\}$  whenever  $i \neq j$ . Assume the following notation: let  $X_n = \bigcup_{i=1}^n P_i$  (for  $n = 1, 2, 3, \dots$ ), let  $r_{ni}$  be the retraction of  $X_n$  onto  $P_i$  (for  $i \leq n$ ) defined by the formula

$$r_{ni}(x) = \begin{cases} x & \text{if } x \in P_i \\ x_0 & \text{if } x \in X_n \setminus P_i \end{cases}$$

and let  $s_{in}$  be the inclusion map of  $P_i$  into  $X_n$  ( $i \leq n$ ). Let  $Z$  denote the group of integers and let  $Z_2$  denote the group of integers reduced modulo 2.

Notice that the groups  $H_1(P, Z)$ ,  $H_1(P, Z_2)$  and  $H_2(P, Z_2)$  are all isomorphic to  $Z_2$ , and therefore the groups  $H_1(X_n, Z)$ ,  $H_1(X_n, Z_2)$  and  $H_2(X_n, Z_2)$  are all isomorphic to the product of  $n$  copies of  $Z_2$ .

If  $\phi$  is a homomorphism between two groups, we say that  $\phi$  is *trivial* and we write  $\phi = 0$  provided that  $\phi(x) = 0$  for every  $x$ . Otherwise we say that  $\phi$  is *non-trivial* and we write  $\phi \neq 0$ .

## 3. Some Homological Properties of the Projective Plane and

### the Bouquets of Projective Planes

*Lemma 1.* For every map  $f: P \rightarrow P$ , the homomorphisms  $f_{*1}: H_1(P, Z_2) \rightarrow H_1(P, Z_2)$  and  $f_{*2}: H_2(P, Z_2) \rightarrow H_2(P, Z_2)$  are either both trivial or both non-trivial.

*Proof.* From the Universal Coefficient Theorem (see [3], p. 160) it follows that there exists a functorial

isomorphism  $\alpha: H_2(P, Z_2) \rightarrow \text{Tor}(H_1(P, Z), Z_2)$ , thus we get the following commutative diagram

$$\begin{array}{ccc} H_2(P, Z_2) & \xrightarrow{\alpha} & \text{Tor}(H_1(P, Z), Z_2) \\ \downarrow f_{*2} & & \downarrow \text{Tor}(f_{*1}, Z_2) \\ H_2(P, Z_2) & \xrightarrow{\alpha} & \text{Tor}(H_1(P, Z), Z_2) \end{array}$$

This yields that  $f_{*2} = 0$  if and only if  $\text{Tor}(f_{*1}, Z_2) = 0$ .

Now, notice that  $\text{Tor}(H_1(P, Z), Z_2) \approx \text{Tor}(Z_2, Z_2) \approx Z_2 \approx H_1(P, Z_2)$ .

Hence,  $f_{*1}$  is either 0 or the identity (these are the only two homomorphisms on  $Z_2$ ). Since  $\text{Tor}(0, Z_2) = 0$  and  $\text{Tor}(\cdot, Z_2)$  is a functor, we get  $\text{Tor}(f_{*1}, Z_2) = 0$  if and only if  $f_{*1} = 0$ , which concludes the proof.

*Lemma 2. Let  $f$  be a map from  $X_n$  to  $X_k$  and denote by  $f_{(ij)}$  the composition  $r_{kj} f s_{in}$ ,  $f_{(ij)}: P_i \rightarrow P_j$ . Then the homomorphism  $f_{*q}: H_q(X_n, G) \rightarrow H_q(X_k, G)$  induced by  $f$  is uniquely determined by the collection  $\{f_{(ij)*q}\}$  of homomorphisms induced by the maps  $f_{(ij)}$ , where  $i = 1, 2, \dots, n$ ,  $j = 1, 2, \dots, k$ .*

We omit the quite simple and natural routine proof of the above lemma.

As we noticed before, the groups  $H_1(X_n, Z_2)$  and  $H_2(X_n, Z_2)$  are isomorphic, namely  $H_1(X_n, Z_2) = \bigoplus_{i=1}^n H_1(P_i, Z_2) \approx \bigoplus_{i=1}^n Z_2$  and  $H_2(X_n, Z_2) = \bigoplus_{i=1}^n H_2(P_i, Z_2) \approx \bigoplus_{i=1}^n Z_2$ .

Let  $h: H_1(X_n, Z_2) \rightarrow H_2(X_n, Z_2)$  be the isomorphism which carries the non-zero element  $e_i$  of  $H_1(P_i, Z_2)$  onto the non-zero element  $E_i$  of  $H_2(P_i, Z_2)$ , for every  $i = 1, 2, \dots, n$ ,  $n = 1, 2, \dots$ .

Lemmas 1 and 2 yield immediately that, for every map  $f: X_k \rightarrow X_n$ , the diagram

$$\begin{array}{ccc}
 H_1(X_k, Z_2) & \xrightarrow{f_{*1}} & H_1(X_n, Z_2) \\
 \downarrow h & & \downarrow h \\
 H_2(X_k, Z_2) & \xrightarrow{f_{*2}} & H_2(X_n, Z_2)
 \end{array}$$

is commutative, for every  $k = 1, 2, \dots$  and every  $n = 1, 2, \dots$ .

**The Example**

Since the inclusion map of  $P_i$  into  $X$  induces a monomorphism of the homology groups in every dimension, we may assume that  $e_i \in H_1(X, Z_2)$  and  $E_i \in H_2(X, Z_2)$ , for every  $i = 1, 2, 3, \dots$ . Let  $a$  be the element of  $H_1(X, Z_2)$  which can be written as  $\sum_{i=1}^{\infty} e_{2i-1}$ , similarly, let  $b = \sum_{i=1}^{\infty} e_{2i}$ , and let  $c = \sum_{i=1}^{\infty} E_i$ . Both  $a$  and  $b$  are cancellable rel.  $\{c\}$ . Indeed, the map of  $X$  onto  $X$  which collapses each  $P_{2i-1}$  into  $x_0$  and maps  $P_{2i}$  homeomorphically onto  $P_i$  keeping  $x_0$  fixed, is a cancellation of  $a$  rel.  $\{c\}$ . A cancellation of  $b$  rel.  $\{c\}$  is constructed in the same fashion. However, as we prove below,  $a + b$  is not cancellable rel.  $\{c\}$ , which solves the problem. Assume then that  $\underline{f}$  is a fundamental sequence from  $X$  to  $X$  with  $\underline{f}_{*1}(a + b) = 0$ . Our claim is that  $\underline{f}_{*2}(c) = 0$ .

For every positive number  $\epsilon$  there exists an integer  $n$  such that, for every  $i > n$ ,  $\text{diam } P_i < \epsilon$ . The map  $r_n: X \rightarrow X_n$  which collapses  $X \setminus X_n$  into  $x_0$  and leaves every point of  $X_n$  fixed is an  $\epsilon$ -displacement. Since  $\epsilon$  is arbitrary, it is sufficient to prove that  $(\underline{r}_n \underline{f})_{*2}(c) = 0$ , where  $\underline{r}_n$  is the fundamental sequence from  $X$  to  $X_n$ , generated by  $r_n$ .

Since  $X_n$  is an ANR-space, the fundamental class  $[\underline{r}_n \underline{f}]$  is generated by a map, say  $g: X \rightarrow X_n$ . Obviously,  $g_{*1}(a + b) = 0$  and it is sufficient to prove that  $g_{*2}(c) = 0$ . Again, since  $X_n$  is an ANR-space, we may assume that, for some integer

$k$ ,  $g$  maps the set  $X \setminus X_k$  into the point  $x_0$ , replacing the map  $g$  by a map homotopic to  $g$ , if necessary. In other words, we assume that  $g = \bar{g} r_k$ , where  $\bar{g}$  is a map from  $X_k$  into  $X_n$  and  $r_k: X \rightarrow X_k$  is a retraction which maps  $X \setminus X_k$  into  $x_0$ . By our assumption, we get  $\bar{g}_{*1} r_{k*1}(a + b) = 0$  which means  $\bar{g}_{*1}(e_1 + e_2 + \dots + e_k) = 0$ . Applying diagram (\*) from Section 3, we get  $\bar{g}_{*2}(E_1 + E_2 + \dots + E_k) = 0$ . This yields  $\bar{g}_{*2} r_{k*2}(c) = 0$  and  $g_{*2}(c) = 0$ , which completes the proof.

### References

- [1] K. Borsuk, *On a new shape invariant*, Topology Proceedings 1 (1976), 1-9.
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- [3] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton, 1952.

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