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1. Introduction

Throughout this note, *space* means compact metric space and *map* means continuous function. If X is a space and G is an Abelian group, then $H_n(X,G)$ denotes the n-dimensional Vietoris-Čech homology group of X with coefficients in G (see for instance [2], p. 36).

Let X and Y be spaces and let $\underline{f}$ be a fundamental sequence from X to Y (see [2] as a reference to this and other notions of shape theory). The homomorphism from $H_n(X,G)$ into $H_n(Y,G)$ induced by $\underline{f}$ is denoted by $\underline{f}_{*n}$.

Let A be a subset of $H_n(X,G)$ and let a be a k-dimensional cycle. Following K. Borsuk [1], we say that the cycle a is *cancellable rel.* A provided that there exists a fundamental sequence $\underline{f}$ from X to X such that $f_{*n}(x) = (x)$ for every $x \in A$ and $\underline{f}_{*k}(a) = 0$. Each $\underline{f}$ with these properties is called a *cancellation of* a *rel.* A.

The notion of cancellability of cycles was used in [1] to establish some facts about simplicity of certain shapes. The aim of this note is to solve the following problem posed in [1]: Let A be a subset of $H_n(X,G)$ and $k \neq n$. Is the set of cycles $a \in H_k(X,G)$ cancellable rel. A always a subgroup of $H_k(X,G)$? We construct an example solving this problem in the negative.

2. Preliminaries

Let P denote the projective plane and let X be the infinite countable "bouquet" of projective planes; that is, let $X = \cup_{i=1}^{\infty} P_i$ where each set P_i is homeomorphic to P, the sequence of diameters of the P_i's converges to zero, and, for some $x_0 \in X$, $P_i \cap P_j = \{x_0\}$ whenever $i \neq j$. Assume the following notation: let $X_n = \cup_{i=1}^{n} P_i$ (for $n = 1,2,3,\cdots$), let r_{ni} be the retraction of X_n onto P_i (for $i \leq n$) defined by the formula

$$r_{ni}(x) = \begin{cases} x \text{ if } x \in P_i \\ x_0 \text{ if } x \in X_n \backslash P_i \end{cases}$$

and let s_{in} be the inclusion map of P_i into $X_n (i \leq n)$. Let Z denote the group of integers and let Z_2 denote the group of integers reduced modulo 2.

Notice that the groups $H_1(P,Z)$, $H_1(P,Z_2)$ and $H_2(P,Z_2)$ are all isomorphic to Z_2, and therefore the groups $H_1(X_n,Z)$, $H_1(X_n,Z_2)$ and $H_2(X_n,Z_2)$ are all isomorphic to the product of n copies of Z_2.

If ϕ is a homomorphism between two groups, we say that ϕ is *trivial* and we write $\phi = 0$ provided that $\phi(x) = 0$ for every x. Otherwise we say that ϕ is non-trivial and we write $\phi \neq 0$.

3. Some Homological Properties of the Projective Plane and the Bouquets of Projective Planes

Lemma 1. *For every map* $f: P \to P$, *the homomorphisms* $f_{*1}: H_1(P,Z_2) \to H_1(P,Z_2)$ *and* $f_{*2}: H_2(P,Z_2) \to H_2(P,Z_2)$ *are either both trivial or both non-trivial.*

Proof. From the Universal Coefficient Theorem (see [3], p. 160) it follows that there exists a functorial isomorphism $\alpha: H_2(P,Z_2) \to Tor(H_1(P,Z),Z_2)$, thus we get the following cummutative diagram

$$\begin{array}{ccc} H_2(P,Z_2) & \xrightarrow{\alpha} & Tor(H_1(P,Z),Z_2) \\ \downarrow f_{*2} & & \downarrow Tor(f_{*1},Z_2) \\ H_2(P,Z_2) & \xrightarrow{\alpha} & Tor(H_1(P,Z),Z_2) \end{array}$$

This yields that $f_{*2} = 0$ if and only if $Tor(f_{*1},Z_2) = 0$. Now, notice that $Tor(H_1(P,Z),Z_2) \approx Tor(Z_2,Z_2) \approx Z_2 \approx H_1(P,Z_2)$. Hence, f_{*1} is either 0 or the identity (these are the only two homomorphisms on Z_2). Since $Tor(0,Z_2) = 0$ and $Tor(\cdot,Z_2)$ is a functor, we get $Tor(f_{*1},Z_2) = 0$ if and only if $f_{*1} = 0$, which concludes the proof.

Lemma 2. *Let* f *be a map from* X_n *to* X_k *and denote by* $f_{(ij)}$ *the composition* $r_{kj}fs_{in}$, $f_{(ij)}: P_i \to P_j$. *Then the homomorphism* $f_{*q}: H_q(X_n,G) \to H_q(X_k,G)$ *induced by* f *is uniquely determined by the collection* $\{f_{(ij)*q}\}$ *of homomorphisms induced by the maps* $f_{(ij)}$, *where* $i = 1,2,\cdots,n$, $j = 1,2,\cdots,k$.

We omit the quite simple and natural routine proof of the above lemma.

As we noticed before, the groups $H_1(X_n,Z_2)$ and $H_2(X_n,Z_2)$ are isomorphic, namely $H_1(X_n,Z_2) = \oplus_{i=1}^n H_1(P_i,Z_2) \approx \oplus_{i=1}^n Z_2$ and $H_2(X_n,Z_2) = \oplus_{i=1}^n H_2(P_i,Z_2) \approx \oplus_{i=1}^n Z_2$.
Let $h: H_1(X_n,Z_2) \to H_2(X_n,Z_2)$ be the isomorphism which carries the non-zero element e_i of $H_1(P_i,Z_2)$ onto the non-zero element E_i of $H_2(P_i,Z_2)$, for every $i = 1,2,\cdots,n$, $n = 1,2,\cdots$.

Lemmas 1 and 2 yield immediately that, for every map $f: X_k \to X_n$, the diagram

$$(*) \qquad \begin{array}{ccc} H_1(X_k, Z_2) & \xrightarrow{f_{*1}} & H_1(X_n, Z_2) \\ \downarrow h & & \downarrow h \\ H_2(X_k, Z_2) & \xrightarrow{f_{*2}} & H_2(X_n, Z_2) \end{array}$$

is commutative, for every $k = 1,2,\cdots$ and every $n = 1,2,\cdots$.

The Example

Since the inclusion map of P_i into X induces a monomorphism of the homology groups in every dimension, we may assume that $e_i \in H_1(X,Z_2)$ and $E_i \in H_2(X,Z_2)$, for every $i = 1,2,3,\cdots$. Let a be the element of $H_1(X,Z_2)$ which can be written as $\sum_{i=1}^{\infty} e_{2i-1}$, similarly, let $b = \sum_{i=1}^{\infty} e_{2i}$, and let $c = \sum_{i=1}^{\infty} E_i$. Both a and b are cancellable rel. $\{c\}$. Indeed, the map of X onto X which collapses each P_{2i-1} into x_0 and maps P_{2i} homeomorphically onto P_i keeping x_0 fixed, is a cancellation of a rel. $\{c\}$. A cancellation of b rel. $\{c\}$ is constructed in the same fashion. However, as we prove below, $a + b$ is not cancellable rel. $\{c\}$, which solves the problem. Assume then that $\underline{f}$ is a fundamental sequence from X to X with $\underline{f}_{*1}(a + b) = 0$. Our claim is that $\underline{f}_{*2}(c) = 0$.

For every positive number ε there exists an integer n such that, for every $i > n$, diam $P_i < \varepsilon$. The map $r_n\colon X \to X_n$ which collapses $X \setminus X_n$ into x_0 and leaves every point of X_n fixed is an ε-displacement. Since ε is arbitrary, it is sufficient to prove that $(\underline{r}_n\underline{f})_{*2}(c) = 0$, where $\underline{r}_n$ is the fundamental sequence from X to X_n, generated by r_n.

Since X_n is an ANR-space, the fundamental class $[\underline{r}_n\underline{f}]$ is generated by a map, say $g\colon X \to X_n$. Obviously, $g_{*1}(a + b) = 0$ and it is sufficient to prove that $g_{*2}(c) = 0$. Again, since X_n is an ANR-space, we may assume that, for some integer k, g maps the set $X \setminus X_k$ into the point x_0, replacing the map g by a map homotopic to g, if necessary. In other words, we assume that $g = \bar{g}\, r_k$, where $\bar{g}$ is a map from X_k into X_n and $r_k: X \to X_k$ is a retraction which maps $X \setminus X_k$ into x_0. By our assumption, we get $\bar{g}_{*1} r_{k*1}(a + b) = 0$ which means $\bar{g}_{*1}(e_1 + e_2 + \cdots + e_k) = 0$. Applying diagram (*) from Section 3, we get $\bar{g}_{*2}(E_1 + E_2 + \cdots + E_k) = 0$. This yields $\bar{g}_{*2} r_{k*2}(c) = 0$ and $g_{*2}(c) = 0$, which completes the proof.

References

[1] K. Borsuk, *On a new shape invariant*, Topology Proceedings 1 (1976), 1-9.

[2] ________, *Theory of shape*, Monografie Matematyczne 59, Warszawa, 1975.

[3] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton, 1952.

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