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1. Introduction

Let $f: X \to X$ be a continuous map on a compact connected ANR X into itself. The Nielsen fixed point theorem says that every map $g: X \to X$ homotopic to f has at least N(f), the Nielsen number of f, fixed points. Thus the Nielsen fixed point theorem is more powerful than the Lefschetz fixed point theorem which only ensures the existence of a single fixed point if the Lefschetz nunber of f, $L(f) \neq 0$. If $L(f) = 0$ then no conclusion can be drawn as to whether or not there exists a fixed point. In fact, recently McCord [9] has constructed a homeomorphism h on a manifold M^n onto itself such that $L(h) = 0$ and $N(h) \geq 2$ in all dimensions n.

The purpose of this paper is to introduce recent developments of the product theorems for the Nielsen numbers of a fiber-preserving map. For the convenience of readers we introduce the Nielsen fixed point theorems from [1] and [4] in section 2. In the third section we study the Jiang's contribution [8] to estimate the Nielsen numbers of a continuous map. In the fourth section we cover some recent results dealing with the product theorems of the Nielsen number of a fiber-preserving map. In particular, we show that the recent product theorem of Giessmann [6] reduces to that of Pak [11].

There are a couple of excellent articles on the Nielsen fixed point theorems by Brown [1] and Fadell [4]. Therefore, we suggest to the readers to these publications for more materials and details with regard to the sections 2 and 3.

2. The Nielsen Fixed Point Theorems

Let $f: X \to X$ be a continuous map on a compact connected ANR X into itself. Let $\Phi(f) = \{x \in X | f(x) = x\}$ be the set of all fixed points of f. Any two elements $x, y \in \Phi(f)$ are said to be f-equivalent if there is a path $C: I \to X$ such that $C(0) = x$, $C(1) = y$, and $C \simeq f(C)$ (homotopic). This relation is an equivalence relation in $\Phi(f)$ and divides $\Phi(f)$ into finite number of equivalence classes F. If the fixed point index $i(F) \neq 0$ then F is called essential Nielsen fixed point class and if $i(F) = 0$ then F is called inessential. It is known that if $i(F) = 0$ then we could remove the fixed points in F by a map g homotopic to f in many cases.

Definition. The Nielsen number N(f) of a map f is defined to be the number of essential fixed point classes of f.

Theorem [Nielsen]. *Let* $f: X \to X$ *be a continuous map from a compact connected* ANR X *into itself. If* $N(f) \neq 0$ *then every map* $g: X \to X$ *homotopic to* f *has at least* N(f) *fixed points.*

In many cases, stronger conclusions can be drawn. For example, if X is a manifold of dimension ≥ 3 then there is a map g homotopic to f which has exactly N(f) fixed points.

3. On the Jiang Spaces

In an effort to compute the Nielsen number of a given map $f: X \to X$, Jiang [8] introduced an interesting subgroup of the fundamental group of X. Let $M(X)$ be the space of all continuous maps from a compact connected ANR X into itself with compact open topology. Let $\alpha: M(X) \to X$ be a map defined by $\alpha(f) = f(x_o)$, i.e., the evaluation map at $x_o \in X$. Then α induces $\alpha_\#: \pi_1(M(X),f) \to \pi_1(X,f(x_o))$. The Jiang subgroup $T(X,f,x_o)$ of f is defined to be the image $\alpha_\#(\pi_1(X),f)$ in $\pi_1(X,f(x_o))$. If we denote $T(X)$ for $T(X,id,x_o)$, then $T(X) \subset T(X,f,x_o) \subset \pi_1(X,f(x_o))$ for all $f \in M(X)$. It is known that $T(X)$ lies in the center of $\pi_1(X,x_o)$ and if $T(X) = \pi_1(X,x_o)$ then $\pi_1(X,x_o)$ abelian, and X said to satisfy the Jiang condition (J-condition). It is well known lens spaces and H-spaces satisfy the J-condition.

Theorem [Jiang]. *If* X *satisfies the* J-*condition then each Nielsen fixed point class* F *of* f *has the same fixed point index* $i(F)$ *and if we denote this number by* $i(f)$ *then* $L(f) = i(f) \cdot N(f)$.

Definition. Let $f_\#: \pi_1(X,x_o) \to \pi_1(X,x_o)$ be a homomorphism. Two elements α and β are said to be f-equivalent if there exists $\gamma \in \pi_1(X,x_o)$ such that $\alpha = \gamma\beta f_\#(\gamma^{-1})$. The Riedeweister number $R(f)$ of f is defined to be the cardinality of the set of equivalence classes in $\pi_1(X,x_o)$.

We apply the f-equivalence relation to $T(f)$ and denote the cardinality of equivalence classes by $J(f)$.

Theorem [Brooks, Brown, Jiang]. *Assume* $L(f) \neq 0$. *Then*

$J(f) \leq N(f) \leq R(f)$.

Theorem [Jiang]. *Assume* $L(f) \neq 0$. *If* $T(X) = \pi_1(X)$, *then* $N(f) = J(f) = R(f)$.

4. On the Fiber-Preserving Maps

To compute the Nielsen number of a given map f, Brown [2] initially studied fiber-preserving maps. Let $\mathcal{J} = \{E,p,B\}$ be an orientable Hurewicz fibering with a regular lifting function λ. If $f: E \to E$ is a fiber-preserving map then f induces $f': B \to B$ and $f_b: P^{-1}(b) \to P^{-1}(b)$ defined by $f_b(e) = \lambda(e,w)(1)$ where w is a path from f(b) to $b \in B$. It is well known that $N(f_b)$ is independent of the choice of w and $b \in B$. The product theorem for the Nielsen number of a fiber-preserving map says $N(f) = N(f') \cdot N(f_b)$, $b \in B$, under the suitable conditions. Thus for a class of fiber-preserving maps we may compute the Nielsen number through the product theorem. Note that $L(f) = L(f') \cdot L(f_b)$ holds for all $b \in B$.

Theorem [Brown-Fadell]. *Let* $\mathcal{J} = \{E,p,B\}$ *be an orientable Hurewicz fibering with fiber* Y *where* E, B *and* Y *are connected finite polyhedra and let* $f: E \to E$ *be a fiber-preserving map. If one of the following conditions is satisfied*

a) $\pi_1(B) = \pi_2(B) = 0$

b) $\pi_1(Y) = 0$

c) $\mathcal{J}$ *is trivial and either* $\pi_1(B) = 0$ *or* $f = f' \times f_b$

then $N(f) = N(f') \cdot N(f_b)$ *for all* f_b, $b \in B$.

Recently Fadell [5] came up with a unifying theorem for a number of existing theorems.

Theorem [Fadell]. *Let* $\mathcal{J} = \{E, p, B\}$ *be an orientable Hurewicz fibering with* E, B *and* Y *compact connected* ANR*'s and let* $f\colon \mathcal{J} \to \mathcal{J}$ *be a fiber-preserving map. Suppose* f *and* $\mathcal{J}$ *satisfy the following conditions:*

a) The sequence

$$0 \to \pi_1(Y) \xrightarrow{i_\#} \pi_1(E) \xrightarrow{p_\#} \pi_1(B) \to 0$$

is exact,

b) $P_\#$ *admits a right inverse* $\sigma\colon \pi_1(B) \to \pi_1(E)$ *such that if* $H = \operatorname{im} \sigma$, *then* H *is normal in* $\pi_1(E)$ *and* $f_\#(H) \subset H$.

Then $N(f) = N(f') \cdot N(f_b)$, $b \in B$.

If spaces involved in $\mathcal{J}$ satisfy the Jiang condition then we have a complete solution to the product theorem. Without loss of generality we may assume $L(f) \neq 0$. Consider the following part of fiber homotopy exact sequence:

$$\begin{array}{ccc} \pi_1(P^{-1}(b)) & \xrightarrow{i_\#} & \pi_1(E) \\ {\scriptstyle 1-f_{b\#}}\downarrow & & \downarrow{\scriptstyle 1-f_\#} \\ \pi_1(P^{-1}(b)) & \xrightarrow{i_\#} & \pi_1(E) \end{array}$$

This diagram induces a homomorphism

$$i^\#\colon \operatorname{coker}(1-f_\#) \to \operatorname{coker}(1-f_\#).$$

Definition. Define $P(f)$ to be the order of $\ker i^\#$.

Theorem [Pak]. *Let* $\mathcal{J} = \{E, P, B\}$ *be an orientable Hurewicz fibering. We assume* E, B *and* Y *satisfy the* J*-condition. If* $f\colon \mathcal{J} \to \mathcal{J}$ *is a fiber-preserving map then* $N(f) \cdot P(f) = N(f') \cdot N(f_b)$, $b \in B$.

The fiber homotopy exact sequence induces the following diagram:

$$\begin{tikzcd}
K\colon 0 \arrow[r] & K_4 \arrow[r] \arrow[d, tail] & K_3 \arrow[r] \arrow[d, tail] & K_2 \arrow[r] \arrow[d, tail] & K_1 \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & d_\#(\pi_2(B)) \arrow[r, "d_\#"] \arrow[d] & \pi_1(P^{-1}(b)) \arrow[r] \arrow[d, "1-f_{b\#}"'] & \pi_1(E) \arrow[r] \arrow[d, "1-f_\#"'] & \pi_1(B) \arrow[r] \arrow[d, "1-f'_\#"'] & 0 \\
0 \arrow[r] & d_\#(\pi_2(B)) \arrow[r, "d_\#"] \arrow[d, two heads] & \pi_1(P^{-1}(b)) \arrow[r] \arrow[d, two heads] & \pi_1(E) \arrow[r] \arrow[d, two heads] & \pi_1(B) \arrow[r] \arrow[d, two heads] & 0 \\
C\colon 0 \arrow[r] & C_4 \arrow[r, "\alpha"] & C_3 \arrow[r, "i^\#"] & C_2 \arrow[r, "\gamma"] & C_1 \arrow[r] & 0
\end{tikzcd}$$

where K = kernel and C = cokernel of corresponding maps, and $d_\#\colon \pi_2(B) \to \pi_1(P^{-1}(b))$ the connecting homomorphism. The sequence C is right exact, i.e., exact at C_1 and C_2 and K is left exact, i.e., exact at K_4 and K_3. Also the homology groups satisfy $H(C_3) = H(K_1)$ and $H(C_4) = H(K_2)$. Let us denote the order of a group G by $O(G)$.

In an effort to generalize Pak's theorem Giessmann [6] proved the following theorem.

Theorem [Giessmann]. *Let* $\mathcal{J} = \{E,p,B\}$ *be an orientable Hurewicz fibering. Let* $f\colon \mathcal{J} \to \mathcal{J}$ *be a fiber-preserving map. If* E *satisfies* J*-condition and if* $\mathrm{Im}(1-f_{b\#})$ *and* $\mathrm{Im}(1-f'_\#)$ *lie in the corresponding Jiang subgroups* $T(f_b)$ *and* $T(f')$ *respectively then* $N(f) \cdot Q(f) = N(f') \cdot N(f_b) \cdot R(f)$, *where* $Q(f) = O(H(K_1)) \cdot O(C_4)$ *and* $R(f) = O(H(K_4))$.

Lemma 1. $O(C_2) \cdot P(f) = O(C_1) \cdot O(C_3)$.

Proof. The above diagram induced by the fiber homotopy exact sequence induces the following short exact sequences:

$$0 \to H(C_4) \to C_4 \to \mathrm{Im}\,\alpha \to 0,$$
$$0 \to \ker i^\# \to C_3 \to \mathrm{Im}\, i^\# \to 0,$$
$$0 \to \ker \gamma \to C_2 \to C_1 \to 0, \text{ and}$$

we have

$$0 \to \ker i^{\#} \to C_3 \to \ker \gamma \to C_2 \to C_1 \to 0.$$

From the above exact sequences we can read off

$$O(C_4) = O(H(C_4) \cdot O(\operatorname{Im} \alpha),$$
$$O(C_3) = O(\ker i^{\#}) \cdot O(\operatorname{Im} i^{\#}),$$
$$O(C_3) = O(C_1) \cdot O(\ker \gamma) \text{ and}$$

we have

$$\frac{O(C_3)}{O(\ker i^{\#})} = O(\operatorname{Im} i^{\#}) = O(\ker \gamma) = \frac{O(C_2)}{O(C_1)}.$$

Therefore we get $O(C_2) \cdot O(\ker i^{\#}) = O(C_1) \cdot O(C_3)$. Since $P(f) = O(\ker i^{\#})$ our lemma is proved.

Theorem 2. *Let* $\mathcal{J} = \{E,p,B\}$ *be an orientable Hurewicz fibering and let* $f\colon \mathcal{J} \to \mathcal{J}$ *be a fiber-preserving map such that* $L(f) \neq 0$. *If* E *satisfies the* J-*condition and if* $\operatorname{Im}(1-f_{b\#})$ *and* $\operatorname{Im}(1-f'_{\#})$ *lie in the corresponding Jiang subgroups* $T(f_b)$ *and* $T(f')$, *then* $N(f) \cdot P(f) = N(f') \cdot N(f_b)$, $b \in B$.

Proof. Giessmann's theorem says $N(f) \cdot Q(f) = N(f') \cdot N(f_b) \cdot R(f)$. Since we know $O(H(K_2)) = O(H(C_4))$ and $O(H(K_1)) = O(H(C_3))$ we have $N(f)\ O(H(C_3)) \cdot O(C_4) = N(f_b)\ N(f')\ O(H(C_4))$. From the proof of lemma 1, we have $N(f)\ O(H(C_3)) \cdot O(H(C_4))\ O(\operatorname{Im} \alpha) = N(f_b) \cdot N(f') \cdot O(H(C_4))$, and this reduced to $N(f)\ O(\frac{\ker i^{\#}}{\operatorname{Im} \alpha}) \cdot O(\operatorname{Im} \alpha) = N(f_b) \cdot N(f')$. Thus we conclude $N(f)\ P(f) = N(f') \cdot N(f_b)$, $b \in B$.

Finally we would like to pose a couple of problems.

Problem 1. If $E = B \times Y$, then Gottlieb [7] has shown that E satisfies the J-condition iff B and Y satisfy the

J-condition. Derivation of theorem 2 from that of Giessmann strongly suggests the following problem:

Let $\mathcal{J} = \{E,p,B,Y\}$ be an orientable Hurewicz fibering. Then E satisfies the J-condition iff B and Y satisfy the J-condition.

Problem 2. It is known that simply connected spaces, lens spaces, and H-spaces satisfy the J-condition. It will be an interesting problem to try to enlarge the class of Jiang spaces.

Added in Proof. Recently we have shown "only if" part of problem 1. That is if E satisfies J-condition then B and Y satisfy the J-condition. This result gives:

Theorem 3. *Let* $\mathcal{J} = \{E,p,B\}$ *be an orientable Hurewicz fibering, where* E *satisfies the* J*-condition. If* $f: \mathcal{J} \to \mathcal{J}$ *is a fiber-preserving map such that* $L(f) \neq 0$*, then* $N(f) \cdot P(f) = N(f') \cdot N(f_b)$*,* $b \in B$.

A proof of this theorem also follows easily from the following theorem of Fadell [5]. We leave the proof to the reader.

Theorem [Fadell]. *Let* $\mathcal{J} = \{E,p,B\}$ *be a Hurewicz fibering with* E *and* B *compact, metric* ANR*'s and* $f: E \to E$ *a fiber-preserving map. Then there is a locally trivial fibering* $\mathcal{J}' = \{E',p',B'\}$ *with fiber* F' *and a fiber-preserving map* $g: E' \to E'$ *with the following properties:*

1) B' *and* F' *are compact polyhedra,*

2) $g': B' \to B'$ *has precisely* $N(g')$ *fixed points, each* *in a maximal simplex,*

3) For each $b \in \Phi(g')$, $g_b\colon F' \to F'$ *has precisely* $N(g_b)$ *fixed points, each in a maximal simplex of* F',

4) $N(f) = N(g)$, $N(f') = N(g')$, *and* $N(f_b) = N(g_b)$.

This theorem implies that without loss of generality we may assume f has all the properties of g. Let $\{F_1,\cdots,F_k\} = \Phi'(f)$ be the Nielsen fixed point classes of f. If $\ell_{i,1}$ and $\ell_{i,2} \in F_i$ then $P(\ell_{i,1}) \sim P(\ell_{i,2})$ in B and from the fact that f' has exactly N(f') fixed points we deduce that all points in F_i lie in the same fiber say $P^{-1}(b)$. Let $F_i \cap P^{-1}(b) = \{\ell_{i,1},\cdots,\ell_{i,k_i}\}$. Define a map $\gamma\colon \Phi'(f_b) \to \Phi'(f)$ for each $b \in B$ by $\gamma(\ell_{i,j}) = F_i$. This is a well-defined map. Let $P(f)_{F_i} = \#(\gamma^{-1}(F_i)) = k_i$. Then the following theorem follows easily.

Theorem 4. *Let* $\mathcal{J} = \{E,p,B\}$ *be an orientable Hurewicz fibering and* $f\colon E \to E$ *a fiber-preserving map. If* $P(f) = \#(\gamma^{-1}(F_i))$ *is a constant for each* i, *then* $N(f)\ P(f) = N(f') \cdot N(f_b)$, $b \in B$.

Finally I would like to state a recent theorem from [15].

Theorem 5. *Let* $\mathcal{J} = \{E,p,B\}$ *be an orientable Hurewicz fibering and* $f\colon E \to E$ *a fiber-preserving map such that* $L(f) \neq 0$. *If the fundamental groups of the spaces involved in* $\mathcal{J}$ *are abelian then* $N(f) \cdot p(f) = N(f') \cdot N(f_b)$, $b \in B$.

Note that this theorem improves theorem 3.

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