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CONVERGENCE OF SEQUENCES OF CLOSED SETS

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The theorem proved in this short note yields new criteria for the convergence of sequences of closed subsets of a locally compact, separable metric space (E,d). This work was motivated by the study of convergence of measurable multifunctions [10] and optimization problems, e.g. [2].

We say that a sequence of closed sets $\{F_n \subset E,\ n \in N\}$ *converges* to the closed set F in E, in which case we write $F = \lim_{n \in N} F_n$, if

$$\limsup F_n = F = \liminf F_n.$$

This is the classical definition (due to Borel). Here

$$\liminf F_n = \{x = \lim x_n \mid x_n \in F_n \text{ for all } n \in N\}$$

and

$$\limsup F_n = \{x = \lim x_m \mid x_m \in F_m \text{ for all } m \in M \subset N\}.$$

By N we always denote a countable ordered index set, typically the natural numbers; M denotes generically a naturally ordered infinite subset of N, i.e. $\{x_m,\ m \in M\}$ is a subsequence of $\{x_n,\ n \in N\}$.

Let $\mathcal{F} = \{F \subset E \mid F \text{ closed}\}$ be the hyperspace of closed subsets of E. There are many ways to topologize $\mathcal{F}$, cf. [6], [8] for example. We equip $\mathcal{F}$ with the topology $\mathcal{T}$ generated by the subbase consisting of all families of sets of the form

$$\{\mathcal{F}^K\} \text{ and } \{\mathcal{F}_G\}$$

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where

$$\mathcal{J}^K = \{F \in \mathcal{J} | F \cap K = \emptyset\} \text{ and } \mathcal{J}_G = \{F \in \mathcal{J} | F \cap G \neq \emptyset\},$$

and, as in the rest of this note, $K, K_1, K_2, \ldots$ are compact, $G, G_1, G_2, \ldots$ are open and $F, F_1, F_2, \ldots$ are closed sets in E. The resulting base consists of all sets of the type:

$$\mathcal{J}^K_{G_1,\ldots,G_p} = \mathcal{J}^K \cap \mathcal{J}_{G_1} \cap \ldots \cap \mathcal{J}_{G_p} \qquad p = 1,2,\ldots$$

This is a variant of the Vietoris topology. We write

$$F = \mathcal{T} - \lim F_n$$

if the elements $\{F_n, n \in N\}$ of $\mathcal{J}$ converge--in this topology--to $F \in \mathcal{J}$. It is easy to verify that the topological space $(\mathcal{J},\mathcal{T})$ is Hausdorff, regular, second countable and thus metrizable. To see that it is compact we can either rely on the fact that the collection $\mathcal{C} = \{\mathcal{J}_K\} \cap \{\mathcal{J}^G\}$ satisfies the finite intersection property or use the equivalence between the notions of $\mathcal{T}$-convergence in $\mathcal{J}$ and convergence for sequences of sets in E (cf. part (i) of the Theorem), this implies that every infinite sequence of elements of the metrizable space $(\mathcal{J},\mathcal{T})$ contains a convergent subsequence since every infinite sequence of sets contains a convergent subsequence [12, Theorem (7.11)]. The space $(\mathcal{J},\mathcal{T})$ being metrizable, convergence could be defined in terms of a metric, for example the metric constructed in the proof of Urysohn's metrization Lemma, cf. [3, p. 24]. Although the Theorem does not identify a new metric compatible with $\mathcal{T}$, it yields criteria for $\mathcal{T}$-convergence that *quantify* the distance between the limit set and the elements in the sequence.

Notations. An arbitrary point of E is designated as the *origin* and denoted O. Let $d(x,F) = \inf [d(x,y) | y \in F]$

be the distance between a closed set F and a point x in E, with the usual convention that $d(x,\emptyset) = +\infty$. In E, the *closed ball* of radius r and center x is denoted by $B_r(x)$ and the *open ball* by $B_r^\circ(x)$. Given any set $D \subset E$ we write simply D^r for $D \cap B_r(0)$, i.e. the intersection of D with the closed ball of radius r centered at the origin. For $\varepsilon > 0$, the (open) ε-*neighborhood* εD of a nonempty set $D \subset E$ is the set $\{y|d(y,D) < \varepsilon\}$; and for the empty set $\emptyset$, $\varepsilon\emptyset = [B_{\varepsilon^{-1}}(0)]^c$.

Note that we have not excluded the possibility that the limit set of a sequence is empty. The Lemma below yields a characterization of convergence to the empty set that is exploited repeatedly in the proof of the Theorem.

Lemma. Suppose that $\{F_n, n \in N\}$ *is a sequence of closed sets in* E. *Then* $\lim F_n = \emptyset$ *if and only if to each* K *there corresponds* n_K *such that*

$$F_n \cap K = \emptyset \text{ for all } n \geq n_K.$$

Equivalently, if and only if to each $r \geq 0$, *there corresponds* $n(r)$ *such that* $F_n^r = \emptyset$ *for all* $n \geq n(r)$.

Proof. The equivalence between these two assertions follows directly from the nature of E.

First, suppose that $\lim F_n = \emptyset$ but there exists K such that $x_m \in F_m \cap K \neq \emptyset$ for all $m \in M \in N$. The sequence $\{x_m\} \subset K$ admits a cluster point which, by construction, is also in $\lim\sup F_m$, contradicting $\lim F_n = \emptyset$.

Since $\lim\inf F_n \subset \lim\sup F_n$, to prove the only if part it suffices to show that $\lim\sup F_n = \emptyset$. Suppose not. Take $x \in \lim\sup F_n$ with $x = \lim\{x_m|x_m \in F_m, m \in M\}$ and let

K be a neighborhood of x. Then $x \in K \cap F_m$ for all m sufficiently large, contradicting the hypothesis.

We are now prepared to state the main theorem. The equivalence between the four first statements was already known or could be deduced from existing results, see for example [1], [6] and [5]. Short (new) elementary proofs are included to make the paper self-contained.

Theorem. Suppose that $\{F; Fn, n \in N\}$ *is a collection of closed sets in* E. *Then* $F = \mathcal{J}\text{-}\lim F_n$ *if and only if, in anyone of the following statements, both parts* a *and* b *are satisfied:*

i_a $F \subset \liminf F_n$,

i_b $\limsup F_n \subset F$;

ii_a if $F \cap G \neq \emptyset$ *then* $F_n \cap G \neq \emptyset$ *for all* $n \geq n_G$,

ii_b if $F \cap K = \emptyset$ *then* $F_n \cap K = \emptyset$ *for all* $n \geq n_K$;

iii_a if $F \cap B^\circ_\varepsilon(x) \neq \emptyset$ *then* $F_n \cap B^\circ_\varepsilon(x) \neq \emptyset$ *for all* $n \geq n(\varepsilon,x)$,

iii_b if $F \cap B_\varepsilon(x) = \emptyset$ *then* $F_n \cap B_\varepsilon(x) = \emptyset$ *for all* $n \geq n'(\varepsilon,x)$;

iv_a *for all* x *in* E, $\limsup d(x,F_n) \leq d(x,F)$,

iv_b *for all* x *in* E, $d(x,F) \leq \liminf d(x,F)$;

v_a $\lim (F \backslash \varepsilon F_n) = \emptyset$ *for all* $\varepsilon > 0$,

v_b $\lim (F_n \backslash \varepsilon F) = \emptyset$ *for all* $\varepsilon > 0$;

vi_a $F^r \subset \varepsilon F_n$ *for all* $\varepsilon > 0$, $r > 0$ *and* $n \geq n(\varepsilon,r)$,

vi_b $F^r_n \subset \varepsilon F$ *for all* $\varepsilon > 0$, $r > 0$ *and* $n \geq n'(\varepsilon,r)$;

vii_a $\quad F \subset \lim_{r\uparrow\infty} \liminf F_n^r$,

vii_b $\quad \lim_{r\uparrow\infty} \limsup F_n^r \subset F$;

Proof. The equivalence between $\mathcal{J}$-convergence and (ii) is an immediate consequence of the base structure of $\mathcal{J}$. Thus the Theorem will be proved if we establish the equivalence between (ii) and the other statements. We only consider the case F nonempty; if $F = \emptyset$ the implications are either trivial or require an elementary appeal to the previous Lemma. The rest is proved in two parts, we show first that all a statements are equivalent; this is done by establishing the following sequence of implications

$$i_a \Rightarrow vii_a \Rightarrow vi_a \Rightarrow v_a \Rightarrow iv_a \Rightarrow iii_a \Rightarrow ii_a \Rightarrow i_a.$$

$i_a \Rightarrow v_{ii}a$. Take any $x \in F = \liminf F_n$, i.e. $x = \lim\{x_n, n \in N | x_n \in F_n\}$. For n sufficiently large and $r \geq s > d(0,x)$, $x_n \in F_n^r$ and thus $x \in \liminf F_n^r \subset \lim_{r\uparrow\infty} \liminf F_n^r$.

$vii_a \Rightarrow vi_a$. If $x \in F^r$ then $x = \lim_{s\uparrow\infty} \lim_n x_n^s$ with $x_n^s \in F_n^s$. Hence for $\varepsilon > 0$ and with $s = r + \varepsilon$, there exists $n'(\varepsilon,s)$ such that $x_n^s \in F_n^s \subset F_n$ for all $n \geq n'(\varepsilon,s)$ or equivalently $x \in \varepsilon F_n$ for all $n \geq n'(\varepsilon,s)$.

$vi_a \Rightarrow v_a$. Apply the Lemma to the sequence $\{F \backslash \varepsilon F_n, n \in N\}$.

$v_a \Rightarrow iv_a$. Suppose not. Then there exists $x \in E$, $\varepsilon > 0$ and $M \subset N$ such that $d(x,F_m) > d(x,F) + 2\varepsilon$ for all $m \in M$ or equivalently $d(x,\varepsilon F_m) > d(x,F) + \varepsilon$. It follows that $\{y | d(x,y) = d(x,F)\} \subset \limsup_n (F \backslash \varepsilon F_n)$ which contradicts v_a.

$iv_a \Rightarrow iii_a$. Note that $\limsup d(x,F_n) \leq d(x,F)$ holds only if for all $\varepsilon > 0$ such that $d(x,F) < \varepsilon$, $d(x,F_n) < \varepsilon$ for n sufficiently large or equivalently if $F \cap B^\circ(x) \neq \emptyset$ implies that $F_n \cap B_\varepsilon^\circ(x) \neq \emptyset$ for n sufficiently large.

$iii_a \Rightarrow ii_a$. Simply note that the properties of E allow us to write every open set as the countable union of open balls.

$ii_a \Rightarrow i_a$. Take $x \in F$ and $\{G_i, i \in I\}$ a fundamental (nested) system of open neighborhoods of x with $G_1 \supset F$. Clearly $G_i \cap F \neq \emptyset$ for all i. By ii_a this implies that $G_i \cap F_n \neq \emptyset$ for $n \geq n_i$ and thus there exists $x_n \in F_n$ such that $x = \lim x_n$, i.e. $F \subset \liminf F_n$.

Next we prove that the b statements are equivalent. But this time we derive a string of implications in the opposite order, i.e.

$$i_b \Rightarrow ii_b \Rightarrow iii_b \Rightarrow iv_b \Rightarrow v_b \Rightarrow vi_b \Rightarrow vii_b \Rightarrow i_b,$$

$i_b \Rightarrow ii_b$. Suppose not. Then there exists a compact K such that $F \cap K = \emptyset$ but for $F_m \cap K \neq \emptyset$ for $m \in M \subset N$. Every sequence $\{x_m \in F_m \cap K\}$ admits a convergent subsequence, say to x. By construction $x \in K \cap \limsup F_n \subset K \cap F$, a contradiction.

$ii_b \Rightarrow iii_b$. Evident.

$iii_b \Rightarrow iv_b$. Since $d(x,F) > \varepsilon$ if and only if $F \cap B_\varepsilon(x) = \emptyset$ which in view of iii_b implies that $F_n \cap B_\varepsilon(x) = \emptyset$, or equivalently $d(x,F_n) > \varepsilon$, for n sufficiently large. From this iv_b follows directly.

$iv_b \Rightarrow v_b$. Suppose not; then $\limsup (F_n \backslash \varepsilon F) \neq \emptyset$, i.e. there exists $\{x_m \in F_m \backslash \varepsilon F, m \in M\}$ such that $\lim x_m = x \in \limsup$

$(F_n \setminus \varepsilon F)$. On one hand we have that for all $m \in M$

$$\varepsilon < d(x_m,F) \leq d(x,F) + d(x,x_m)$$

and thus $d(x,F) > \varepsilon - d(x,x_m)$, on the other hand $d(x,F_m) \leq d(x_m,F_m) + d(x,x_m)$. This implies that $\liminf d(x,F_m) = 0 < \varepsilon - \lim d(x,x_m)$, contradicting iv_b.

$v_b \Rightarrow vi_b$. Apply the Lemma to the sequence $\{F_n \setminus \varepsilon F), n \in N\}$.

$vi_b \Rightarrow vii_b$. Since for all $\varepsilon > 0$, $r > 0$ there exists $n(\varepsilon,r)$ such that for all $n \geq n(\varepsilon,r)$ it follows that $\limsup_n F_n^r \subset \varepsilon F$. Since this holds for all $\varepsilon > 0$ and $\limsup_n F_n^r$ is closed we also have that $\limsup_n F_n^r \subset F$. From this the assertion follows directly.

$vii_b \Rightarrow i_b$. If $x \in \limsup_n F_n$ then $x = \lim x_m$, $x_m \in F_m$, $m \in M \subset N$. Take $s > d(0,x)$, then the $x_m \in F_m^s$ for m sufficiently large. Thus $x \in \limsup F_m^s \subset \lim_{r\uparrow\infty} \limsup F_m^r = F$.

In the following Corollary we give an abbreviated version of a number of statements appearing in the Theorem. The proofs follow directly from the inclusion lim inf $\subset$ lim sup for any sequence of subsets of E and the inequality lim inf $\leq$ lim sup for any sequence of numbers.

Corollary 1. *Suppose that* $\{F; F_n, n \in N\}$ *is a collection of closed subsets of* E. *Then the following are equivalent*

(i) $F = \lim F_n$

(ii) $F = \mathcal{J}\text{-}\lim F_n$

(iii) *for all* x *in* E, $d(x,F) = \lim_n d(x,F_n)$

(iv) $\lim_n [(F \setminus \varepsilon F_n) \cup (F_n \setminus \varepsilon F)] = \emptyset$

(v) $F = \lim_{r\uparrow\infty} \liminf F_n^r = \lim_{r\uparrow\infty} \limsup F_n^r$.

The last one of the above statements suggest the definition of an r-distance between subsets of E. This has been done for linear subspaces [4], convex cones [11] and for convex subsets of $R^p = E$ [7], [9]. Let C,D be two closed convex subsets of R^p, then the r-distance between C and D is by definition

$$h_r(C,D) = \begin{cases} 0 \text{ if } C^r = D^r = \emptyset \\ +\infty \text{ if } C^r = \emptyset \text{ or } D^r = \emptyset \text{ but } C^r \neq D^r \\ h(C^r,D^r) \text{ otherwise.} \end{cases}$$

Corollary 2. [7] *Suppose that* $\{F;F_n, n \in N\}$ *is a collection of closed convex subsets of* R^p. *Then* $F = \lim F_n$ *if and only if there exists* $r_0 > 0$ *such that for all* $r \geq r_0$, $\lim h_r(F,F_n) = 0$.

In [9] it was conjectured that a statement similar to that of Corollary 2 holds for closed sets if the r-distance is defined as follows $p_r(C,D) = \lim_{s\downarrow r} h_s(C,D)$. It can be shown that if $\lim p_r(F,F_n) = 0$ for all $r \geq r_0 > 0$ then $F = \lim F_n$. Unfortunately the converse does not hold as can be seen from the following example: Consider $E = R^2$ with d the l^∞-distance, i.e., $B_1(0)$ is the unit square. Let

$$F = \{[(-1,k), (1,k)], k \in K\},$$

with $K = \{1,2,\dots\}$ and

$$F_n = \{[(-1, k + n^{-1}), (0,k)] \cup [(0,k), (-1, k + n^{-1})], k \in K\}.$$

Clearly $F = \lim F_n$ but for all k in K and all n in N, $p_k(F,F_n) = 1$.

Finally note that the main theorem could be proved with E locally compact, Hausdorff and second countable.

But then E is metrizable and separable. After defining a metric on E compatible with the underlying topology, we would be in the setting considered in this paper.

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