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The combined results of Gleason [2] and Montgomery and Zippin [4] characterize Lie groups as the locally connected, locally compact metrizable topological groups of finite dimension. The assumption of finite-dimensionality is essential here, as the example of infinite product of circles shows. In this note we remark that a theorem of Iwasawa and results from Q-manifold theory allow this assumption to be replaced by the one that the space underlying the group be an Absolute Neighbourhood Retract:

Theorem. Let X *be a locally compact, metrizable topological group. If* $X \in \text{ANR}$ *then* $\dim X < \infty$ *and hence* X *is a Lie group.*

In the proof we need the following (we write $I = [0,1]$):

Lemma. Let $Y \in \text{ANR}$. *If there is an open cover* $\mathcal{U}$ *of* Y *consisting of sets having the disjoint* n-*cube property, then* Y *has the disjoint* n-*cube property*, i.e. *any map* $I^n \times \{1,2\} \to Y$ *is the uniform limit of maps sending* $I^n \times 1$ *and* $I^n \times 2$ *to disjoint sets.*

Proof. (By induction on n). Fix $f: I^n \times \{1,2\} \to Y$, choose $\varepsilon > 0$ so that each subset of im(f) of diameter $< \varepsilon$ is contained in an element of $\mathcal{U}$ and let $\mathcal{J}$ be a triangulation of I^n such that diam $f(\sigma) < \varepsilon/3$ for $\sigma \in \mathcal{J} \times \{1,2\}$. By inductive assumption and properties of ANR's we may assume that $f(A \times 1) \cap f(A \times 2) = \phi$, where A is the (n-1)-skeleton of $\mathcal{J}$. Let $\{\sigma_1,\ldots,\sigma_k\}$ be all the n-simplices in $\mathcal{J}$. We construct maps $f_1,\ldots,f_k\colon I^n \times \{1,2\} \to Y$ such that, for $i \le k$, the following holds

$(1)_i$ $f_i((\sigma_1 \cup \ldots \cup \sigma_i) \times 1) \cap f_i(I^n \times 2) = \phi$,

$(2)_i$ $f_i(x) = f(x)$ for $x \in A \times \{1,2\}$,

$(3)_i$ $\mathrm{dist}(f_i,f_{i-1}) < \delta/k$,

where $f_0 = f$. Then f_k sends $I^n \times 1$ and $I^n \times 2$ to disjoint sets and approximates f within a given $\delta > 0$.

The construction of f_i (We assume $\delta < \varepsilon/3$): Consider the set $J = \{j\colon f_{i-1}(\sigma \times 2) \cap f_{i-1}(\sigma_i \times 1) \neq \phi\}$. By (3) we have $\mathrm{dist}(f,f_{i-1}) < \varepsilon/3$ whence $\mathrm{diam}\, f(\sigma) < \varepsilon/3$ for $\sigma \in \mathcal{J} \times \{1,2\}$ and, with $F = \cup_{j\in J} f_{i-1}(\sigma_j \times 2) \cup f_{i-1}(\sigma_i \times 1)$ we have $\mathrm{diam}\, F < \varepsilon$. Thus F is contained in a member of $\mathcal{U}$ and we may alter f_{i-1} on $\cup_{j\in J}\sigma_j \times 2 \cup \sigma_i \times 1$ modulo $A \times \{1,2\}$ by so small an amount that the resulting map satisfies $(1)_i$ and $(3)_i$.

Proof of the Theorem. Assume that $X \in$ ANR and $\dim X = \infty$. Given an integer n it follows from a theorem of Iwasawa that each point $x \in X$ has a neighbourhood homeomorphic to $V_x \times R^{2n+1}$, for some space V_x. (See [3] or [5], p. 184). Since $V_x \times R^{2n+1}$ has the disjoint cube property (by general position applied to R^{2n+1}) it follows from the Lemma that X also has this property. A locally compact ANR having the disjoint n-cube property for each n is a manifold modeled on the Hilbert cube Q [6], and hence X is a Q-manifold. This, however, is impossible since no Q-manifold carries a topological group structure (see [1]). Thus either $X \notin$ ANR or $\dim X < \infty$.

The above result shows in particular that, among locally compact metrizable topological groups, the property of being an ANR forces the underlying space to have a manifold structure. It is unknown if the same is true for complete metrizable groups X, where by a manifold we now mean a space locally homeomorphic to a Hilbert space of infinite dimension. (Even the case of linear metric spaces is not settled and in general it is known only that $X \times \ell_2$ is a manifold, cf[7]). It is also unknown if the assumption "$X \in$ ANR" in the theorem can be replaced by "X is locally contractible".

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