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The combined results of Gleason [2] and Montgomery and Zippin [4] characterize Lie groups as the locally connected, locally compact metrizable topological groups of finite dimension. The assumption of finite-dimensionality is essential here, as the example of infinite product of circles shows. In this note we remark that a theorem of Iwasawa and results from Q -manifold theory allow this assumption to be replaced by the one that the space underlying the group be an Absolute Neighbourhood Retract:

Theorem. Let X be a locally compact, metrizable topological group. If $X \in \text{ANR}$ then $\dim X < \infty$ and hence X is a Lie group.

In the proof we need the following (we write $I = [0,1]$):

Lemma. Let $Y \in \text{ANR}$. If there is an open cover \mathcal{U} of Y consisting of sets having the disjoint n -cube property, then Y has the disjoint n -cube property, i.e. any map $I^n \times \{1,2\} \rightarrow Y$ is the uniform limit of maps sending $I^n \times 1$ and $I^n \times 2$ to disjoint sets.

Proof. (By induction on n). Fix $f: I^n \times \{1,2\} \rightarrow Y$, choose $\varepsilon > 0$ so that each subset of $\text{im}(f)$ of diameter $< \varepsilon$ is contained in an element of \mathcal{U} and let \mathcal{J} be a triangulation of I^n such that $\text{diam } f(\sigma) < \varepsilon/3$ for $\sigma \in \mathcal{J} \times \{1,2\}$. By

inductive assumption and properties of ANR's we may assume that $f(A \times 1) \cap f(A \times 2) = \emptyset$, where A is the $(n-1)$ -skeleton of J . Let $\{\sigma_1, \dots, \sigma_k\}$ be all the n -simplices in J . We construct maps $f_1, \dots, f_k: I^n \times \{1, 2\} \rightarrow Y$ such that, for $i \leq k$, the following holds

$$(1)_i \quad f_i((\sigma_1 \cup \dots \cup \sigma_i) \times 1) \cap f_i(I^n \times 2) = \emptyset,$$

$$(2)_i \quad f_i(x) = f(x) \text{ for } x \in A \times \{1, 2\},$$

$$(3)_i \quad \text{dist}(f_i, f_{i-1}) < \delta/k,$$

where $f_0 = f$. Then f_k sends $I^n \times 1$ and $I^n \times 2$ to disjoint sets and approximates f within a given $\delta > 0$.

The construction of f_i (We assume $\delta < \varepsilon/3$): Consider the set $J = \{j: f_{i-1}(\sigma \times 2) \cap f_{i-1}(\sigma_i \times 1) \neq \emptyset\}$. By (3) we have $\text{dist}(f, f_{i-1}) < \varepsilon/3$ whence $\text{diam } f(\sigma) < \varepsilon/3$ for $\sigma \in J \times \{1, 2\}$ and, with $F = \bigcup_{j \in J} f_{i-1}(\sigma_j \times 2) \cup f_{i-1}(\sigma_i \times 1)$ we have $\text{diam } F < \varepsilon$. Thus F is contained in a member of \mathcal{U} and we may alter f_{i-1} on $\bigcup_{j \in J} \sigma_j \times 2 \cup \sigma_i \times 1$ modulo $A \times \{1, 2\}$ by so small an amount that the resulting map satisfies $(1)_i$ and $(3)_i$.

Proof of the Theorem. Assume that $X \in \text{ANR}$ and $\dim X = \infty$. Given an integer n it follows from a theorem of Iwasawa that each point $x \in X$ has a neighbourhood homeomorphic to $V_x \times \mathbb{R}^{2n+1}$, for some space V_x . (See [3] or [5], p. 184). Since $V_x \times \mathbb{R}^{2n+1}$ has the disjoint cube property (by general position applied to \mathbb{R}^{2n+1}) it follows from the Lemma that X also has this property. A locally compact ANR having the disjoint n -cube property for each n is a manifold modeled on the Hilbert cube Q [6], and hence X is a Q -manifold. This, however, is impossible since no Q -manifold carries a

topological group structure (see [1]). Thus either $X \not\in \text{ANR}$ or $\dim X < \infty$.

The above result shows in particular that, among locally compact metrizable topological groups, the property of being an ANR forces the underlying space to have a manifold structure. It is unknown if the same is true for complete metrizable groups X , where by a manifold we now mean a space locally homeomorphic to a Hilbert space of infinite dimension. (Even the case of linear metric spaces is not settled and in general it is known only that $X \times \ell_2$ is a manifold, cf[7]). It is also unknown if the assumption " $X \in \text{ANR}$ " in the theorem can be replaced by " X is locally contractible".

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