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INTERSECTIONS OF COUNTABLY COMPACT SUBSPACES OF βX

Charles Waiveris[1]

1. Introduction

We will assume all spaces to be completely regular Hausdorff. The term "countable" will mean countably infinite. Throughout this paper we adopt the basic terminology and definitions of [GJ]. In particular, for a space X, C(X) denotes the collection of continuous real-valued functions on X and βX is the Stone-Čech compactification of X. For a function $f \in C(X)$ the zero set, Z(f), of f is defined by $Z(f) = \{x \in X: f(x) = 0\}$.

A topological space X is *countably compact* if and only if every countable open cover of X has a finite subcover. In our context, this is equivalent to the condition that every countable subset of X has an accumulation point in X.

It is well-known that the product of two countably compact spaces need not be countably compact. Terasaka [T] and Novák [N] first proved this by finding two countably compact subspaces P,Q of $\beta\mathbf{N}$ whose intersection is $\mathbf{N}$, the set of positive integers. Later it was shown by Frolík that every discrete space X can be written as the intersection of two countably compact subspaces P,Q of βX. Frolík [F_2] has also shown that for an arbitrary separable metric space X there exists a family $\{P_\xi: \xi < 2^c\}$ of countably compact subspaces

[1]These results are taken from Section 2 of the author's doctoral dissertation (Wesleyan University, May 1979), directed by W. W. Comfort.

of βX such that $P_\xi \cap P_{\xi'} = X$ whenever $\xi < \xi' < 2^c$.

Not every space X, however, can be written as an intersection of countably compact subspaces of βX. If X is any space such that βX contains a sequence $\{p_n: n < \omega\}$ which converges to a point $p \in \beta X \backslash X$, with $p_n \neq p$ for each $n < \omega$, then $Y = X \cup \{p_n: n < \omega\}$ is a space which cannot be written as an intersection of any family of countably compact subspaces of βY. For $\{p_n: n < \omega\}$ is countable, closed and discrete in Y; therefore any countably compact subspace of βY that contains Y must also contain the point p. Another example of such a space is the Tychonoff Plank

$$((\omega \cup \{p\}) \times (\omega_1 \cup \{q\})) \backslash \{(p,q)\}$$

where $\omega \cup \{p\}$ and $\omega_1 \cup \{q\}$ are the one-point compactifications of the ordinal spaces ω and ω_1 (in their usual order topologies) respectively.

We will show that there is a large class of spaces X for which we can construct a family of countably compact subspaces of βX which pairwise intersect in X. This will be done by adding accumulation points to X for each countable subset of βX. Since every countable subset D of βX contains a countable discrete subset (cf. [CN_1, Lemma 2.10] or [W, Lemma 2.1]) we need only add accumulation points for the countable discrete subsets of βX.

For proofs in greater detail and for additional related results, see [CW] and [W].

2. X as an Intersection of Countably Compact Subspaces of βX

Definition. A space X is an *extra countably compact* subspace of Y if every countable subset of Y has an accumulation point in X.

When X is extra countably compact in βX we say simply that X is *extra countably compact.* In particular, since $\beta P = \beta X$ whenever $X \subset P \subset \beta X$ we have: if $X \subset P \subset \beta X$ then P is extra countably compact if and only if P is extra countably compact in βX.

When we say X is an intersection of extra countably compact spaces $P_\xi, \xi < \alpha$, we are assuming $X \subset P_\xi \subset \beta X$ for each $\xi < \alpha$.

Notation. Let X be a space and $D \subset \beta X$. We set

$$\overline{D} = \mathrm{cl}_{\beta X} D \text{ and } D^* = \overline{D} \backslash D.$$

For any subset A of βX set

$$N(A) = \{D: D \text{ is a countable discrete subset of } A\}.$$

As we mentioned in the introduction, the countable discrete space $\mathbf{N}$ can be written as an intersection of two countably compact subspaces of $\beta\mathbf{N}$. We can in fact show more. There exists a family of $2^{\mathfrak{c}}$ extra countably compact spaces which pairwise intersect in $\mathbf{N}$. The properties of $\beta\mathbf{N}$ which are needed for this proof are that every countable subset of $\beta\mathbf{N}$ has $2^{\mathfrak{c}}$ accumulation points in $\beta\mathbf{N}$ and that $|N(\beta\mathbf{N})| = 2^{\mathfrak{c}}$. We will establish the following more general statement which will be necessary for the proof of Theorem 2.3

Lemma 2.1. *Let* X *be a space and* $D \in N(\beta X)$. *If* $|\overline{V}| = 2^{\mathfrak{c}}$ *for each* $V \in N(\overline{D})$ *then there exists a family* $\{B_\xi: \xi < 2^{\mathfrak{c}}\}$ *of pairwise disjoint subsets of* D^* *such that for each* $V \in N(\overline{D})$ *and* $\xi < 2^{\mathfrak{c}}$ *there exists an accumulation point* $b_{\xi,V}$ *of* V *in* B_ξ.

Proof. Let $\{V_\eta: \eta < 2^{\mathfrak{c}}\}$ be a well-ordering of $N(\overline{D})$ and let $b_{0,0} \in V_0^*$ be an accumulation point of V_0. (Note that since D is discrete in $\overline{D}$ we have $V_\eta^* \subset D^*$ for each $\eta < 2^c$).

Now let $\zeta < 2^c$ and suppose for each $\xi < \zeta$ and $\eta < \zeta$ points $b_{\xi,\eta}$ have been chosen such that

(i) $b_{\xi,\eta}$ is an accumulation point of V_η and $b_{\xi,\eta} \in V_\eta^*$,

(ii) $b_{\xi,\eta} \neq b_{\xi',\eta'}$ if either $\xi < \xi' < \zeta$ or $\eta < \eta' < \zeta$.

Then since

$$|\{b_{\xi,\eta}: \xi < \zeta, \eta < \zeta\}| < 2^c$$

and

$$|V_\zeta^*| = 2^c$$

we can choose distinct points $b_{\xi,\zeta}$ and $b_{\zeta,\eta}$ for each $\xi < \zeta$ and $\eta \leq \zeta$ contained in

$$D^* \setminus \{b_{\xi,\eta}: \xi < \zeta, \eta < \zeta\}$$

such that

$$b_{\xi,\zeta} \in V_\zeta^* \text{ and } b_{\zeta,\eta} \in V_\eta^*$$

for each $\xi < \zeta$ and $\eta \leq \zeta$. Then clearly the hypotheses (i) and (ii) are satisfied by the points

$$\{b_{\xi,\eta}: \xi \leq \zeta, \eta \leq \zeta\}.$$

Let

$$B_\xi = \{b_{\xi,\eta}: \eta < 2^c\}$$

for each $\xi < 2^c$. Obviously

$$B_{\xi'} \cap B_\xi = \emptyset$$

if $\xi' < \xi < 2^c$. If $V \in N(\overline{D})$ then $V = V_\eta$ for some $\eta < 2^c$. Then $b_{\xi,\eta} \in B_\xi$ is an accumulation point of V_η for each $\xi < 2^c$. Thus the sets $B_\xi, \xi < 2^c$, are as required.

Remark. Whenever we refer to Lemma 2.1 we will adopt the notation

$$B_\xi = \{b_{\xi,V}: V \in N(\overline{D})\}, \xi < 2^c$$

where each $b_{\xi,V}$ is an accumulation point of V.

Corollary 2.2. *There exists a family of* 2^c *extra countably compact spaces which pairwise intersect in* $\mathbf{N}$.

Proof. Let $X = D = \mathbf{N}$ in Lemma 2.2. Then the sets $B_\xi, \xi < 2^c$, are extra countably compact subspaces of $\beta\mathbf{N}$.

Let $P_\xi = \mathbf{N} \cup B_\xi$ for each $\xi < 2^c$. Then $\{P_\xi: \xi < 2^c\}$ is a family of extra countably compact spaces which pairwise intersect in $\mathbf{N}$.

A mild separation hypothesis and a bookkeeping argument will enable us to prove Theorem 2.5 below and give us a sufficient condition for a space X to be written as an intersection of two extra countably compact spaces.

For a space X let

$$\mathcal{A}(X) = N(\beta X)\setminus\{D \in N(\beta X): D^* \cap X \neq \emptyset\}.$$

Thus $\mathcal{A}(X)$ is the set of all countable discrete subsets of βX which have no accumulation points in X.

Definition. A collection $\mathcal{A}' \subset \mathcal{A}(X)$ is *cofinal in* $\mathcal{A}(X)$ if for each $D \in \mathcal{A}(X)$ there exists a set $D' \in \mathcal{A}'$ such that $D' \subset D$.

Definition. Let X be a space and $\mathcal{A}' \subset \mathcal{A}(X)$. Then X has property $\underline{(*)(\mathcal{A}')}$ if:

for each $D,E \in \mathcal{A}'$ and $V \in N(\overline{E})$ with

$|\overline{D} \cap V| < \omega$ and $|D \cap \overline{V}| < \omega$ we have

$D^* \cap V^* = \emptyset$.

That is, X has $(*)(\mathcal{A}')$ if for any D,V as above the remainders, D^* and V^*, are disjoint whenever the intersections $\overline{D} \cap V$ and $D \cap \overline{V}$ are finite.

Theorem 2.3. *Let* X *be a space. If* $\mathcal{A}(X)$ *contains a cofinal collection* $\mathcal{A}'$ *such that* X *has* $(*)(\mathcal{A}')$ *and* $|\overline{V}| = 2^c$ *for each* $V \in N(\overline{D})$ *and* $D \in \mathcal{A}'$*, then there exists a family* $\{P_\xi: \xi < 2^c\}$ *of extra countably compact spaces such that* $P_\xi \cap P_{\xi'} = X$ *whenever* $\xi < \xi' < 2^c$.

Proof. Let $\mathcal{A}'$ be a cofinal subset of $\mathcal{A}(X)$ with the properties listed above.

We want to construct extra countably compact spaces which pairwise intersect in X. For this purpose we need only to add accumulation points x_D to X for each $D \in \mathcal{A}(X)$. Since $\mathcal{A}'$ is cofinal in $\mathcal{A}(X)$ we can restrict our attention to $\mathcal{A}'$. That is, we need only choose accumulation points x_D for each D such that $D \in \mathcal{A}'$.

Let $\{D_\eta: \eta < \alpha\}$ be a well-ordering of $\mathcal{A}'$. For each $\eta < \alpha$ let

$$\{B_\xi(\eta): \xi < 2^c\}$$

be a family of pairwise disjoint subsets of D^*_η as in Lemma 2.1. That is, for each $\xi < 2^c$, $\eta < \alpha$ and $V \in N(\overline{D}_\eta)$ let

$$b_{\xi,V,\eta} \in D^*_\eta$$

be an accumulation point of V and

$$B_\xi(\eta) = \{b_{\xi,V,\eta}: V \in N(\overline{D}_\eta)\}, \xi < 2^c, \eta < \alpha.$$

We define the points of the sets $P_\xi, \xi < 2^c$, recursively as follows:

For each $\xi < 2^c$ and $V \in N(\overline{D}_0)$ let

$$p_{\xi,V,0} = b_{\xi,V,0}.$$

Now let $\zeta < \alpha$ and suppose for each $\xi < 2^c$, $\eta < \zeta$ and $V \in N(\overline{D}_\eta)$ points $p_{\xi,V,\eta}$ have been defined such that

(i) $p_{\xi,V,\eta} \neq p_{\xi',V',\eta'}$ if $\xi < \xi' < 2^c$, $\eta, \eta' < \zeta$, $V \in N(\overline{D}_\eta)$ and $V' \in N(\overline{D}_{\eta'})$ and

(ii) $p_{\xi,V,\eta}$ is an accumulation point of V for each $\xi < 2^c$, $\eta < \zeta$ and $V \in N(\overline{D}_\eta)$.

We must define $p_{\xi,V,\zeta}$ for each $\xi < 2^c$ and $V \in N(\overline{D}_\zeta)$.

Let $V \in N(\overline{D}_\zeta)$. We consider three cases.

Case 1. There exists $\eta < \zeta$ such that $|\overline{D}_\eta \cap V| = \omega$.

Let γ be the least such ordinal η and set

$$p_{\xi,V,\zeta} = p_{\xi,\overline{D}_\gamma \cap V,\gamma}$$

for each $\xi < 2^c$.

Case 2. Case 1 fails but there exists $\eta < \zeta$ such that

$$|D_\eta \cap \overline{V}| = \omega.$$

Let γ be the least such ordinal η and set

$$p_{\xi,V,\zeta} = p_{\xi,D_\gamma \cap \overline{V},\gamma}$$

for each $\xi < 2^c$.

Case 3. For each $\eta < \zeta$ we have

$$|\overline{D}_\eta \cap V| < \omega$$

and

$$|D_\eta \cap \overline{V}| < \omega.$$

Since X has $(*)(\mathcal{A}')$ we have

$$D_\eta^* \cap V^* = \emptyset$$

for each $\eta < \zeta$. Since all accumulation points of V belong to V^* we set

$$p_{\xi,V,\zeta} = b_{\xi,V,\zeta}$$

for each $\xi < 2^c$.

The definition of $p_{\xi,V,\eta}$ for $\xi < 2^c$, $\eta < \alpha$ and $V \in N(\overline{D}_\eta)$ is now complete.

Now for each $\xi < 2^c$ let

$$P_\xi = X \cup \{p_{\xi,V,\eta}\colon \eta < \alpha, V \in N(\overline{D}_\eta)\}.$$

Then $P_\xi \cap P_{\xi'} = X$ whenever $\xi < \xi' < 2^c$. If not, then for $p \in (P_\xi \cap P_{\xi'})\setminus X$ there exist $\eta, \eta' < \alpha$ and $V \in N(\overline{D}_\eta)$, $V' \in N(\overline{D}_{\eta'})$ such that

$$p = p_{\xi,V,\eta} \text{ and } p = p_{\xi',V',\eta'}.$$

But

$$p_{\xi,V,\eta} \neq p_{\xi',V',\eta'}$$

since $\xi < \xi' < 2^c$.

To see that each P_ξ is extra countably compact let $D \in N(\beta X)$. If $D^* \cap X \neq \emptyset$ then we are done. Otherwise $D \in \mathcal{A}(X)$ and therefore there exists $\eta < \alpha$ such that $D_\eta \subset D$. Then

$$p_{\xi,D_\eta,\eta} \in P_\xi$$

is an accumulation point of D for each $\xi < 2^c$.

The hypotheses of Theorem 2.5 may seem complicated but they are actually generalizations of properties satisfied by many kinds of spaces.

A space X is an F-*space* if every finitely generated ideal of C(X) is principal. It is noted in [GJ, 14.N.4, 14.25] that every countable subset of an F-space is C*-embedded and that a space X is an F-space if and only if βX is an F-space.

Corollary 2.4. *If* X *is an* F-*space then there exists a family* $\{P_\xi : \xi < 2^c\}$ *of extra countably compact spaces such that* $P_\xi \cap P_{\xi'} = X$ *whenever* $\xi < \xi' < 2^c$.

Proof. For each $D \in N(\beta X)$ we have $|\overline{D}| = 2^c$ since D is countable discrete and C*-embedded in βX.

Obviously $\mathcal{A}(X)$ is cofinal in $\mathcal{A}(X)$. We show that X has $(*)(\mathcal{A}(X))$.

Let $D,E \in \mathcal{A}(X)$ and $V \in N(\overline{E})$ such that

$$|\overline{D} \cap V| < \omega \text{ and } |D \cap \overline{V}| < \omega.$$

Let

$$D' = D\setminus\overline{V} \text{ and } V' = V\setminus\overline{D}.$$

Then $D' \cup V'$ is countable discrete and C*-embedded in βX. But $D' \cap V' = \emptyset$. Therefore we have

$$(D')^* \cap (V')^* = \emptyset.$$

Now $D' \subset D$ and $V' \subset V$ and D,V are both C*-embedded in βX. Since

$$|D\setminus D'| < \omega \text{ and } |V\setminus V'| < \omega$$

we have

$$D^* = (D')^* \text{ and } V^* = (V')^*.$$

Therefore

$$D^* \cap V^* = \emptyset.$$

Thus X has $(*)(\mathcal{A}(X))$.

The result now follows from Theorem 2.3.

It is well-known (see for example [GJ] or [CN_2]) that a space X is realcompact if and only if for every $p \in \beta X\setminus X$ there exists a zero set Z of βX such that $p \in Z \subset \beta X\setminus X$. We will show that realcompact spaces satisfy the hypotheses of Theorem 2.3. We need first the following three lemmas. We prove only the third.

Lemma 2.5. (cf. [GJ, 9.4]) *Let* X *be realcompact and* $D \in \mathcal{A}(X)$. *Then there exists a set* $E \in N(D)$ *such that* E *is* C-*embedded in* $X \cup E$.

Lemma 2.6. (cf. [Wa, 4.8]) *Let* X *be a space and* $D \in \mathcal{A}(X)$ *such that* D *is* C-*embedded in* $X \cup D$. *If* $V \in N(\beta X)$

is such that

$$\overline{D} \cap V = \emptyset \text{ and } D \cap \overline{V} = \emptyset$$

then

$$\overline{D} \cap \overline{V} = \emptyset.$$

Lemma 2.7. *Let* X *be a space and let* $D \in \mathcal{A}(X)$ *be such that* D *is* C-*embedded in* $X \cup D$. *If* $V \in N(\beta X)$ *is such that*

$$|\overline{V} \cap D| < \omega \text{ and } |\overline{D} \cap V| < \omega$$

then

$$V^* \cap D^* = \emptyset.$$

Proof. Let D,V be as above and set

$$D' = D \setminus \overline{V}$$

and

$$V' = V \setminus \overline{D}.$$

Since D' is C-embedded in D we have that D' is C-embedded in $X \cup D$.

Now,

$$\overline{D'} \cap V' = \emptyset \text{ and } D' \cap \overline{V'} = \emptyset$$

so that we may apply Lemma 2.6 to conclude

$$\overline{D'} \cap \overline{V'} = \emptyset.$$

But

$$D^* \subset \overline{D'} \text{ and } V^* \subset \overline{V'};$$

therefore

$$V^* \cap D^* = \emptyset.$$

Theorem 2.8. *If* X *is a realcompact space then there exists a family* $\{P_\xi : \xi < 2^c\}$ *of extra countably compact spaces such that* $P_\xi \cap P_{\xi'} = X$ *whenever* $\xi < \xi' < 2^c$.

Proof. We show the conditions of Theorem 2.3 are satisfied.

Let $\mathcal{A}' = \{D \in \mathcal{A}(X)\colon D \text{ is C-embedded in } X \cup D\}$. Then for each $D \in \mathcal{A}'$ we have D is C*-embedded in $X \cup D$ and therefore C*-embedded in βX. Since D is homeomorphic with $\mathbf{N}$ it follows that $\overline{D}$ is homeomorphic with $\beta\mathbf{N}$. Thus for each $V \in N(\overline{D})$ we have $|\overline{V}| = 2^{\mathfrak{c}}$.

From Lemma 2.5 we have that $\mathcal{A}'$ is cofinal in $\mathcal{A}(X)$; and from Lemma 2.7 we have that X has $(*)(\mathcal{A}')$.

The result now follows from Theorem 2.3.

A space X is *topologically complete* if and only if X can be embedded as a closed subset of a product of metric spaces. A topologically complete space X is realcompact if and only if every closed discrete subset of X has cardinality which is not Ulam-measurable, [CN_1, 6.3].

It follows that if one assumes Ulam-measurable cardinals do not exist then Theorem 2.8 remains valid with realcompact replaced by topologically complete. We will show below, however, that the conclusion fails for appropriately chosen topologically complete spaces if one assumes the existence of Ulam-measurable cardinals. We show, in fact, that if there exists an Ulam-measurable cardinal then there exists a metric space X which cannot be written as the intersection of any family of extra countably compact spaces. To see this we will need the following result, communicated in correspondence to W. W. Comfort independently by van Douwen and Kato.

For a space X and $p \in X$ the *character of* p *in* X is defined by $\mathcal{L}(P,X) = \min\{|\mathcal{N}|\colon \mathcal{N} \text{ is a neighborhood base at } p\}$.

Lemma 2.9. [C. O'C.] *Let* α *be the least Ulam-measurable cardinal and let* K *be any compact space such that* $\mathcal{L}(x,K) < \alpha$

for all $x \in K$. *Then* $\beta(K \times \alpha)\setminus(K \times \alpha)$ *contains a homeomorphic copy of* K.

Theorem 2.10. *Assume there is an Ulam-measurable cardinal. Then there exist metrizable spaces* X *which cannot be written as the intersection of any family of extra countably compact subspaces of* βX.

Proof. Let $K = \omega \cup \{p\}$ be the one-point compactification of ω, let α be the least Ulam-measurable cardinal with the discrete topology and set $X = K \times \alpha$. As a product of two metric spaces X is surely metrizable. Then $\beta X\setminus X$ contains a homeomorphic copy

$$h[K] = \{h(n)\colon n < \omega\} \cup \{h(p)\}$$

of K. Now $h[\omega]$ is a sequence in βX converging to its unique accumulation point $h(p)$; and $h(p) \in \beta X\setminus X$. Therefore any extra countably compact subspace of βX must contain $h(p)$. It follows that there is no family $\mathcal{P}$ of extra countably compact subspaces of βX such that $\cap\mathcal{P} = X$.

This last result suggests the following theorem which was pointed out to the author by Professor M. Henriksen.

Theorem 2.11. *A space* X *can be written as an intersection of extra countably compact subspaces of* βX *if and only if no sequence of* βX *converges to a point of* $\beta X\setminus X$.

Proof. If no sequence of βX converges to a point of $\beta X\setminus X$ then $\beta X\setminus\{p\}$ is extra countably compact in βX for each $p \in \beta X\setminus X$. Then the relation

$$X = \cap_{p\in\beta X\setminus X}(\beta X\setminus\{p\})$$

expresses X as an intersection of extra countably compact spaces.

On the other hand, if $\{p_n: n < \omega\} \subset \beta X$ and $\{p_n: n < \omega\}$ converges to $p \in \beta X \setminus X$ with $p_n \neq p$ for each $n < \omega$ then every extra countably compact subspace of βX must contain p. Thus X cannot be written as an intersection of extra countably compact subspaces of βX.

Despite Theorem 2.10 we can still show that every topologically complete space X may be written as the intersection of two countably compact spaces. This in fact will follow from Theorem 2.15 and a simple result due to Kato. We let γX denote the topological completion of X.

Theorem 2.12. [K] *Let* F *be a subset of a space* X *such that* $|F|$ *is not Ulam-measurable. Then*

$$\overline{F} \cap (\upsilon X \setminus \gamma X) = \emptyset.$$

Definition. Let X be a space and define

$$\omega^+ X = \cup\{\overline{D}: D \text{ is a countable subset of } X\}.$$

By Theorem 2.12 we have

$$\omega^+ X \cap (\upsilon X \setminus \gamma X) = \emptyset$$

for every space X.

The following result is probably well-known although we have been unable to find a reference for it.

Lemma 2.13. *Every countably compact topologically complete space* X *is compact.*

Theorem 2.14. *Let* X *be a noncompact, topologically complete space. Then there exists a family* $\{Q_\xi: \xi < 2^c\}$ *of countably compact subspaces of* βX *such that* $Q_\xi \cap Q_{\xi'} = X$ *whenever* $\xi < \xi' < 2^c$.

Proof. For a space X the space υX is realcompact. Then from Theorem 2.5 there exists a family $\{P_\xi : \xi < 2^c\}$ of extra countably compact spaces such that $P_\xi \cap P_{\xi'} = \upsilon X$ whenever $\xi < \xi' < 2^c$.

Let $Q_\xi = P_\xi \cap \omega^+ X$ for each $\xi < 2^c$. Then $Q_\xi \cap Q_{\xi'} = X$ whenever $\xi < \xi' < 2^c$ since $\omega^+ X \cap (\upsilon X \setminus X) = \emptyset$. ($X = \gamma X$ since X is topologically complete.)

To see that each Q_ξ is countably compact let D be a countable discrete subset of Q_ξ. Then since $D \subset \omega^+ X$ there is for each $d \in D$ a countable set $E_d \subset X$ such that $d \in \overline{E}_d$. Let $E = \cup\{E_d : d \in D\}$. Then $\overline{D} \subset \overline{E} \subset \omega^+ X$. Now P_ξ is extra countably compact in βX, so there exists $p_D \in P_\xi$ such that p_D is an accumulation point of D. But $p_D \in \overline{D} \subset \overline{E} \subset \omega^+ X$. Therefore $p_D \in P_\xi \cap \omega^+ X = Q_\xi$. Thus Q_ξ is countably compact.

Finally we note that if $\xi < \xi' < 2^c$ then $Q_\xi \neq Q_{\xi'}$. Otherwise we would have $Q_\xi = Q_{\xi'}$ for some $\xi < \xi' < 2^c$. Since $Q_\xi \cap Q_{\xi'} = X$ we must have $Q_\xi = Q_{\xi'} = X$. So that X is topologically complete and countably compact and therefore compact from Lemma 2.13. This, however, is a contradiction as X was assumed to be noncompact.

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