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## Research Announcement:

### A SUMMARY OF RESULTS ON ORDER-CAUCHY COMPLETIONS OF RINGS AND VECTOR LATTICES OF CONTINUOUS FUNCTIONS

by

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## A SUMMARY OF RESULTS ON ORDER-CAUCHY COMPLETIONS OF RINGS AND VECTOR LATTICES OF CONTINUOUS FUNCTIONS

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This paper is a summary of joint research by F. Dashiell, A. Hager and the present author. Proofs are largely omitted. A complete version will appear in the Canadian Journal of Mathematics. It is devoted to a study of sequential order-Cauchy convergence and the associated completion in vector lattices of continuous functions. Such a completion for lattices  $C(X)$  is related to certain topological properties of the space  $X$  and to ring properties of  $C(X)$ . The appropriate topological condition on the space  $X$  equivalent to this type of completeness for the lattice  $C(X)$  was first identified for compact spaces  $X$  in [D]. This condition is that every dense cozero set  $S$  in  $X$  should be  $C^*$ -embedded in  $X$  (that is, all bounded continuous functions on  $S$  extend to  $X$ ). We call Tychonoff spaces  $X$  with this property quasi-F spaces (since they generalize the  $F$ -spaces of [GH]).

In Section 1, the notion of a completion with respect to sequential order convergence is first described in the setting of a commutative lattice group  $G$ . A sequence  $\{g_n\}$  in  $G$  is said to be o-Cauchy if there exists a decreasing sequence  $\{u_n\}$  with  $\wedge u_n = 0$  in  $G$  and  $|g_n - g_{n+p}| \leq u_n$  for all  $n, p$ . If there exist such a sequence  $\{u_n\}$  and a  $g \in G$  with  $|g_n - g| \leq u_n$ , then  $\{g_n\}$  o-converges to  $g$ .  $G$  is o-Cauchy complete if each o-Cauchy sequence o-converges to

some  $g \in G$ . We give an abstract characterization of this completion and show how it applies to vector lattices and to certain lattice-ordered rings (including function rings) which satisfy a mild continuity condition for the multiplication.

In Section 2, the discussion of Section 1 is specialized to the lattice-ordered algebra  $C(X)$  of all continuous real-valued functions on a Tychonoff space  $X$ . In Theorem 2.1, the  $o$ -Cauchy completion of  $C(X)$  is described as the algebra of all bounded continuous functions defined on some countable intersection of dense cozero sets in  $\beta X$  (the domain depending on the function).

In Section 3, the description of the  $o$ -Cauchy completion of  $C(X)$  and of the subalgebra  $C^*(X)$  of bounded functions is made more explicit. It is described in terms of the uniform completion of certain algebras of functions defined on dense cozero subsets of  $X$  or of  $\beta X$  (see Corollary 3.5). It is shown in Theorem 3.7 that for any Tychonoff space  $X$ ,  $C(X)$  is  $o$ -Cauchy complete if and only if  $X$  is a quasi-F space (as defined above). For every space  $X$ , the  $o$ -Cauchy completion of  $C^*(X)$  takes the form  $C(K(X))$  for a certain compact space  $K(X)$  (which is necessarily a quasi-F-space). We show how to construct  $K(X)$  as the inverse limit space of  $\{\beta S: S \text{ is a dense cozero set in } X\}$ , as well as two other equivalent inverse limit constructions. An example is given of a  $C(X)$  whose  $o$ -Cauchy completion is not a  $C(Y)$ .

In Section 4, the above mentioned space  $K(X)$ , for compact  $X$ , is characterized by the property of being a quasi-F

space admitting a continuous irreducible surjection onto  $X$  which is minimal in a certain natural sense. Accordingly, we call  $K(X)$  the minimal quasi-F cover of  $X$ . This is similar to the description of Gleason's minimal projective cover  $G(X)$  for a compact  $X[\beta X]$  as being the only extremally disconnected space admitting a continuous irreducible surjection onto  $X$ . We show that, for an arbitrary  $X$ , the  $\sigma$ -Cauchy completion of  $C(X)$  coincides with the Dedekind completion if and only if  $K(X) = G(\beta X)$ , and this is true whenever every dense open subset of  $X$  contains dense cozero set.

In Section 5, we study quasi-F spaces per se and characterize them in terms of the ring  $C(X)$ . If  $\beta X$  is zero-dimensional, then  $X$  is a quasi-F space if and only if every non-divisor of zero in  $C(X)$  is a multiple of its absolute value, but the sufficiency can fail if  $X$  is not strongly zero-dimensional. A  $\sigma$ -compact space is a quasi-F-space if and only if each of its dense Baire sets is  $C^*$ -embedded. First countable quasi-F spaces are discrete. Every Tychonoff space is a closed subspace of some quasi-F space. We conclude with some results on products of quasi-F spaces.

### 1. Order-Cauchy Completions of $\ell$ -Groups

The term " $\ell$ -group" will be used to denote a commutative lattice-ordered group  $G(+, \vee, \wedge)$ , where, as usual  $a \vee b$ , respectively,  $a \wedge b$  denote the least upper bound and the greatest lower bound of  $a$  and  $b$ . We let  $a^+ = a \vee 0$ ,  $a^- = (-a) \vee 0$  and  $|a| = a^+ + a^-$ . We will write  $g_n \downarrow$  if the sequence  $\{g_n\}$  is decreasing; if in addition  $\wedge g_n = g$ , we will write  $g_n \downarrow g$ . Increasing sequences are handled similarly. An

embedding  $G \hookrightarrow H$  of  $G$  into an  $\ell$ -group  $H$  is called  $\sigma$ -regular if it preserves all existing countable suprema and infima in  $G$ ; that is if  $g_n \downarrow 0$  in  $G$  implies  $g_n \downarrow 0$  in  $H$ .

1.1. *Definitions.* Suppose  $G$  is an  $\ell$ -group,  $\{g_n\}$  is a sequence in  $G$ , and  $g \in G$ .

- (a) The sequence  $\{g_n\}$  order-converges (or  $o$ -converges) to  $g$ , written  $g_n \xrightarrow{o} g$  or  $o\text{-}\lim g_n = g$ , if  $|g_n - g| \leq u_n$ , for  $n = 1, 2, 3, \dots$ , for some  $u_n \downarrow 0$  in  $G$ . (Such limits are unique.)
- (b) The sequence  $\{g_n\}$  is order-Cauchy (or  $o$ -Cauchy) if, for some  $u_n \downarrow 0$  in  $G$ ,  $|g_n - g_{n+p}| \leq u_n$  for all  $n, p$ .
- (c)  $G$  is called order-Cauchy complete (or  $o$ -Cauchy complete) if each  $o$ -Cauchy sequence in  $G$   $o$ -converges to a limit in  $G$ .

We are interested in constructing a minimal "completion" of  $\ell$ -groups  $G$  with respect to  $o$ -Cauchy sequences. Our applications to follow are concerned with richer structure (i.e.,  $G = C(X)$ ), and it is pertinent to ascertain what algebraic structure is preserved by this completion process. Accordingly, we take the following as our definition of completion.

1.2. *Definition.* Let  $L$  denote any subcategory of  $\ell$ -groups (e.g.,  $\ell$ -groups, vector lattices,  $\ell$ -rings,  $\ell$ -algebras). For  $G$  in  $L$ , an  $o$ -Cauchy completion of  $G$  (in  $L$ ) is an  $H$  in  $L$  together with an  $L$ -embedding  $G \hookrightarrow H$  satisfying:

- (a)  $H$  is  $o$ -Cauchy complete;
- (b)  $G$  is  $\sigma$ -regular in  $H$ ; and
- (c) for each  $h \in H$  there exist sequences  $\{g_n\}, \{u_n\}$  in

$G$  with  $u_n \neq 0$  and  $|g_n - h| \leq u_n, n = 1, 2, \dots$ .

Such an  $H$  is called *essentially unique (in  $L$ )* if, for every  $H'$  which is an  $\sigma$ -Cauchy completion of  $G$  in  $L$ , there is an  $L$ -isomorphism from  $H$  onto  $H'$  which restricts to the identity on  $G$ .

We record below (1.3 and 1.5) two lemmas due to Papangelou which are used in the construction of a completion, in the proof of its uniqueness, and in subsequent material.

1.3. *Lemma.* [P, 2.10]. *A sequence  $\{g_n\}$  in an  $\ell$ -group  $G$  is  $\sigma$ -Cauchy if and only if there exist sequences  $\{u_n\}, \{v_n\}$  in  $G$  such that  $u_n \leq g_n < v_n$  for all  $n$ ,  $\{u_n\}$  is increasing,  $\{v_n\}$  is decreasing, and  $\wedge (v_n - u_n) = 0$  in  $G$ .*

The following is immediate.

1.4. *Corollary.* *The  $\ell$ -group  $G$  is  $\sigma$ -Cauchy complete if and only if for every increasing sequence  $\{u_n\}$  in  $G$  sitting below a decreasing sequence  $\{v_n\}$  with  $\wedge (v_n - u_n) = 0$ , there exists  $g \in G$  with  $u_n \leq g \leq v_n$  for all  $n$  (and hence  $g = \vee u_n = \wedge v_n$ ).*

Given  $\ell$ -groups  $G$  and  $H$  and an  $\ell$ -group embedding  $G \hookrightarrow H$ , let  $G_1^H$  consist of all  $h \in H$  for which there exist sequences  $\{g_n\}, \{u_n\}$  in  $G$  such that  $u_n \neq 0$  in  $G$  and  $|g_n - h| \leq u_n$ .

1.5. *Lemma.* [P, 3.3]. *Suppose  $G$  is  $\sigma$ -regular in  $H$  and  $\{v_n\}$  is a decreasing sequence in  $G_1^H$  with  $v_n \neq \vee$  for some  $\vee \in H$ . Then there exists a decreasing sequence  $\{u_n\}$  in  $G$  with  $u_n \geq v_n$  for all  $n$  and  $u_n \neq \vee$ . The corresponding*

statement for increasing sequences also holds.

By "*l*-ring" we mean an *l*-group  $G$  with a multiplication making  $G$  into a ring satisfying  $xy \geq 0$  whenever  $x \geq 0$  and  $y \geq 0$  in  $G$  (see [F] or [BKW]). In order to construct an *o*-Cauchy completion for *l*-rings  $G$ , it seems necessary to assume some kind of order continuity for the multiplication in  $G$ , for example:

(\*) If  $u_n \downarrow 0$  in  $G$  and  $h \geq 0$  then  $hu_n \downarrow 0$  and  $u_nh \downarrow 0$ .

1.6. Lemma. Suppose  $G$  is an *l*-ring satisfying (\*).

(a) If  $g_n \xrightarrow{0} g$  and  $h_n \xrightarrow{0} h$ , then  $g_n h_n \xrightarrow{0} gh$ .

(b) If  $H$  is an *l*-ring and  $G$  is embedded as a  $\sigma$ -regular sub-*l*-ring of  $H$ , then  $G_1^H$  is a  $\sigma$ -regular sub-*l*-ring of  $H$ , and  $G_1^H$  satisfies (\*).

We can now state the main theorem of this section.

1.7. Theorem. Suppose  $G$  is an *l*-group (resp. vector lattice, *l*-ring satisfying (\*), *l*-algebra satisfying (\*)). Then  $G$  has an essentially unique *o*-Cauchy completion  $H$  among *l*-groups (resp. vector lattices, *l*-rings, *l*-algebras). Moreover,  $H$  is minimal in the sense that if  $H'$  is an *o*-Cauchy complete *l*-group and  $\phi: G \rightarrow H'$  is a  $\sigma$ -regular *l*-group embedding, then there is a unique order-preserving  $\tilde{\phi}: H \rightarrow H'$  extending  $\phi$ , and  $\tilde{\phi}(H) = \phi(G)_1^{H'}$ . The map  $\tilde{\phi}$  is necessarily an *l*-group embedding, and if in addition  $\phi: G \rightarrow H$  is an embedding of vector lattices (*l*-rings, *l*-algebras), then so is  $\tilde{\phi}$ .

An *l*-ring is called an *f*-ring if  $g \wedge h = 0$  and  $f \geq 0$

imply  $fg \wedge h = gh \wedge h = 0$ , or equivalently if it is a subdirect sum of totally ordered rings [F]. The following is due independently to Bernau [B, p. 622] and Johnson [J].

1.8. *Lemma. Every Archimedean f-ring G satisfies property (\*)*.

1.9. *Corollary. Every lattice-ordered ring (respectively, lattice-ordered-algebra) of real-valued functions on some set (with pointwise operations) has an essentially unique o-Cauchy completion in  $\ell$ -rings (respectively, in  $\ell$ -algebras)*.

We close this section with two remarks.

(i) It can be shown if  $G = Ba_1 [0,1]$  and  $H = Ba_2 [0,1]$  denote, respectively, the functions of Baire class 1 and 2 on  $[0,1]$ , then each of  $G$  and  $H$  are o-Cauchy complete, and the natural embedding of  $G$  into  $H$  is  $\sigma$ -regular. Hence condition definition (c) of Definition 1.2 cannot be replaced by the requirement that each  $h$  in  $H$  be the o-limit in  $H$  of a sequence in  $G$ .

(ii) Condition (\*) is not always necessary for the existence of an order-Cauchy completion of an  $\ell$ -ring. For example if  $G = R[x]$  is the ring of polynomials with real coefficients lexicographically ordered by terms of highest degree, then  $G$  is o-Cauchy complete but fails to satisfy (\*).

We do not know of any necessary and sufficient condition on an  $\ell$ -ring to guarantee that multiplication is preserved under the embedding described in Theorem 1.7.



## 2. The $\mathfrak{o}$ -Cauchy Completion of $C(X)$

We now specialize the discussion of §1 to the  $\ell$ -algebra  $C(X)$  of all continuous real-valued functions on the Tychonoff space  $X$  (equipped with pointwise operations). The sub- $\ell$ -algebra of bounded functions is denoted  $C^*(X)$ . In this section and the next we describe the  $\mathfrak{o}$ -Cauchy completion of  $C(X)$  (see 1.9) in several ways as  $\ell$ -algebras of functions, and for compact  $X$  we obtain in fact a  $C(K)$  for a certain compact space  $K$ .

Some terminology: For  $f: X \rightarrow \mathbb{R}$ , the *cozero set* of  $f$  is  $\text{coz } f = \{x \mid f(x) \neq 0\}$  and the *zeroset* is  $Z(f) = X - \text{coz } f$ . In a topological space  $X$ , a *cozero set* is a set  $\text{coz } f$  for some  $f \in C(X)$ . For  $X$  compact Hausdorff (or just normal), the cozero sets are exactly the open  $F_\sigma$ 's.  $\beta X$  will denote the Stone-Ćech compactification of  $X$ . For its properties, see [GJ, Chapter 6].

The method of construction employed here is quite similar to the method of [FGL, §2.4 and §4.1]. We first recall the generalities. Suppose  $\mathcal{J}$  is a filter base of dense subsets of a topological space  $X$ , i.e.,  $\mathcal{J}$  is a family of dense, nonempty subsets of  $X$  closed under finite intersections. Consider the set of all functions  $f \in C(S)$  for some  $S \in \mathcal{J}$ , and identify  $f \in C(S)$  with  $g \in C(T)$  if and only if  $f = g$  on  $S \cap T$ . Denote the set of all equivalence classes by  $C[\mathcal{J}]$ , and let  $C^*[\mathcal{J}]$  denote all the equivalence classes containing bounded functions. Alternatively, observe that  $\{C(S) : S \in \mathcal{J}\}$  or  $\{C^*(S) : S \in \mathcal{J}\}$  form directed systems, where  $S \supset T$  in  $\mathcal{J}$  yields the bonding homomorphism  $f \rightarrow f|_T$  for  $f \in C(S)$  (or  $f \in C^*(S)$ ). Then  $C[\mathcal{J}]$  and  $C^*[\mathcal{J}]$  are the

direct limits  $\varinjlim \{C(S) : S \in \mathcal{F}\}$  and  $\varinjlim \{C^*(S) : S \in \mathcal{F}\}$ .

One easily checks that  $C[\mathcal{F}]$  and  $C^*[\mathcal{F}]$  are  $\ell$ -algebras under the operations canonically induced by the  $C(S)$ . Furthermore, each  $C(S)$  or  $C^*(S)$  for  $S \in \mathcal{F}$  is isomorphically embedded as an  $\ell$ -algebra into  $C[\mathcal{F}]$  or  $C^*[\mathcal{F}]$ , since each  $S$  is dense. In particular, if  $X \in \mathcal{F}$ , then  $C(X)$  and  $C^*(X)$  are sub- $\ell$ -algebras of  $C[\mathcal{F}]$  and  $C^*[\mathcal{F}]$ .

As a notational convenience, we shall write  $f \in C[\mathcal{F}]$  if  $f \in C(S)$  for some  $S \in \mathcal{F}$ , thus ignoring the distinction between equivalence classes and representatives. In this case, we write  $S = \text{dom } f$ .

If  $\mathcal{F}$  is a filter base of dense sets in  $\beta X$  and  $\mathcal{F}$  contains all the dense cozero sets of  $\beta X$ , then there is a natural embedding of  $C(X)$  into  $C[\mathcal{F}]$ , as follows. Each  $f \in C(X)$  has a unique Stone-Ćech extension  $\beta f: \beta X \rightarrow R \cup \{\infty\}$  (the one-point compactification of  $R$ ), and if  $\text{fin}(f) = (\beta f)^{-1}(R)$  then  $\text{fin}(f) \in \mathcal{F}$  and  $(\beta f)|_{\text{fin}(f)} \in C(\text{fin}(f))$ . This provides the canonical  $\ell$ -algebra embedding  $C(X) \rightarrow C[\mathcal{F}]$ . Moreover, this embedding induces an embedding  $C^*(X) \rightarrow C^*[\mathcal{F}]$ .

If  $\mathcal{F}$  is a filter base of dense sets in  $\beta X$  containing all the dense cozero sets in  $\beta X$ , there is an  $\ell$ -algebra intermediate between  $C^*[\mathcal{F}]$  and  $C[\mathcal{F}]$  which is central to our subject. This is defined to be

$$C^\#[\mathcal{F}, X] = \{h \in C[\mathcal{F}] : |h| \leq f \text{ for some } f \in C(X)\},$$

where we have assumed  $C(X) \subset C[\mathcal{F}]$  by the above canonical embedding. Thus,  $C^*[\mathcal{F}] \subset C^\#[\mathcal{F}, X] \subset C[\mathcal{F}]$ , and in case  $X$  is compact, then  $C^*[\mathcal{F}] = C^\#[\mathcal{F}, X]$ . In the following, the dependence of  $C^\#[\mathcal{F}, X]$  on the space  $X$  will be implicitly

understood, and we shall for convenience suppress explicit mention of  $X$  and write simply  $C^\#[\mathcal{J}]$ .

The results of [FGL] deal primarily with the case where  $\mathcal{J}$  is taken to be either the family of dense open sets or the family of dense  $G_\delta$  sets in  $X$  (for the latter,  $X$  is assumed compact, and closure under finite intersections follows from the Baire category theorem). It turns out that the structure required for the present purposes is obtained by taking for  $\mathcal{J}$  either the family of dense cozero sets or the family of countable intersections of dense cozero sets. These families are denoted  $C(X)$  and  $C_\delta(X)$ , respectively. Some of the results here are exactly analogous to the corresponding results in [FGL], but the proofs are different, apparently of necessity.

The main result of this section now follows. It is analogous to the representation of the Dedekind MacNeille completion (by cuts) of  $C(X)$  as  $C^\#[\mathcal{G}_\delta]$ , where  $\mathcal{G}_\delta$  is the class of all dense  $G_\delta$ -sets in  $\beta X$  (see [FGL], 4.11 and 4.6)].

2.1. *Theorem.* *The  $o$ -Cauchy completion of  $C(X)$  (as an  $\ell$ -algebra) is  $C^\#[C_\delta(\beta X)]$ .*

Some preliminary facts are needed to outline the proof. Recall that a subgroup  $G$  of an  $\ell$ -group  $H$  is called *order-convex* if  $0 \leq h \leq g$  and  $g \in G$  imply  $h \in G$ .

2.2. *Lemma.* *Any order-convex sub- $\ell$ -group  $G$  of an  $o$ -Cauchy complete  $\ell$ -group  $H$  is itself  $o$ -Cauchy complete.*

2.3. A subspace  $S$  of a space  $X$  is called  *$z$ -embedded* if whenever  $Z$  is a zero-set in  $S$ , then  $Z = Z' \cap S$  for some

zero-set  $Z'$  in  $X$ . Since every  $S \in \mathcal{C}_\delta(\beta X)$  is a Baire set in  $\beta X$  and is therefore Lindelöf [CN, p. 77], and a Lindelöf subspace is always  $z$ -embedded [CN, p. 79], each  $S \in \mathcal{C}_\delta(\beta X)$  is  $z$ -embedded in every superspace.

The following approximation property characterizes  $z$ -embedded subspaces. See [H] or [BH] for a proof.

2.4. *Lemma.*  $S$  is  $z$ -embedded in  $X$  if and only if given  $h \in C(S)$  and  $\epsilon > 0$  there exist a cozero set  $T$  in  $X$  with  $S \subset T$  and  $g \in C(T)$  such that  $|h(x) - g(x)| < \epsilon$  for  $x \in S$ .

By using 2.4, we can establish:

2.5. *Lemma.* Suppose  $S$  is  $z$ -embedded in  $X$ ,  $h \in C(S)$ , and there exists  $f \in C(X)$  such that  $|h(x)| \leq f(x)$  for all  $x \in S$ . Then there exist sequences  $\{u_n\}$  and  $\{v_n\}$  in  $C(X)$  such that  $u_1 \leq u_2 \leq \dots \leq v_2 \leq v_1$ , and for each  $x \in S$

$$h(x) = \sup u_n(x) = \inf v_n(x).$$

*Proof of 2.1.* (Outline) We need to show that the  $\ell$ -algebras  $G = C(X)$  and  $H = C^\#[\mathcal{C}_\delta(\beta X)]$  satisfy the three conditions of Definition 1.2.

To prove conditions (a) and (b) of Definition 1.2, we show first the following:

(+) If  $S \in \mathcal{C}_\delta(\beta X)$ ,  $w_n \rightarrow 0$  in  $C(S)$ , and  $T = \{x \in S : w_n(x) \rightarrow 0\}$ , then  $T \in \mathcal{C}_\delta(\beta X)$ .

To show (a) that  $H$  is  $o$ -Cauchy complete, it suffices by Lemma 2.2 to show that  $C[\mathcal{C}_\delta(\beta X)]$  is  $o$ -Cauchy complete with the aid of Corollary 1.4 and (+). To show (b), we use (+) again, and to obtain (b), use is made of Lemma 2.5.

2.6. *Corollary.* The  $\mathfrak{o}$ -Cauchy completion of  $C^*(X)$  is  $C^*[\mathcal{C}_\delta(\beta X)]$ . In particular, for compact  $X$ , the  $\mathfrak{o}$ -Cauchy completion of  $C(X)$  is  $C[\mathcal{C}_\delta(X)]$ .

### 3. More on the $\mathfrak{o}$ -Cauchy Completion of $C(X)$

In order to amplify the description of the  $\mathfrak{o}$ -Cauchy completion of  $C(X)$  given in 2.1, we need to study the relationship between the  $\mathfrak{l}$ -algebra  $C[\mathcal{J}]$  for various filter-bases  $\mathcal{J}$  of dense sets in  $X$  or in  $\beta X$ . We will be specifically concerned with  $\mathcal{C}(X)$ ,  $\mathcal{C}(\beta X)$ , and  $\mathcal{C}_\delta(\beta X)$ .

Observe first that  $C[\mathcal{C}(\beta X)]$  is embedded as a sub- $\mathfrak{l}$ -algebra of  $C[\mathcal{C}(X)]$  by restriction: if  $S \in \mathcal{C}(\beta X)$  and  $f \in C(S)$  then  $S \cap X \in \mathcal{C}(X)$  and  $f|_{(S \cap X)} \in C[\mathcal{C}(X)]$ . By abuse of notation we write  $C[\mathcal{C}(\beta X)] \subset C[\mathcal{C}(X)]$ . This relation is in fact an equality, as the following lemma will show. The essence of this result is contained in [FGL, 3.8]. Recall that a subspace  $S$  of  $X$  is  $C^*$ -embedded if every  $f \in C^*(S)$  extends to some  $\bar{f} \in C^*(X)$ .

3.1. *Lemma.* Let  $X$  be a  $C^*$ -embedded subspace of a Tychonoff space  $Y$ . Every continuous function on a cozero subset of  $X$  extends continuously to a cozero subset of  $Y$ .

3.2. *Corollary.*  $C[\mathcal{C}(\beta X)] = C[\mathcal{C}(X)]$ .  
 $C^\#[\mathcal{C}(\beta X)] = C^\#[\mathcal{C}(X)]$ .  
 $C^*[\mathcal{C}(\beta X)] = C^*[\mathcal{C}(X)]$ .

If  $\mathcal{J}$  is a filter base of dense sets in  $X$ ,  $C[\mathcal{J}]$  has a natural metric topology: the topology of uniform convergence, in which a sequence  $\{f_n\}$  converges to  $f$  if and only if for

each  $\epsilon > 0$ , eventually  $|f_n - f| \leq \epsilon \cdot 1$  in the lattice  $C[\mathcal{F}]$ .

The following lemma is [FGL, 4.5].

3.3. *Lemma.* *If  $\mathcal{F}$  is closed under countable intersections then  $C[\mathcal{F}]$  is uniformly complete.*

3.4. *Proposition.*  *$C[\mathcal{C}(BX)]$  is uniformly dense in  $C[\mathcal{C}_\delta(BX)]$ , so that  $C[\mathcal{C}_\delta(BX)]$  is the uniform completion of  $C[\mathcal{C}(X)]$  (or of  $C[\mathcal{C}(BX)]$ ).*

3.5. *Corollary.* *The  $o$ -Cauchy completion of  $C(X)$  (i.e.,  $C^\#[\mathcal{C}_\delta(BX)]$ ) is the uniform completion of  $C^\#[\mathcal{C}(X)] = C^\#[\mathcal{C}(BX)]$ . The  $o$ -Cauchy completion of  $C^*(X)$  (i.e.,  $C^*[\mathcal{C}_\delta(BX)]$ ) is the uniform completion of  $C^*[\mathcal{C}(X)] = C^*[\mathcal{C}(BX)]$ .*

The analogues of 3.2 and 3.4 for dense open sets are proved in [FGL].

Recall that  $X$  is an  $F$ -space if each of its cozero sets is  $C^*$ -embedded (See [GJ, 14.25]).

3.6. *Definition.* A Tychonoff space is called a quasi- $F$ -space if each dense cozero set is  $C^*$ -embedded.

The following result was originally proved in [D] for the case of compact  $X$  by a rather more direct argument. An extensive description of quasi- $F$  spaces is given in Section 5.

3.7. *Theorem.* *For an arbitrary Tychonoff space  $X$ ,  $C(X)$  is  $o$ -Cauchy complete if and only if  $X$  is a quasi- $F$  space.*

We recall some generalities about direct and inverse limits. Let  $\{K_a\}$  be any inverse system of compact Hausdorff spaces with respect to surjections  $\pi_a^b: K_b \rightarrow K_a$  for  $a \leq b$ . Then  $\{C(K_a)\}$  is a direct system of  $\ell$ -algebras with respect to the embeddings  $f_a \rightarrow f_b = f_a \circ \pi_a^b$ ,  $a \leq b$ . The *inverse limit space*  $K = \varprojlim_a K_a$  is a compact Hausdorff space and the direct limit  $A = \varinjlim_a C(K_a)$  is an  $\ell$ -algebra.

3.8. *Theorem.* [FGL, 6.8]. *The  $\ell$ -algebra  $A = \varinjlim_a C(K_a)$  is isomorphic with a uniformly dense sub- $\ell$ -algebra of  $C(K)$ , where  $K = \varprojlim_a K_a$ , and  $K$  is the maximal ideal space of  $A$ .*

For a Tychonoff space  $X$ , we consider the directed systems  $\mathcal{C}(X)$  of dense cozero sets in  $X$  and  $\mathcal{C}_\delta(\beta X)$  of dense countable interesections of cozero sets in  $\beta X$ . The system  $\{\beta S: S \in \mathcal{C}(X)\}$  is an inverse system of compact spaces with respect to the surjections  $\pi_T^A: \beta S \rightarrow \beta T$  which extend the inclusions  $S \subset T$ . Similarly,  $\{\beta S: S \in \mathcal{C}_\delta(\beta X)\}$  is an inverse system of compact spaces. We now define the inverse limit spaces

$$K(X) = \varprojlim \{\beta S: S \in \mathcal{C}(X)\}$$

and

$$K_\delta(X) = \varprojlim \{\beta S: S \in \mathcal{C}_\delta(\beta X)\}.$$

Since  $K(X)$  is a certain subset of  $\prod\{\beta S: S \in \mathcal{C}(X)\}$ , there exists a natural, continuous surjection and projection  $\pi_X: K(X) \rightarrow \beta X$ . This induces a natural embedding of  $C(\beta X)$  into  $C(K(X))$  by  $f \rightarrow f \circ \pi_X$ . Since  $C^*(X)$  is isomorphic with  $C(\beta X)$ ,  $C^*(X)$  is naturally embedded as a sub- $\ell$ -algebra of  $C(K(X))$ .

3.9. *Theorem.* (a) The spaces  $K(X)$ ,  $K(\beta X)$ , and  $K_\delta(X)$  are all homeomorphic and are quasi-F spaces.

(b) The natural embedding  $C^*(X) \rightarrow C(K(X))$  is a realization of the  $o$ -Cauchy completion of  $C^*(X)$  as the space  $C(K(X))$ .

In contrast to 3.9, if  $X$  fails to be compact, the  $o$ -Cauchy completion of  $C(X)$  need not be a  $C(Y)$ . Such an example may be constructed with the aid of Proposition 4.6 below (which gives a sufficient condition for  $K(X)$  to coincide with the Gleason cover) and enables us to modify an example given in [MJ] of a space  $X$  such that the Dedekind-MacNeille completion of  $C(X)$  is not a  $C(Y)$ .

#### 4. The Quasi-F Cover

Next, we examine some of the properties of the pair  $(K(X), \pi_X)$ , which we shall call the *minimal quasi-F cover* of  $X$ , where  $\pi_X: K(X) \rightarrow X$  is the canonical projection (see 4.3).

Recall that a map  $\pi: X \rightarrow Y$  is *irreducible* if  $X$  is the only closed subspace of  $X$  whose image under  $\pi$  is all of  $Y$ . A subset  $G$  of an  $\ell$ -group  $H$  is *order-dense* if for each nonzero  $h \geq 0$  in  $H$  there exists a nonzero  $g \in G$  with  $0 \leq g \leq h$ . The following lemma appears in [We, p. 17].

4.1. *Lemma.* If  $X$  and  $Y$  are compact then a map  $\pi: X \rightarrow Y$  is irreducible if and only if the dual embedding  $\pi^\circ: C(Y) \rightarrow C(X)$  has an order-dense image in  $C(X)$ .

4.2. *Definition.* A minimal quasi-F cover for a compact space  $X$  is a pair  $(K, \pi)$  such that:



- (a)  $K$  is a compact quasi-F space;
- (b)  $\pi: K \rightarrow X$  is a continuous irreducible surjection;
- (c) if  $(K_1, \pi_1)$  is a pair satisfying (a) and (b) then there exists a continuous surjection  $\tau: K_1 \rightarrow K$  such that  $\pi_1 = \pi \circ \tau$ .

4.3. *Theorem.* If  $X$  is compact, then  $(K(X), \pi_X)$  is a minimal quasi-F-cover which is unique in the sense that if  $(K, \pi)$  is a minimal quasi-F-cover, then there exists a unique homeomorphism  $\tau: K \rightarrow K(X)$  such that  $\pi = \pi_X \circ \tau$ .

4.4. *Remarks.* As continuous surjection  $\pi: K \rightarrow X$  is called *strongly irreducible* if for every cozero set  $V \subset K$ , there is a cozero set  $W \subset X$  such that  $\pi^{-1}[W]$  is dense in  $V$ . F. Dashiell has shown that the projection map  $\pi: K(X) \rightarrow X$  is strongly irreducible if  $X$  is compact and that a strongly irreducible map of a compact space onto a quasi-F-space is a homeomorphism.

In as yet unpublished work, Charles Neville has determined some classes of mappings for which quasi-F-spaces become projective in the sense of [Gℓ].

For any  $\ell$ -group  $G$ , the  $o$ -Cauchy completion and the Dedekind-MacNeille completion by cuts are the same if and only if the  $o$ -Cauchy completion (as given in Theorem 1.7) is Dedekind complete.

4.5. *Proposition.* Let  $X$  be a Tychonoff space. The following are equivalent:

- (1) The  $o$ -Cauchy completion of  $C(X)$  is Dedekind complete.
- (2) The  $o$ -Cauchy completion of  $C^*(X)$  is Dedekind complete.

(3)  $K(X)$  is extremally disconnected.

(4) The minimal quasi-F cover of  $\beta X$  is the same as Gleason's minimal projective cover of  $\beta X$ .

4.6. *Proposition.* If  $X$  is a Tychonoff space and every dense open set of  $X$  contains a dense cozero set of  $X$ , then  $K(X)$  is extremally disconnected, and the  $\circ$ -Cauchy completion of  $C(X)$  is Dedekind complete.

Recall from [CHN] that a space  $X$  is called *weakly Lindelöf* if each of its open covers contain a countable subfamily whose union is dense in  $X$ .

4.7. *Corollary.* If  $X$  satisfies any one of the conditions:

(1)  $X$  is perfectly normal (in particular if  $X$  is metrizable);

(2)  $X$  has the countable chain condition;

(3) every dense (open) subset of  $X$  is weakly Lindelöf;

then  $K(X)$  is extremally disconnected and the  $\circ$ -Cauchy completion of  $C(X)$  is the Dedekind-MacNeille completion.

Note that (2) implies (3).

4.8. *Corollary.* If  $X$  is a quasi-F space in which every dense open subset contains a dense cozero set (in particular, if any of the conditions of 4.7 hold), then  $X$  is extremally disconnected.

4.8 also follows immediately from the definition of quasi-F-spaces, since  $X$  is extremally disconnected whenever every dense open set is  $C^*$ -embedded.

### 5. Characterizations of Quasi-F-Spaces

In this section, quasi-F-spaces are characterized in a number of ways both topologically and in terms of the ring of continuous real-valued functions on the space. These characterizations are used in a number of ways; in particular to study when a finite product of quasi-F-spaces is a quasi-F-space.

Recall that an element  $r$  of a commutative ring  $A$  is called *regular* if  $ra = 0$  for  $a \in A$  implies that  $a = 0$ . An ideal of  $A$  is called *regular* if it contains a regular element. Note that an  $r \in C(X)$  is regular if and only if  $\text{coz}(r)$  is dense in  $X$ .

If  $A$  and  $A'$  are lattice-ordered and  $\phi: A \rightarrow A'$  is a ring homomorphism that preserves the partial ordering on  $A$ , then we call the kernel of  $\phi$  an *order-convex ideal* of  $A$ . If  $\phi$  also preserves the lattice operations of  $A$ , we call its kernel an  $\ell$ -*ideal* of  $A$ . It is well-known that a ring ideal  $I$  is order-convex [resp. an  $\ell$ -ideal] if and only if  $0 \leq a \leq b$  [resp.  $|a| \leq |b|$  and  $b \in I$  imply that  $a \in I$ ] [F]. (In [GJ] our order-convex ideals are called convex ideals, and our  $\ell$ -ideals are called absolutely convex ideals.)

5.1. *Theorem.* *If  $X$  is a Tychonoff space, then the following are equivalent.*

- (a)  $X$  is a quasi-F space.
- (b) Every dense  $z$ -embedded subspace of  $X$  is  $C^*$ -embedded.
- (c) Whenever  $f$  and  $r$  are elements of  $C(X)$  such that  $|f| \leq |r|$  and  $r$  is regular, then  $f$  is a multiple of  $r$ .

- (d) Every regular ideal of  $C(X)$  is order-convex.
- (e) Every regular ideal of  $C(X)$  is an  $\mathfrak{l}$ -ideal.
- (f) Every finitely generated regular ideal of  $C(X)$  (with generators  $f_1, \dots, f_n$ ) is principal (with generator  $|f_1| + \dots + |f_n|$ ).
- (f') Every regular ideal of  $C(X)$  with two nonnegative generators is principal.
- (g)  $C(X)$  is  $\sigma$ -Cauchy complete as a vector lattice.
- (h)  $\beta X$  is a quasi-F-space.

Furthermore, an equivalent condition is obtained if  $C(X)$  is replaced by  $C^*(X)$  in any of the preceding conditions.

Suppose  $X$  is a topological space. The members of the  $\sigma$ -field of subsets of  $X$  generated by the cozero sets of  $X$  are called *Baire sets*.

5.2. *Corollary.* Consider the following properties of a Tychonoff space  $X$ .

- (a) Every dense Baire set in  $X$  is  $C^*$ -embedded.
  - (b)  $X$  is a quasi-F-space.
  - (c) Every dense Lindelöf subspace of  $X$  is  $C^*$ -embedded.
- Then (a) implies (b), (b) implies (c), and if  $X$  is  $\sigma$ -compact then (a), (b), and (c) are equivalent.

We call a space  $X$  *strongly zero-dimensional* if  $\beta X$  has a base for its topology consisting of sets that are closed (and open). In [He], L. Heider showed that  $X$  is strongly zero-dimensional if and only if each of its zerosets is a countable intersection of open and closed sets.

5.3. *Lemma.* Suppose  $X$  is strongly zero-dimensional.

- (a) Every  $z$ -embedded subspace of  $X$  is strongly zero-dimensional.
- (b) If  $Z_1$  and  $Z_2$  are disjoint zerosets of  $X$ , then there is open and closed set  $U$  in  $X$  such that  $Z_1 \subset U$  and  $Z_2 \subset X \setminus U$ .

Next, some known properties of  $F$ -spaces are generalized.

5.4. *Theorem.* Consider the following conditions of a Tychonoff space  $X$ .

- (a)  $X$  is a quasi- $F$ -space.
- (b) If  $f \in C(X)$  is regular, then there is a  $k \in C(X)$  such that  $f = k|f|$ .
- (c) If  $f \in C(X)$  is regular, then  $\text{pos } f$  and  $\text{neg } f$  are completely separated.

Then (a) implies (b), (b) and (c) are equivalent, and if  $X$  is strongly zero-dimensional, then (a), (b), and (c) are equivalent.

5.5. *Proposition.* If  $X$  is a quasi- $F$ -space, and  $x \in X$  has a countable base of neighborhoods, then  $x$  is an isolated point. In particular, any quasi- $F$ -space satisfying the first axiom of countability is discrete.

Next we give an example to show that neither the assumption that  $X$  is strongly zero-dimensional in Theorem 5.4 nor the assumption of  $\sigma$ -compactness in Corollary 5.2 can be deleted. First we describe a way of constructing certain kinds of topological spaces.

Let  $D$  denote an uncountable discrete space and let  $\alpha D = D \cup \{\infty\}$  denote its one-point compactification. Suppose

$Y$  is a subspace of a Tychonoff space  $X$ , let  $\Delta = \Delta(Y, X) = (X \times \alpha D) \setminus \{(p, q) : p \notin Y \text{ and } q \neq \infty\}$ , and refine the product topology on  $\Delta$  by letting any point whose second coordinate is not  $\infty$  be isolated. Then  $\Delta$  is said to be *the space obtained by attaching a copy of  $\alpha D$  to each point of  $Y$* .

5.6. *Example.* A Tychonoff space satisfying (c) of Theorem 5.4 that is not a quasi-F-space.

$\Delta_1 = \Delta([0, 1], [0, 1])$ , the space obtained by attaching a copy of  $\alpha D$  to the closed unit interval  $[0, 1]$  at each point of  $[0, 1]$  is such a space.

Note also that  $\Delta_1$  contains no dense Lindelöf subspace, so the implication (c) implies (b) of Corollary 5.2 need not hold if the hypothesis of  $\sigma$ -compactness is deleted.

Recall from [GJ, Chapter 14] that a Tychonoff space  $X$  is called a *P-space* if every zero set of  $X$  is open and from [L], that  $X$  is called an *almost-P-space* if each of its zero sets has a nonempty interior. Clearly every almost-P-space is a quasi-F-space. If  $X$  is any noncompact realcompact space, then  $\Delta(X, \beta X)$  is a quasi-F-space that is not an almost-P-space. A space with this latter property is called a *proper quasi-F-space*.

A closed subspace of a quasi-F-space need not be a quasi-F-space. In fact, since  $X$  is a closed subspace of  $\Delta(X, X)$  we have:

5.7. *Proposition.* *Every Tychonoff space  $X$  is homeomorphic to a closed subspace of an almost-P-space.*

$X$  is called an *F'-space* if for every  $f \in C(X)$ ,  $\text{pos } f$

and  $\text{neg } f$  have disjoint closures. Every normal  $F'$ -space is an  $F$ -space, but there are  $F'$ -spaces that are not  $F$ -spaces [GH, 8.14] and [CHN]. In [CHN, Theorem 1.1] it is shown that  $X$  is an  $F'$ -space if and only if every cozero set in  $X$  is  $C^*$ -embedded in its closure.

5.8. *Proposition.* Consider the following properties of a Tychonoff space  $X$ .

(a)  $X$  is an  $F'$ -space.

(b) The closure of any cozero set of  $X$  is a quasi- $F$ -space.

(c)  $X$  is an  $F$ -space.

(d) Every closed subset of  $X$  is a quasi- $F$ -space.

Properties (a) and (b) are equivalent. If  $X$  is normal, then (a), (b), (c), and (d) are equivalent.

In [K, Example 3], Carl Kohls gives an example of an (extremally disconnected)  $F$ -space  $X$  with a closed subspace  $Y$  that is not an  $F'$ -space; indeed,  $Y$  is not a quasi- $F$ -space.

Next we consider conditions under which the property of being a quasi- $F$ -space is preserved under finite products.

5.9. *Proposition.* Suppose  $X_1$  and  $X_2$  are Tychonoff spaces.

(a)  $X_1 \times X_2$  is an almost- $P$ -space if and only if both  $X_1$  and  $X_2$  are almost- $P$ -spaces.

(b) If  $X_1 \times X_2$  is a quasi- $F$ -space, then so are  $X_1$  and  $X_2$ .

(c) If  $X_1$  and  $X_2$  are strongly zero-dimensional and  $X_1 \times X_2$  is a quasi- $F$ -space, then  $X_1$  or  $X_2$  is an

*almost P-space.*

(d) *If  $X_1 \times X_2$  is a quasi-F-space and  $X_2$  is a compact proper quasi-F-space, then  $X_1$  is a P-space.*

5.10. *Corollary. The product of two infinite compact spaces is never a proper quasi-F-space.*

We do not know if the requirements that  $X_1$  and  $X_2$  be strongly zero-dimensional in the statement of Proposition 5.9(c), or the requirement that  $X_2$  be compact in the statement of Proposition 5.9(d) are necessary.

In [N, Theorem 6.5], S. Negrepointis shows that  $X$  is a P-space if and only if  $X \times \beta X$  is an F-space. An analog of this result follows.

5.11. *Corollary. For any Tychonoff space  $X$ ,  $X \times \beta X$  is a quasi-F-space if and only if  $X$  is a P-space or  $\beta X$  is an almost-P-space.*

In [G], an example of an extremally disconnected and a P-space whose product is not an F-space is given. By modifying Gillman's argument, it can be shown that this latter product is not even a quasi-F-space.

The problem of determining exactly when a product of two spaces is a quasi-F-space seems to be at least as complicated as the corresponding one for F-spaces. See [CHN].

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