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**Research Announcement:**  
A METRIZATION THEOREM FOR  
GENERALIZED ORDERED SPACES

by

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## **A METRIZATION THEOREM FOR GENERALIZED ORDERED SPACES**

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This research report describes joint work with H. R. Bennett; details will appear in [BnL].

Most metrization theory for GO spaces (=generalized ordered spaces; cf. [L] for definitions and terminology) is based on Bing's theorem [Bi] that a space is metrizable iff it is collectionwise normal and developable. The theorem which will appear in [BnL] rests on a different base, namely a theorem of Bennett [Bn] which asserts that a space is metrizable iff it is a quasi-developable paracompact  $p$ -space. Here we use " $p$ -space" in the sense of Arhangel'skii [A<sub>1</sub>] and we say that a space  $X$  is *quasi-developable* if  $X$  has a base  $\beta = \cup\{\beta(n) : n \geq 1\}$  with the property that whenever  $p$  is a point of an open set  $U$ , there is an  $n \geq 1$  such that  $p \in \text{St}(p, \beta(n)) \subset U$ . (It is important to note that the collections  $\beta(n)$  are not required to be coverings of  $X$ .) Other definitions used in this report appear in [BkL]. The central result in [BnL] is

*Main Theorem.* *If  $X$  is a GO-space, the following are equivalent:*

- (a)  $X$  is metrizable;
- (b)  $X$  is hereditarily a  $p$ -space (i. e., each subspace of  $X$  is a  $p$ -space) [A<sub>2</sub>];
- (c)  $X$  is hereditarily an  $M$ -space [Mo];

- (d)  $X$  is hereditarily a  $w\Delta$ -space [Bo];  
 (e)  $X$  is hereditarily quasi-complete [C].

Our proof of that theorem is broken into several steps. The first step involves an extension of a result of van Wouwe [vW].

*Proposition.* The following properties of a GO-space  $X$  are equivalent:

- (a)  $X$  is an M-space;  
 (b)  $X$  is a  $w\Delta$ -space;  
 (c)  $X$  is quasi-complete.

(The equivalence of (a) and (b) in this proposition was due to van Wouwe.)

We next show that no stationary set in a regular uncountable cardinal can be hereditarily quasi-complete. In view of Theorem 2.4 in [EL], that is enough to show that a hereditarily quasi-complete GO-space must be hereditarily paracompact. But it is known that any paracompact M-space is a p-space [BkL] so that we now have the equivalence of (b), (c), (d) and (e) in the main theorem above. Thus it remains only to show that (b) implies (a). In the light of Bennett's metrization theorem, it is enough to show that a GO-space  $X$  which is hereditarily a p-space must be quasi-developable. In our proof, we decompose  $X$  into five special subspaces, each of which can be proved to be quasi-developable. But then their union will be quasi-developable since we can show that any GO-space which hereditarily a p-space must be first countable and that any first countable GO-space

which is the union of countably many quasi-developable subspaces must itself be quasi-developable. Our proof that the five special subspaces are quasi-developable rests on a technical lemma about linear orders. Our proof of that lemma is unreasonably long and messy; Brian Scott has told us that he has another proof, but it is also messy.

*Lemma.* *Let  $Y$  be any subset of a linearly ordered set  $X$ . Then there are disjoint subsets  $D, E \subset Y$  such that:*

- (a) *if  $p \in X$  and if  $]p, q[ \cap Y$  is infinite for each  $q > p$ , then for each  $q > p$  both sets  $]p, q[ \cap D$  and  $]p, q[ \cap E$  are also infinite;*
- (b) *if  $p \in X$  and if  $]q, p[ \cap Y$  is infinite for each  $q < p$ , then for each  $q < p$  both sets  $]q, p[ \cap D$  and  $]q, p[ \cap E$  are also infinite.*

The results described in this announcement are easy enough to state, but all known proofs are long and technical. Indeed, it was reassuring when van Wouwe informed us that he could give a different proof of our main theorem [vW] even though his proof, like ours, lacks the simplicity that we have come to expect in metrization theory for  $GO$  spaces.

There is a way to view our main theorem as a partial answer to a much more general problem which asks for a small, easily studied class  $\mathcal{C}$  of  $GO$  spaces such that every non-metrizable  $GO$ -space must contain a (homeomorphic copy of a) member of  $\mathcal{C}$  as a subspace. The analogous problem for paracompactness in  $GO$ -spaces is solved and can be used as a paradigm for the solution of the metrizability problem: one can take  $\mathcal{C}$  to be the class of stationary subsets of regular

uncountable cardinals and prove that any non-paracompact GO-space must contain a closed subspace belonging to  $\mathcal{C}$ . The main theorem, above, asserts that if we take  $\mathcal{C}$  to be the family of GO-spaces which are not p-spaces, then every non-metrizable GO-space contains a member of  $\mathcal{C}$ . However, that is not a satisfactory solution since there is no particularly good way to recognize GO-spaces which are not p-spaces, and no particularly useful tools for studying a GO-space even when one knows it is not a p-space.

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