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In his classic paper "Mappings and Spaces" [A] A. V. Arhangel'skiǐ introduced the class MOBI. In [B_1] it was shown that a space X is in MOBI if there is a metric space M and a finite set $\{\phi_1, \phi_2, \cdots, \phi_n\}$ of open-compact maps (= open continuous functions with compact fibers) such that

$$X = \phi_n \circ \cdots \circ \phi_1(M).$$

This characterization greatly facilitated the study of the class MOBI.

In [B_1] and [BB_1] many of the questions asked by Arhangel'skiǐ were answered negatively. Unfortunately a non-regular T_2 space was used. It became clear that the study of MOBI becomes much more interesting (and difficult) if all the spaces in MOBI must be at least regular T_1-spaces. The use of non-regular spaces did, however, give a great deal of useful insight into what could be expected to be found in the class MOBI. In [C] Chaber answered many of Arhangel'skiǐ's original questions using spaces that are at least regular T_1-spaces. He exploited a construction first used by Tall [T] in answering questions in completeness.

Recall that a space X is weak θ-refinable if for each open covering $\mathcal{U}$ of X there is an open refinement $\mathcal{G} = \cup\{\mathcal{G}_n | n = 1,2,\cdots\}$ such that each $\mathcal{G}_n$ is a collection of open sets and if $x \in X$, then there is a natural number n such that x is in only finitely many mmebers of $\mathcal{G}_n$ [BL].

The study of MOBI would be greatly advanced if it could be proved that the oc-image (= open compact image) of a hereditarily weak θ-refinable space is hereditarily weak θ-refinable. If this were the case, then, since each metric space is hereditarily weak θ-refinable, it would follow that each space in MOBI would be hereditarily weak θ-refinable. Then, since each space in MOBI has a base of countable order [WW], it would follow that each space in MOBI was quasi-developable [B_2]. This last result follows since each hereditarily weak θ-refinable space with a base of countable order is quasi-developable [BB_2].

Unfortunately, as the next example illustrates, it is not true that the oc-image of a hereditarily metacompact space is even weak θ-refinable. However, since spaces in MOBI have additional structure, it still may be the case that each space in MOBI is hereditarily weak θ-refinable (and, hence, quasi-developable). The example does use a non-regular domain space but it still gives valuable insight into the problem.

Before describing the example some notation is needed. Let Z be a linearly ordered topological space with order $\leq$. Then, if $a < b$,

$$[a,b] = \{x \in Z \mid a \leq x \leq b\},$$

$$]a,b[= \{x \in Z \mid a < x < b\}, \text{ and}$$

$$[a,b[= \{x \in Z \mid a \leq x < b\}.$$

Example. There is a hereditarily metacompact, non-regular T_2-space X and a linearly ordered topological space Y that is not weakly θ-refinable such that Y is the oc-image of X.

Let $X = \{(\alpha,\beta) \in [0,\omega_1[\times[0,\omega_1] \mid \beta > \alpha\}$ where, as usual, ω_1 denotes the first uncountable ordinal. For each $\beta < \omega_1$, topologize $L_\beta = \{(\alpha,\beta) \mid 0 \leq \alpha < \beta\}$ so L_β is homeomorphic to $[0,\beta[$ with usual interval topology. Since L_β is countable it can be indexed by the natural numbers. If α is a limit ordinal construct a local base for (α,β) such that each member of the local base is a convex set that does not contain a point of L_β that precedes (α,β) in the indexing of L_β. If α is a non-limit ordinal, let $\{(\alpha,\beta)\}$ be the only member of the local base for (α,β).

Let the local base for (α,ω_1) consist of all sets of

$$\{(\alpha,\omega_1)\} \cup (\cup\{U(\alpha,\beta) \mid \beta \in]\alpha,\omega_1[\setminus F\})$$

where F is a finite subset of $]\alpha,\omega_1[$ and $U(\alpha,\beta)$ is a member of the local base for (α,β).

It is easy to see that X topologized in this fashion is a T_2-space.

Let α be a limit ordinal and let U be a member of the local base for (α,ω_1). If V is an open set such that $x \in V \subset U$, then there is an ordinal $\eta < \alpha$ such that $(\delta,\beta) \in V$ for uncountably many $\beta > \alpha$. Thus $(\delta,\omega_1) \in \overline{V}$. Since $(\delta,\omega_1) \notin U$ it follows that X cannot be regular.

Let $\mathcal{U}$ be an open covering of X. For each $\alpha < \omega_1$ let U_α be a basic open set that refines some member of $\mathcal{U}$ such that $(\alpha,\omega_1) \in U_\alpha$ and, if $\alpha \neq \beta$, then $(\beta,\omega_1) \notin U_\alpha$. The collection $\mathcal{U}_1 = \{U_\alpha \mid \alpha < \omega_1\}$ is obviously a point finite at each (α,ω_1). If $\beta \neq \omega_1$ and $(\alpha,\beta) \in \cup\mathcal{U}_1$ then, by construction, $(\alpha,\beta) \in U_\eta$ only if (η,β) precedes (α,β) in the indexing of

L_β. Since only finitely many (η,β) precede (α,β) in the indexing it follows that $\mathcal{U}_1$ is point finite for each point in $\cup\mathcal{U}_1$. Since $X\setminus\cup\mathcal{U}_1$ is a metric space, an open (in X) refinement $\mathcal{U}_2$ of $\mathcal{U}$ may be found that is point-finite and covers $X - \cup\mathcal{U}_1$. Hence $\mathcal{U}_1 \cup \mathcal{U}_2$ is a point-finite open refinement of $\mathcal{U}$. Thus X is metacompact. Arguing in similar fashion it is readily seen that X is, in fact, hereditarily metacompact.

Let $Y = [0,\omega_1[$ with the usual linear order topology. It was shown in [BL] that Y is not weakly θ-refinable.

Let ϕ: $X \to Y$ be defined by $\phi((\alpha,\beta)) = \alpha$. It is easy to check that ϕ is an open-compact map of X into Y.

This example also answers some questions in [G].

References

[A] A. V. Arhangel'skiǐ, *Mappings and spaces*, Usp. Mat. Nouk 21 (1966), 133-184 (Russian).

[B_1] H. R. Bennett, *On Arhangel'skiǐ's class* MOBI, PAMS 26 (1970), 178-180.

[B_2] ________, *On quasi-developable spaces*, Gen. Top. and Appl. 2 (1972), 49-54.

[BB_1] ________ and E. S. Berney, *Subparacompactness and* G_δ*-diagonals in Arhangel'skiǐ's class* MOBI, PAMS 30 (1971), 573-581.

[BB_2] ________, *On certain generalizations of developable spaces*, Gen. Top. and Appl. 4 (1974), 48-50.

[BL] H. R. Bennett and D. J. Lutzer, *A note on weak* θ*-refinability*, Gen. Top and Appl. 2 (1972), 49-54.

[C] J. Chaber, *Metacompactness and the class* MOBI, Fund. Math. XCI (1976), 211-217.

[G] R. F. Gittings, *Open mapping theory, Set-theoretic topology*, Academic Press, 1977, 141-191.

[T] F. Tall, *On the existence of normal metacompact Moore spaces which are not metrizable*, Canad. J. Math. 26 (1974), 1-6.

[WW] H. H. Wicke and J. M. Worrell, *Open continuous mappings of spaces having bases of countable order*, Duke Math. J. 34 (1967), 255-271.

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