TOPOLOGY PROCEEDINGS
Volume 4, 1979
Pages 371–384

http://topology.auburn.edu/tp/

TOPOLOGIES DETERMINED BY PATHS

by

Stanley P. Franklin and Barbara V. Smith-Thomas

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGIES DETERMINED BY PATHS

Stanley P. Franklin and Barbara V. Smith-Thomas

1. Introduction

We began this study by re-examining Franklin's proposition ([F_4], p. 56) relating $\mathcal{A}$-generated spaces and the coreflective hull of $\mathcal{A}$ in TOP, where $\mathcal{A}$ is some class of spaces. Recall that a space X is *$\mathcal{A}$-generated* if it has the property (analogous with k-spaces) that a subset of X is closed if and only if its intersection with every $\mathcal{A}$-subspace of X is closed, and that X is in *the coreflective hull* of $\mathcal{A}$ if it is a quotient of a disjoint sum of members of $\mathcal{A}$. Franklin showed that if $\mathcal{A}$ is map-invariant, that is continuous images of $\mathcal{A}$-spaces are $\mathcal{A}$-spaces, then the $\mathcal{A}$-generated spaces and the coreflective hull of $\mathcal{A}$ coincide. (The essential ideas may be found in [F_1].) From now on we will denote the subcategory of $\mathcal{A}$-generated spaces by $\mathcal{A}$GS, and the coreflective hull of $\mathcal{A}$ by TOP($\mathcal{A}$). In the remainder of this section we make some observations about $\mathcal{A}$GS for some connectedness properties $\mathcal{A}$. We specialize in the next section to the path-generated spaces. The third section is devoted to the construction of an example of a space which is locally path connected and sequential, but which is not a quotient of paths.

Two common examples of map invariant classes are C, the class of connected spaces, and PC, the class of path connected spaces. From Franklin's theorem above we have CGS = TOP(C) and PCGS = TOP(PC). We first consider CGS: the connectedly generated spaces are characterized as those spaces whose components are open sets. This tells us that spaces in TOP(C) are not just quotients of disjoint sums of connected spaces, they are actually disjoint sums of connected spaces. It also tells us that each locally connected space is connectedly generated. It is well-known that in TOP a coreflection is simply a strengthening of the topology; the characterization above tells us that the proper strengthening for the TOP(C) coreflection is obtained by declaring components to be open. Since this process generates no new components we are done after one step. Some other observations which can be made at this point are that being connectedly generated is not hereditary or even closed hereditary, but that open subsets are CGS if and only if the space is locally connected; CGS is finitely productive, and totally disconnected members of CGS are discrete.

We now turn to PCGS. The situation is similar to CGS. A space in PCGS is one whose path components are open; such spaces are disjoint sums of path connected spaces. The coreflection is obtained by declaring path components open. Since every path connected space is connected

$$PCGS = TOP(PC) \subseteq TOP(C) = CGS.$$

That the inclusion is proper is demonstrated by any connected space whose path components are not open, for example the positive $\sin(\frac{1}{x})$ curve together with the non-positive x-axis in $\mathbb{R}^2$. Finally, we note that local path connectivity implies PCGS (since locally path connected implies path components of open sets are open), and that PCGS is finitely productive.

2. Characterization of T2 ($\mathcal{P}$) = T2 (II)

If we restrict our consideration to Hausdorff images the

Peano spaces, $\mathcal{P}$, are map invariant. Members of PGS $\cap$ T2 = T2($\mathcal{P}$), Hausdorff quotients of disjoint sums of Peano spaces, have the property that a subset is closed if and only if its intersection with every Peano subspace is closed. Since each Peano space is a quotient of the closed unit interval $\mathbb{I}$, we see that elements of T2($\mathcal{P}$) are actually quotients of disjoint sums of closed intervals, that is, T2($\mathcal{P}$) = T2($\mathbb{I}$). The coreflection in T2($\mathbb{I}$) of an arbitrary space is obtained by declaring all path closed subsets to be closed; this process does not disturb any paths so does not have to be iterated. Each member of T2($\mathbb{I}$) is clearly sequential, and since local path connectivity is preserved under taking disjoint sums and quotients, path generated implies locally path connected. (Characterize local path connectivity as "path components of open sets are open.") We have argued:

2.1 *Proposition.* T2($\mathbb{I}$) $\subseteq$ LPC $\cap$ Seq $\cap$ T2.

That this inclusion is proper is shown in the example of section 3.

We next bracket T2($\mathbb{I}$) on the other side with a proper inclusion.

2.2 *Proposition.* LPC $\cap$ 1st *countable* $\cap$T2 $\subsetneq$ T2($\mathbb{I}$)

Proof. First, $\mathbb{R}$ with the integers identified to a point is a space in T2($\mathbb{I}$) which is not first countable.

Now, suppose X is a first countable, locally path connected Hausdorff space and that F $\subseteq$ X is path closed. Let x $\in$ $cl_X F$. Choose a countable nested neighborhood base at x consisting of path connected sets, and a sequence in F, one term from each of the neighborhoods, say $x_i \in U_i$, $i = 1,2,\cdots$. For each i, choose a path $p_i[\frac{1}{i}, \frac{1}{i+1}] \to X$ such that $p_i(\frac{1}{i}) = x_i$, $p_i(\frac{1}{i+1}) = x_{i+1}$, and $p_i([\frac{1}{i}, \frac{1}{i+1}]) \subseteq U_i$. Now define a path p: $[0,1] \to X$ by

$$p(t) = \begin{cases} p_i(t) & t \in [\frac{1}{i}, \frac{1}{i+1}] \\ x & t = 0 \end{cases}.$$

Since the U_i form a nested neighborhood base p is continuous. Since F is path closed $F \cap p[0,1]$ is closed; hence, since each $x_i \in F$, so also $x \in F$. Thus F is closed.

This ability to run a path through (a subsequence of) a convergent sequence is precisely what is needed to go from sequential and locally path connected to path generated. Let us say that a space *has the subsequence-path property* if every convergent sequence contains a subsequence through which a path can be run.

2.3 *Theorem. A Hausdorff sequential space is in* T2(Π) *if and only if it has the subsequence-path property.*

Proof. Suppose X is sequential and has the subsequence-path property. Suppose also that $F \subseteq X$ is path closed; it suffices to show that F is sequentially closed. So let (x_i) be a sequence in F converging to a point $x \in X$. Take a subsequence (x_{i_k}) and let p: $\Pi \to X$ be the path through the subsequence. Then $F \cap p[\Pi]$ is closed in $p[\Pi]$ and contains each x_{i_k}. It follows that $x \in F$.

Conversely suppose X is in T2(Π), and suppose the sequence (x_i) converges to x. If (x_i) contains a constant subsequence we're done, so we may assume that (x_i) consists of distinct points, none of them equal to x. What we are asserting is that there is a path containing infinitely many of the points x_i. If this were not the case then $\{x_i\} \cap p[\mathrm{II}]$ would be closed for every path p, and thus $\{x_i\}$ would be closed, which it is not. (By abuse of notation, $\{x_i\}$ is shorthand for the set $\{x_i\}_{i=1}^{\infty}$ of terms of the sequence (x_i).)

Observe that the assumption that X be sequential cannot be dropped from this theorem. For, II^c is a non-sequential space with the subsequence-path property, as is any locally convex topological vector space which is not sequential.

3. Example

We are now ready to show that the inclusion $T2(\mathrm{II}) \subseteq$ LPC $\cap$ Seq $\cap$ T2 is proper. We start with the space S_2 of Arens [A]. This space consists of a sequence (s_i), the level one points, converging to the level zero point, s_o, and for each s_i, a sequence (s_{ij}) converging to s_i. The points s_{ij} are called the level two points. S_2 carries the quotient topology: the points s_{ij} are isolated; a basic neighborhood of a point s_i contains a tail of the sequence (s_{ij}); and a basic neighborhood, U, of s_o contains a tail of the sequence (s_i), plus for each $s_i \in U$ a tail of the sequence (s_{ij}). This space is the canonical sequential but not Fréchet space $[F_3]$. Observe that S_2 fails to be first countable at s_o. Index a neighborhood base at s_o as $\{U_\alpha | \alpha \in A\}$. (It will be convenient to assume S_2 itself is in this collection; we so assume.) For each α let C_α be the pathwise connectification (see [W], problem 27C) of U_α. It consists of U_α with a closed interval attached to each point and with all the opposite ends of all the intervals identified. We call C_α the pseudocone on U_α, and we call its vertex v_α. Let X be the quotient set of the disjoint sum of the C_α's obtained by identifying various occurrences of the same point of S_2. We will topologize X by describing basic neighborhoods of various types.

a) A point interior to an interval of a pseudocone has a usual basis of open intervals.

b) A basic neighborhood of a vertex v_α will contain an interval $(x, v_\alpha]$ down each interval terminating at v_α. (Note that the x's are independent.)

c) A level 2 point of S_2 will have basic neighborhoods containing intervals $[s_{ij}, y)$ (independent y's) up each interval emerging from s_{ij}.

d) A level one point s_i will have basic neighborhoods containing a tail of the sequence (s_{ij}) and for each s_{ij} in that tail, and also for s_i, an interval $[s_{ij}, y)$ (resp. $[s_i, y)$) along each interval emerging from s_{ij}, (resp. s_i).

e) A basic neighborhood of s_o will contain some U_α, the whole pseudocone C_β for each β such that $U_\beta \subseteq U_\alpha$, and for each $x \in U_\alpha$, a "whisker" $[x, y)$ up each of the remaining intervals attached to x.

The set X with this topology is path connected, and locally path connected except at the level one points of S_2. To make it locally path connected at the level one points we add countably many closed unit intervals I_i, identifying $0 \in I_i$ with s_i and each $\frac{1}{j} \in I_i$ with s_{ij}, giving the resulting set, T, the quotient topology.

Now a routine case-by-case consideration shows that T is Hausdorff. That T fails to be sequential is seen by considering the collection $\{v_\alpha | \alpha \in A\}$. This set has s_o in its closure, and is thus not closed, but meets every convergent sequence in a closed set.

We ignore for the moment the fact that T is not sequential, and argue that it fails to have the subsequence-path property. The sequence we want is (s_i), the level one points. We show that any function $p: [0,1] \to T$ which has infinitely many level one points in its image must fail to be continuous. So suppose there is a subsequence (s_{i_k}) of (s_i) such that for each k, $s_{i_k} = p(t_k)$, where t_k is some point of Π. Without loss of generality we may assume that the t_k are a convergent sequence say with limit t. Now, if p is to have any hope of being continuous, between t_k and t_{k+1} there must be an r_k such that $p(r_k) = v_{\alpha_k}$; the r_k will also be a convergent sequence with t as their limit. Since $\{v_{\alpha_k}\}$ is closed in T, if p were continuous we would have $p(t) \in \{v_{\alpha_k}\}$. But, $(s_{i_k}) = (p(t_k))$ converges to s_o, so we would also have $p(t) = s_o$, a contradiction.

To make T sequential we now introduce another layer of points and paths "between" the pseudocones C_α and s_o. The basic idea is to provide sequential limits for the countable subsets of T which should but do not have sequential limits, and to do so in such a way that there is still no path through (s_i), and so that the resulting space HE will be sequential. The construction is modeled on Isbell's space Ψ (see [G-J], problem 5I). Let S denote the subset of T consisting of S_2 and the closed intervals I_i running through the sequences (s_{ij}) which provided the local path connectivity at the level one points s_i. T is not sequential because a subset of $T\setminus S$ which consists of exactly one point from each of uncountably many pseudocones C_α can have s_o in its closure, but any countable subset of such a set is closed.

Denote the interval connecting $x \in S_2$ with v_α by $I_{x\alpha}$ and consider the collection of all countable subsets $\{y_k | y_k \in I_{x_k\alpha_k}\}$ of $T\setminus S$ with the following properties: all the $x_k \in S_2\setminus\{s_o\}$ are distinct (that is, if $k \neq k'$ then $x_k \neq x_{k'}$ unless both are s_o), all the α_k are distinct, and for each i, $(\{s_i\} \cup \{s_{ij}\}_{j=1}^{\infty}) \cap \{x_k\}$ is finite. Let $\mathcal{D}$ be a maximal almost disjoint subset of this collection. Elements of $\mathcal{D}$ will be denoted by upper case Roman letters, D,E,F, etc. They have the following properties: $\forall$ D,E $\in \mathcal{D}$, $D \cap E$ is finite; if Y is a subset of $T\setminus S$ which meets infinitely many $I_{x_k\alpha_k}$ with distinct α_k, with $x_k \neq x_{k'}$ unless both equal s_o, and where, unless all but finitely many of the $x_k = s_o$, they are taken from infinitely many $(\{s_i\} \cup \{s_{ij}\}_{j=1}^{\infty})$, then there is a $D \in \mathcal{D}$ such that $D \cap Y$ is infinite. To T we add $\mathcal{D}$ as a new set of points. For each $D \in \mathcal{D}$ we add an arc A_D connecting D with s_o. We index D as $D = \{z_1, z_2, \cdots\}$ and add a path p_D through D to the point D so that $p_D(\frac{1}{i}) = z_i$ and $p_D(0) = D$. Except at s_o we give this new space, HE, the quotient topology. Specifically we have the following types of basic neighborhoods at the following types of points:

a) Points of $S\setminus S_2$, points in the interiors of the arcs A_D, and points in $P_D\setminus(\{z_i\} \cup \{D\})$ have a base of usual interval neighborhoods.

b) To a T-neighborhood of a point s_{ij}, which consists of an interval in I_i and intervals $[s_{ij}, Y)$ in each pseudocone C_α containing s_{ij}, we must add an interval in every path

p_D passing through each $z \in [s_{ij}, y)$ for every such $[s_{ij}, y)$.

c) To a T-neighborhood of a point s_i, we must add an interval in every p_D passing through each z in any interval $[s_{ij}, y)$ or $[s_i, y)$ already in the neighborhood.

d) For points interior to the intervals of the pseudocones C_α we add to the interval neighborhoods of T similar intervals in each path p_D which passes through the T-neighborhood.

e) Similarly, for a vertex v_α we must add to each T-neighborhood a subinterval in each path p_D passing through the neighborhood.

f) For a basic neighborhood of a point $D \in \mathcal{D}$ take a tail of the path p_D plus for every $z_i = p_D(\frac{1}{i})$ a basic neighborhood of type d) or e) above plus an interval which contains D in the arc A_D.

Note that since $\mathcal{D}$ is an almost disjoint collection, points of $\mathcal{D}$ have disjoint neighborhoods in HE, and that $\mathcal{D}$ with the subspace topology is discrete.

Finally, we describe the basic neighborhoods of s_o in HE. We want to make HE sequential, to preserve the "no path through (s_i)" property, and at the same time to make the neighborhoods path connected. So let

g) A basic neighborhood of s_o be a T-neighborhood of s_o plus the following additional points: all but finitely many $D \in \mathcal{D}$, for each such D the whole arc A_D plus a tail of the path p_D, for the finitely many excluded $E \in \mathcal{D}$ a "whisker" $(t_E, s_o]$ in A_E, and for any z which is now in the neighborhood a subinterval containing z in any path P_F which passes through z.

Let us first show that the "no path through (s_i)" property is retained. As before suppose that $p: [0,1] \to HE$ with $p(t_k) = s_{i_k}$ and $t_k \to t$. One cannot now argue that to have any hope of continuity p must pass through vertices of pseudocones, but it is true that p would have to assume some value z_k in some pseudocone C_{α_k}, say $z_k = p(r_k)$ where r_k is between t_k and t_{k+1}, and so again $r_k \to t$. We can assume, without loss of generality that z_k is either in the interior of the interval connecting s_{i_k} with v_{α_k}, or else in the interior of the interval connecting some $s_{i_k j}$ with v_{α_k}. If $\{z_i\}$ meets infinitely many pseudocones, then $\{z_i\}$ has infinite intersection with some $D \in \mathcal{D}$ so there is a subsequence of $\{z_i\}$ which converges to D. If, on the other hand $\{z_i\}$ meets only finitely many pseudocones, then $cl\{z_i\}$ is a subset of their union. In either case it follows that p cannot be continuous.

Since each of the neighborhoods in a) through g) above is path connected, HE *is locally path connected.* Also it is clear that HE *is* T1. A 27 case analysis shows that HE *is* T2. We omit the proof; it is tedious but routine.

We now turn to showing that HE *is sequential.* So let $\mathcal{F} \subseteq HE$ be sequentially closed, i.e. if $x_i \to x$ with each $x_i \in \mathcal{F}$, then $x \in \mathcal{F}$; and let $x \in cl_{HE}\mathcal{F}$. If x is a point of type a), that is $x \in S \setminus S_2$, $x \in A_D \setminus \{D, s_o\}$, or $x \in p_D \setminus (\{z_i\} \cup \{D\})$, then x is a point of first countability so there is a sequence in $\mathcal{F}$ converging to x, and thus $x \in \mathcal{F}$.

If x is a point of type b)-f), then a basic neighborhood at x is a quotient of open and half-open intervals, such a neighborhood U is sequential. Thus $\mathcal{F} \cap U$ is closed in U.

It follows that $x \in \mathcal{J}$. It remains to consider the case where $x = s_o$. We first take care of the two simplest possibilities $s_o \in cl(\mathcal{J} \cap I_{s_o\alpha})$ for some α or $s_o \in cl(\mathcal{J} \cap A_D)$ for some D; that is, $s_o \in cl\mathcal{J}$ because $\mathcal{J}$ "goes all the way down" some interval attached to s_o. In this case there is a sequence in $\mathcal{J}$ converging to s_o. The next possibility is that $\mathcal{J} \cap D$ is infinite. In this case, since $\mathcal{D} \cup \{s_o\}$ is the one-point compactification of $\mathcal{D}$ any sequence of distinct points from $\mathcal{J} \cap \mathcal{D}$ converges to s_o. Similarly, if $\mathcal{J}$ meets infinitely many of the arcs A_D, then there is a sequence in $\mathcal{J}$ converging to s_o. If $s_o \in cl_{HE}(\mathcal{J} \cap S) = cl_S(\mathcal{J} \cap S)$ then $s_o \in \mathcal{J}$ since S is sequential and $\mathcal{J}$ is sequentially closed.

We will conclude by arguing that if none of the five possibilities listed above occurs, and if $\mathcal{J}$ is sequentially closed then $s_o \notin cl\mathcal{J}$. Observe first that since $s_o \notin cl(\mathcal{J} \cap S)$ there is a U_{α_o} such that $\mathcal{J} \cap T$ is bounded away from s_i or s_{ij} in every $I_{s_i,\beta}$ or $I_{s_{ij},\beta}$ for all β and all s_i, s_{ij} in U_{α_o}. Next observe that $\mathcal{J}$ can travel to the D end of at most finitely many paths p_D. We claim that this implies there is a U_{α_1} so that $\mathcal{J}$ meets only countably many pseudocones C_α with $U_\alpha \subseteq U_{\alpha_1}$, call them C_{α_i}, $i = 2,3,4,\cdots$. For if not the subset $Y = \{y \in S_2 | \mathcal{J} \cap I_{y\alpha} \neq \phi$ for uncountably many $\alpha\}$ contains s_o in its closure. And then, if $s_o \in Y$, choose a point in $\mathcal{J} \cap I_{s_o\alpha}$ for each α such that $\mathcal{J} \cap I_{s_o\alpha} \neq \phi$. This set travels to the D end of uncountably many p_D, and hence so does $\mathcal{J}$. While if $s_o \notin Y$ then there is a sequence of integers i_k so that $Y \cap (\{s_{i_k}\} \cup \{s_{i_kj}\}_{j=1}^{\infty}) \neq \phi$. Let $x_k \in Y \cap (\{s_{i_k}\} \cup \{s_{i_kj}\})$. Then for each k the collection of α such that $I_{x_k\alpha} \cap \mathcal{J} \neq \phi$ is uncountable so we can choose uncountably many sequences (α_k) so that all the α_k's are distinct and $I_{x_k\alpha_k} \cap \mathcal{J} \neq \phi$. Hence we can choose uncountably many pairwise disjoint sequences (z_k) so that for each k, $z_k \in (I_{x_k\alpha_k} \cap \mathcal{J})$. Now these sequences satisfy the defining properties of the maximal almost disjoint family $\mathcal{D}$ so each travels to the end of some p_D, that is, again $\mathcal{J}$ travels to the D end of infinitely many paths p_D.

Choose $U_\gamma \subseteq U_{\alpha_0} \cap U_{\alpha_1}$ so that U_γ contains none of the neighborhoods U_{α_i}, $i = 2,3,4,\cdots$. We build a neighborhood of s_0 missing $\mathcal{J}$: The T-neighborhood we start with is based on U_γ. The pseudocones C_β with $U_\beta \subseteq U_\gamma$ are all distinct from the C_{α_i} meeting $\mathcal{J}$. Since $\mathcal{J} \cap T$ is bounded away from s_0, s_i, S_{ij} for each such point in U_γ we can add the necessary whiskers $[s_0, y)$, $[s_i, y)$, and $[s_{ij}, y)$. To this T-neighborhood we add all points of $\mathcal{D}$ except the finitely many for which $\mathcal{J}$ travels to the D end of p_D or meets A_D. For each D so included we can add the arc A_D and a tail of the path p_D. Since $\mathcal{J}$ is bounded away from s_0 in all the remaining arcs A_E we can add a whisker $(t_E, s_0]$ in each such A_E. Finally, since $\mathcal{J}$ is sequentially closed, around any z now in the neighborhood we can find a subinterval missing $\mathcal{J}$ and containing z in any path p_F passing through z, and add all these subintervals to the neighborhood. The neighborhood so constructed misses $\mathcal{J}$ entirely. It follows that HE is sequential, and is thus the promised example of a sequential and locally path connected, Hausdorff space which is not in TOP(Π).

4. Some Further Observations and Questions

Observation 4.1. The space HE fails to be Fréchet since it contains S_2 as a closed subspace. In fact, the sequential order [A-F] of HE is 4. For each point z over some s_{ij} in some given $C_\alpha \backslash S$ choose a deleted z-neighborhood in some p_D passing through z so that no other point of T is in the closure of the chosen neighborhood. The union of these deleted z-neighborhoods has s_o in its closure; it takes four steps to reach s_o by sequential limits.

This leads to

Question 4.2. Does locally path connected plus Fréchet imply path generated?

We note that there are path generated spaces which are not Fréchet, so equality cannot hold. Take the space S, run a path through (s_i) to s_o and give the resulting space the quotient topology. This produces an interval version of S_2 which is in TOP(Π) but is not Fréchet.

Observation 4.3. HE fails to be regular: Let $\{s_{i_k j_k}\}$ be an infinite subset of S_2 such that if $k \neq k'$, $i_k \neq i_{k'}$ and such that each $s_{i_k j_k}$ is in uncountably many U_α. Then any HE neighborhood of $\{s_{i_k j_k}\}$ has s_o in its closure.

Question 4.4. Is it true that every regular (normal, compact) sequential, locally path connected space is in TOP(Π)? To make HE Hausdorff we had to allow many countable subsets of T to fail to have cluster points, so perhaps the right question is, "Is every Hausdorff, countably compact, sequential, and locally path connected space in TOP(Π)?"

The following questions were suggested by the referee:

Question 4.5. Is there a locally pathwise connected regular space which does not have the subsequence-path property?

Question 4.6. Is there a topological vector space (necessarily *not* locally convex) which does not have the subsequence-path property?

References

[A] R. Arens, *Note on convergence in topology*, Math. Mag. 28 (1950), 229-234.

[A-F] A. V. Arhangel'skiǐ and S. P. Franklin, *Ordinal invariants for topological spaces*, Michigan Math. J. 15 (1968), 313-320.

[F_1] S. P. Franklin, *Natural covers*, Compositio Math. 21 (1969), 253-261.

[F_2] ________, *Spaces in which sequences suffice*, Fund. Math. 57 (1965), 107-115.

[F_3] ________, *Spaces in which sequences suffice II*, Fund. Math. 61 (1967), 51-56.

[F_4] ________, *Topics in categorical topology*, class notes, Carnegie-Mellon Univ., 1969-70.

[G-J] L. Gillman and M. Jerrison, *Rings of continuous functions*, The University Series in Higher Mathematics, Van Nostrand, Princeton, N.J., 1960.

[W] S. Willard, *General topology*, Addison Wesley, Reading, Mass., 1970.

Memphis State University

Memphis, Tennessee 38159