CONCERNING EQUIVALENCIES OF ALMOST CONTINUOUS AND CONNECTIVITY FUNCTIONS OF BAIRE CLASS 1 ON PEANO CONTINUA

by

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Kuratowski and Sierpinski showed, in 1922 [6], that whenever a function of Baire class 1, f: I → I, has the Darboux property, then that function is a connected subset of the plane. For real functions with domain a connected subset of the real line, the property of a function being a connected subset of the plane, and the property of a function being a connectivity function are the same. Stating the Kuratowski-Sierpinski result in this terminology we have: a necessary and sufficient condition that the Baire class 1 function f: I → I, be a Darboux function is that it be a connectivity function. Intuitively, when considering various generalizations of continuity, there are three classes of discontinuous functions which seem to be inherently comparable or interrelated: Darboux functions, connectivity functions, and almost continuous functions. Thus there was the obvious question of whether or not the Kuratowski-Sierpinski characterization could be extended to include the almost continuous functions. Brown has answered that question affirmatively [1]. Herein we are concerned with the extension of such properties to cases where the domain space of the functions is other than a subset of the real line.

We will consider only real functions having domain a Peano Continuum (a compact separable locally connected metric

space). For the definitions or properties of such things as
"hereditarily locally connected space," "continuum of con-
vergence," etc., see [10]. Most of the notation and termi-
nology, such as R being the real line or I being the closed
unit interval, is standard or it will be clear from context.
When each of X and Y is a topological space and M is a subset
of the product space X × Y, then (M)$_X$ denotes the projection
of M into X.

   Definition 1. The function f: X → Y is said to be a
Darboux function or to have the *Darboux property* if, when-
ever C is a connected subset of X, then f(C) is connected.

   Definition 2. The function f: X → Y is said to be a
connectivity function provided that, whenever C is a connected
subset of X, then f|$_C$ is connected.

   Definition 3. The function f: X → Y is said to be an
almost continuous function provided that, whenever 0 is an
open set in X × Y which contains f, then 0 contains a con-
tinuous function from X to Y.

   There are examples in the literature which can be used
to show that, for functions f: X → Y, where X is a Euclidean
space other than the real line, equivalences like those found
by Kuratowski and Sierpinski or by Brown do not hold [1]
[5]. Using the Theorem below, Example 1 of [5] can be shown
to be a Baire class 1, almost continuous, Darboux function
which is not a connectivity function. There is some character
to the real line not possessed by the other Euclidean spaces
which allows those equivalences, or, looking at it from the

other side, such equivalences impose restrictions on the

domain of the functions which rule out the other Euclidean

spaces. Clearly though, there are other spaces (such as a

simple triod) where those same equivalences hold.

It is interesting to consider problems of this type.

Generally, if the Baire class 1 function f: X → Y belongs to

some one of the classes determined by Definition 1, Defini-

tion 2, or Definition 3 only in case it also belongs to some

other of those classes, then what sort of space is X?

After building (or in other cases being unable to

build) a number of examples, a conjecture resulted: for

connectivity and almost continuity to be equivalent for real

functions of Baire class 1, having domain a topological

space X, it is necessary and sufficient that X be hereditarily

locally connected. The question of the sufficiency of this

conjecture remains open; however, we show that the necessity

is true whenever X is a Peano Continuum.

Theorem. Suppose X *is a Peano Continuum and the func-*

tion f: X → R *is of Baire class* 1. *In order that it be true*

that f *is almost continuous if and only if* f *is a connec-*

tivity function, then X *must be hereditarily locally connected.*

Proof. Under the hypotheses, assume that X is not

hereditarily locally connected and so has a continuum of

convergence. Then X has a nondegenerate subcontinuum M and

there is a sequence $N_1, N_2, N_3, \cdots$ of subcontinua of X such

that no two of them intersect and no one of them intersects

M, and the sequence of continua converges to M [10]. Denote

by P a point of M and by T an open set of X containing P but

not all of M. Select a subsequence $M_1, M_2, M_3, \cdots$, of $N_1, N_2, N_3, \cdots$, each term of which intersects T.

There is a monotonic collection $(D_1, D_2, D_3, \cdots)$ of open subsets of X such that

(1) D_1 is X,

(2) for each integer $i \geq 2$, $\overline{D}_i$ is a subset of D_{i-1},

(3) M is the common part of all of them, and

(4) for each integer $i \geq 1$, M_i is a subset of D_i but has no point in $\overline{D}_{i+1}$.

For each positive integer n, A_n is the set $M + Bd(D_2) + Bd(D_3) + Bd(D_4) + \cdots + Bd(D_{n-1})$ and B_n is the set $M_1 + M_2 + M_3 + \cdots + M_n$. Because each of these is closed, by using Urysohn's lemma, there is a continuous function $g_n: X \to R$ such that g_n is 0 on A_n and 1 on B_n. It is clear that this sequence of g_n's need not converge; however, suppose $f_1, f_2, f_3, \cdots$, is a sequence such that: f_1 is g_1, and, if n is a positive integer greater than 1, f_n is f_{n-1} on $X - D_n$ and is g_n on D_n. Each of these f_n's will be continuous on all of X and whenever x is in X, there is an integer N such that, if n and m are greater than N, $f_n(x) = f_m(x)$. Denote by f the function which is the limit of the sequence of f_n's so that we have a function $f: X \to R$ which is of Baire class 1.

We will show that this Baire class 1 function f is almost continuous, which, by hypothesis, requires it to be a connectivity function and then reach a contradiction by find-ing a connected subset C of X such that $f|_C$ is not connected.

To show that f is almost continuous, consider X and its various subsets as objects in $X \times R$, i.e., M is $M \times 0$, D_1 is $D_1 \times 0$, and so on. Thus, if U is an open set in $X \times R$

containing M × 0 (that is, U contains M), then U·(X × 0) is

open in D_n plus its boundary relative to X. Suppose V is an

open subset of X × R containing f, and suppose the subset U

of V is open in X × R and contains M × 0. There is a posi-

tive integer k such that k is in U·(X × 0).

   Define g: X → R as follows: g(x) = 0 if x is in $\overline{D}_k$

and g(x) = f_k(x) for x not in $\overline{D}_k$. Notice that since f_k is

0 on $\overline{D}_k$ - D_k, then this function is continuous. Clearly

each point (x,g(x)), with x in $\overline{D}_k$ is in U and therefore in

V, by the construction of the f_i's, outside of $\overline{D}_k$, g is f

so that this function g: X → R is continuous and is a subset

of V. Thus f is almost continuous and must be a connectivity

function.

   We can show that f cannot be a connectivity function if

we can find a connected subset C of X such that $f|_C$ is not

connected. Describe such a connected set as follows. Sup-

pose A is an arc containing the point P of M, intersecting

each M_i, and which is a subset of the open set T of X con-

taining P but not all of M. To the set A + M_1 + M_2 + M_3 +

··· add a single point q of M which is not in $\overline{T}$ and call

the resulting connected set C. It is easy to see that each

sequence of points of f which converges in X × R to a point

(q,y) is not the point (q,f(q)). The point (q,f(q)) is an

isolated point of $f|_C$. Thus $f|_C$ cannot be connected and so

f is not a connectivity function even though it is almost

continuous. With this contradiction we have proved that X

must be hereditarily locally connected.

   In order to more concisely state some problems concerning

the possibilities of extending the Kuratowski-Sierpinski and
the Brown results, we use the following definitions.

   Definition 4. Suppose X is a topological space.

   (1) If for each Baire class 1 function f: X → R, f is
a connectivity function if and only if f is a Darboux func-
tion, then X is said to be a *KS-space* or to have *property
KS*.

   (2) If for each Baire class 1 function f: X → R, f is
a Darboux function if and only if f is almost continuous,
then X is said to be a *B-space* or to have *property B*.

   (3) If for each Baire class 1 function f: X → R, f is
a connectivity function if and only if it is almost continu-
ous, X is said to be a *G-space* or to have *property G*.

Problems:

   (1) We know that I is a KS-space, a B-space and a
G-space, and we have shown above that other n-cells are not
G-spaces. From known examples it seems to be true that if
$n > 1$, then I^n would not be either a B-space of a KS-space.
What sort of spaces will be those defined in definition 4?

   (2) Does hereditary local connectedness imply property
G?

   (3) Are the properties defined in definition 4 equiva-
lent?

   (4) Stallings [9] showed that f: I^n → R is almost con-
tinuous if it is a connectivity function, for $n > 1$, and f
is a connectivity function if it is an almost continuous
function, for $n = 1$. Would it be true in general that, in
a G-space almost continuity implies connectivity, and in a

non-G-space connectivity implies almost continuity?   (What about B-spaces and KS-spaces?)

     (5) Extending local characterizations [3] [4] of the Darboux property and connectivity to n-cells other than I has been successful only with severe restrictions [2] [5], apparently because I is the only one of them which is hereditarily locally connected.   It seems reasonable to conjecture that for G-spaces, and probably KS-spaces and B-spaces, extensions of local properties such as are found in [3], [4], [7], and [8] could be proved.

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