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PARACOMPACTNESS IN NORMAL, LOCALLY CONNECTED, LOCALLY COMPACT SPACES

Gary Gruenhage

Introduction

G. M. Reed and P. Zenor [RZ] have shown that every perfectly normal, locally connected, locally compact Moore space is metrizable. This followed from their more general result that every perfectly normal, locally connected, locally compact, θ-refinable space is paracompact. Later, J. Chaber and Zenor [CZ] showed that every perfectly normal, locally connected, rim-compact subparacompact space is paracompact. It is the purpose of this paper to prove that the assumption of "perfectly normal" in the above two theorems can be reduced to "normal".

Definitions and Main Results

All our spaces are assumed to be regular. A space X is θ-*refinable* if and only if every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V} = \cup_{n\in\omega}\mathcal{V}_n$ such that

(i) for each $n \in \omega$, $\mathcal{V}_n$ is a cover of X; and

(ii) for each $x \in X$, there exists $n(x) \in \omega$ such that x is an element of only finitely many members of $\mathcal{V}_{n(x)}$.

A space X is said to be *subparacompact* if and only if every open cover of X has a σ-discrete refinement. It is not hard to show that subparacompact spaces are θ-refinable.

A space X is *rim-compact* if and only if X has a base of open sets with compact boundaries.

We intend to prove the following two results:

Theorem 1. *Every normal, locally connected, locally compact, θ-refinable space is paracompact.*

Theorem 2. *Every normal, locally connected, rim-compact, subparacompact space is paracompact.*

The proofs of these theorems consist of two main parts. One part is to show that certain discrete collections of cardinality not greater than the continuum c can be separated by disjoint open sets. The other part is to show that every open cover of a component of the space has a subcover of cardinality not greater than ω_1. Then to reach the final conclusion, we follow the usual proof that θ-refinable (or subparacompact), collectionwise normal spaces are paracompact.

Let κ be a cardinal number. A space X is κ-*collectionwise* T_2 if and only if whenever A is a closed discrete subset of X with $|A| \leq \kappa$, then there exists a pairwise-disjoint collection $\{U_a: a \in A\}$ of open sets with $a \in U_a$ for each $a \in A$.

Lemma 1. *If* X *is normal, locally connected, rim-compact, and* θ-*refinable, then* X *is* c-*collectionwise* T_2.

Proof. Let $A \subset X$ be closed discrete, with $|A| \leq c$. Since there exists a one-to-one function of A into the irrationals, we see that there is a sequence $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \cdots$ such that

(i) $\mathcal{B}_i = \{B_{i,1}, B_{i,2}, \cdots, B_{i,k_i}\}$ is a partition of A;

(ii) $B_{n+1,k} \cap B_{n,k'} \neq \emptyset$ implies $B_{n+1,k} \subset B_{n,k'}$;

and

(iii) if $x, y \in A$, then there are $n, k \in \omega$ with

$x \in B_{n,k}$ and $y \notin B_{n,k}$.

By normality, there exist open sets $B^*_{n,k}$ containing $B_{n,k}$ such that $B^*_{n,k} \cap B^*_{n,k'} = \emptyset$ if $k \neq k'$.

For each $x \in A$, let 0_x be an open set containing x such that the boundary $\partial 0_x$ is compact and $\overline{0}_x \cap A = \{x\}$. Let $\{\mathcal{U}_n: n = 1, 2, \cdots\}$ be a θ-refinement of the open cover $\{0_x: x \in A\} \cup \{X - A\}$.

For each $x \in A$, choose a sequence $N_0(x), N_1(x), \cdots$ of connected open sets containing x with the following properties:

(i) $0_x \supset \overline{N_1(x)} \supset N_1(x) \supset \overline{N_2(x)} \supset \cdots$;

(ii) $N_n(x)$ is contained in some element of $\mathcal{U}_n$;

(iii) $N_n(x) \subset \cap\{B^*_{m,k}: x \in B^*_{m,k}$ and $m \leq n\}$

(iv) $Cl(\cup\{N_{n+1}(x): x \in B_{n+1,k}\}) \subset \cup\{N_n(x): x \in B_{n+1,k}\}$.

To get (iv), one must proceed inductively; i.e., first choose the $N_0(x)$'s satisfying (i)-(iii). Then, with the help of normality, choose the $N_1(x)$'s satisfying (i)-(iv), and so on.

Claim. For each $x \in A$, there exists n_x such that $N_{n_x}(x) \cap N_{n_x}(y) = \phi$ if $y \in A$, $y \neq x$. If this claim is true, then $\{N_{n_x}(x): x \in A\}$ is the desired collection of open sets separating the points of A.

Suppose the claim is not true. Then there exist $x \in A$ and $y_n \in A$, $y_n \neq x$, such that $N_n(x) \cap N_n(y_n) \neq \phi$ for each $n \in \omega$. Choose a point $z_n \in (\partial 0_x) \cap N_n(y_n)$. This is possible since $N_n(y_n)$ is connected and meets 0_x. Let z be a cluster point of $\{z_n: n \in \omega\}$. Then $z \in \partial 0_x$, hence not in $N_0(x)$. There exists $m_1 \in \omega$ such that z is in only finitely many elements of $\mathcal{U}_{m_1}$. Then $A' = \{y \in A: y \in U$ for some $U \in \mathcal{U}_{m_1}$ with $z \in U\}$ is finite, so there exist $m_2 \geq m_1$ and $k \in \omega$ such that $x \in B_{m_2,k}$ and $B_{m_2,k} \cap A' = \phi$. Then $z \notin \cup\{N_{m_2}(y): y \in B_{m_2,k}\}$, as is seen from properties (i) and (ii) above. Let $x \in B_{m_2+1,k'} \subset B_{m_2,k}$. Then by (iv), $z \notin \mathrm{Cl}(\cup\{N_{m_2+1}(y): y \in B_{m_2+1,k'}\}) = H$.

Pick $n_0 > m_2 + 1$ such that $z_{n_0} \notin H$. If $y_{n_0} \in B_{m_2+1,k'}$, then $z_{n_0} \in N_{n_0}(y_{n_0}) \subset N_{m_2+1}(y_{n_0})$, contradiction. Thus $y_{n_0} \notin B_{m_2+1,k'}$. But then $N_{n_0}(y_{n_0}) \subset N_{m_2+1}(y_{n_0}) \subset B^*_{m_2+1,k''}$ with $k' \neq k''$ and $N_{n_0}(x) \subset B^*_{m_2+1,k'}$, contradicting $N_{n_0}(y_{n_0}) \cap N_{n_0}(x) \neq \emptyset$. Thus the claim is true and Lemma 1 is proved.

Let κ be a cardinal. A space X is κ-*collectionwise-normal with respect to compact sets* if and only if whenever $\{F_\alpha: \alpha \in A\}$ is a discrete collection of compact sets with $|A| \leq \kappa$, there exists a pairwise-disjoint collection $\{U_\alpha: \alpha \in A\}$ of open sets with $F_\alpha \subset U_\alpha$ for each $\alpha \in A$.

Lemma 2. *If* X *is normal, locally connected, locally compact, and* θ-*refinable, then* X *is* c-*collectionwise-normal with respect to compact sets.*

Proof. Suppose X satisfies the hypotheses of the lemma. Let $\{F_\alpha: \alpha \in A\}$ be a discrete collection of compact subsets of X with $|A| \leq c$. The space Y obtained from X by collapsing each F_α to a point satisfies the hypotheses of Lemma 2 (since all the properties are preserved by closed maps), and hence of Lemma 1. Thus Y is c-collectionwise T_2, so the F_α's can be separated in Y, hence also in X.

Lemma 3. *If* X *is normal, locally connected, locally compact,* θ*-refinable, and connected, then* X *is* ω_1*-Lindelöf, i.e., every open cover of* X *has a subcover of cardinality* $\leq \omega_1$.

Proof. Let $\mathcal{U} = \cup_{n\in\omega}\mathcal{U}_n$ be a θ-refinement of a cover of X by open sets with compact closures. Pick $U_0 \in \mathcal{U}$, and let $\mathcal{V}_0 = \{U_0\}$. Suppose a countable subset $\mathcal{V}_\alpha \subset \mathcal{U}$ has been defined for all $\alpha < \beta < \omega_1$, with $\mathcal{V}_\alpha \subset \mathcal{V}_{\alpha'}$ and $\overline{\cup\mathcal{V}_\alpha} \subset \cup\mathcal{V}_{\alpha'}$ whenever $\alpha < \alpha' < \beta$.

If β is a limit ordinal, define $\mathcal{V}_\beta = \cup_{\alpha<\beta}\mathcal{V}_\alpha$. If β is not a limit ordinal, first we claim that for each $n \in \omega$, there is a countable subcover from $\mathcal{U}_n$ of $F_n \cap (\overline{\cup\mathcal{V}_{\beta-1}})$, where $F_n = \{x \in X\colon x$ is in only finitely many members of $U_n\}$. To see this, for each finite subset $\mathcal{F}$ of $\mathcal{U}_n$, define $H(\mathcal{F}) = \{x \in X\colon x \in U \in \mathcal{U}_n$ if and only if $U \in \mathcal{F}\}$. Then $\{H(\{U\})\colon U \in \mathcal{U}_n\}$ is a closed discrete collection. Only countably many elements of this collection meet $\overline{\cup\mathcal{V}_{\beta-1}}$, for otherwise we could, by Lemma 2 and normality, separate ω_1 such elements by a discrete collection of open sets, uncountably many of which would meet some element V of $\mathcal{V}_{\beta-1}$, contradicting the fact that $\overline{V}$ is compact.

Let $V_{00}, V_{01}, V_{02}, \cdots$ be the elements U of $\mathcal{U}_n$ such that $H(\{U\}) \cap (\overline{\cup\mathcal{V}_{\beta-1}}) \neq \phi$. Then $\{[H(\mathcal{F}) \cap (\overline{\cup\mathcal{V}_{\beta-1}})]\setminus\cup_{i\in\omega}V_{oi}\colon \mathcal{F} \subset \mathcal{U}_n, |\mathcal{F}| = 2\}$ is closed discrete, and thus countable for the same reason as above. Let $V_{10}, V_{11}, V_{12}, \cdots$ be the elements of the sets $\mathcal{J} \subset \mathcal{U}_n$ with $|\mathcal{J}| = 2$ and $[H(\mathcal{J}) \cap (\overline{\cup\mathcal{V}_{\beta-1}})] \setminus \cup_{i\in\omega} V_{oi} \neq \phi$. Continuing in like manner, we construct $\mathcal{V}_{\beta,n} = \{V_{ij}: i,j \in \omega\}$, which is a countable cover of $F_n \cap (\overline{\cup\mathcal{V}_{\beta-1}})$, and $\mathcal{V}_{\beta,n} \subset \mathcal{U}_n$. Define $\mathcal{V}_\beta = \mathcal{V}_{\beta-1} \cup (\cup_{n\in\omega}\mathcal{V}_{\beta,n})$. Then $\mathcal{V}_{\beta-1} \subset \mathcal{V}_\beta$ and $\overline{\cup\mathcal{V}_{\beta-1}} \subset \cup\mathcal{V}_\beta$ as desired.

We now claim that $W = \cup_{\beta<\omega_1}(\cup\mathcal{V}_\beta)$ is a clopen set, hence $W = X$. To see this, suppose $x_0 \in \overline{W}\setminus W$. Let V be an open set containing x_0, with compact closure. Let $\mathcal{U}_n' = \mathcal{U}_n \cap (\cup_{\alpha<\omega_1}\mathcal{V}_\alpha)$. Since $\{H(\mathcal{J}) \cap \overline{V}: \mathcal{J} \subset \mathcal{U}_n', |\mathcal{J}| = 1\}$ is closed discrete in W, it must be finite, for otherwise there is $\alpha < \omega_1$ such that $\{\overline{\cup\mathcal{V}_\alpha} \cap H(\mathcal{J}) \cap \overline{V}: \mathcal{J} \subset \mathcal{U}_n', |\mathcal{J}| = 1\}$ is infinite.

Thus there is $\alpha_{n0} < \omega_1$ such that $x \in \overline{V} \cap W\setminus\cup\mathcal{V}_{\alpha_{n0}}$ implies x is in at least two elements of $\mathcal{U}_n'$. Thus $\{H(\mathcal{J}) \cap \overline{V}\setminus\cup\mathcal{V}_{\alpha_{n0}}: \mathcal{J} \subset \mathcal{U}_n', |\mathcal{J}| = 2\}$ is finite, and there exists $\alpha_{n1} < \omega_1$ such that $x \in \overline{V} \cap W\setminus\cup\mathcal{V}_{\alpha_{n1}}$ implies x is in at least three elements of $\mathcal{U}_n'$. Continuing in like manner, and doing this for each $n \in \omega$, we see there exists $\alpha < \omega_1$ such that $x \in \overline{V} \cap W\setminus\cup\mathcal{V}_\alpha$ implies x is in infinitely many elements of $\mathcal{U}_n'$ for each $n \in \omega$. But this means $\overline{V} \cap W\setminus\cup\mathcal{V}_\alpha = \phi$, and so $(X\setminus\overline{\cup\mathcal{V}_\alpha}) \cap V$ is an open set containing x_0 which does not meet W, contradicting $x_0 \in \overline{W}$. This finishes the proof of Lemma 3.

Proof of Theorem 1. We need only prove each component is paracompact, so assume X is connected. We follow the standard proof that θ-refinable collectionwise-normal spaces are paracompact. Let $\mathcal{U}$ be an open cover of X by sets with compact closures. Let $\{\mathcal{U}_n\}_{n\in\omega}$ be a θ-refinement of $\mathcal{U}$. By Lemma 3, we may assume $|\mathcal{U}_n| \leq \omega_1$. The set $\{H(\mathcal{J}): \mathcal{J} \subset \mathcal{U}_n,$

$|\mathcal{F}| = 1\}$ is closed discrete. From Lemma 2, we see there is a discrete collection $\mathcal{V}$ of open sets separating the elements of this collection and refining $\mathcal{U}$. Now the collection $\{H(\mathcal{F})\setminus\cup\mathcal{V}: \mathcal{F} \subset \mathcal{U}_n, |\mathcal{F}| = 2\}$ is closed discrete, and so on. Continuing in this way, we produce a σ-discrete open refinement of $\mathcal{U}$ covering X. Thus X is paracompact.

Now we go on to the proof of Theorem 2. The proof of this theorem is very similar to the proof of Theorem 1, though more complicated. Observe that the property of subparacompactness is assumed in Theorem 2, instead of the weaker property of θ-refinability. I do not know if Theorem 2, or Lemma 4, are true if subparacompactness is replaced by θ-refinability, or even metacompactness, but I suspect they are.

Lemma 4. *If* X *is normal, locally connected, rim-compact, subparacompact, and connected, then* S *is* ω_1*-Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X by connected rim-compact open sets. Let $\mathcal{F} = \cup_{n\in\omega}\mathcal{F}_n$ be a σ-discrete refinement of $\mathcal{U}$. For each $F \in \mathcal{F}$, let $U_F \in \mathcal{U}$ be such that $F \subset U_F$.

For $A \subset X$, define $\mathcal{F}(A) = \{F \in \mathcal{F}: F \cap A \neq \phi\}$. Pick $U_0 \in \mathcal{U}$, and let $\mathcal{V}_0 = \{U_0\}$. Since ∂U_0 is compact, $|\mathcal{F}(\partial U_0)| \leq \omega$. Let $\mathcal{V}_1 = \mathcal{V}_0 \cup \{U_F: F \in \mathcal{F}(\partial U_0)\}$. Then $\overline{\cup\mathcal{V}_0} \subset \cup\mathcal{V}_1$. Suppose a countable subset $\mathcal{V}_\alpha$ of $\mathcal{U}$ has been defined for all $\alpha < \beta$, $\beta < \omega_1$, with $\mathcal{V}_\alpha \subset \mathcal{V}_{\alpha'}$, and $\overline{\cup\mathcal{V}_\alpha} \subset \cup\mathcal{V}_{\alpha'}$, whenever $\alpha < \alpha' < \beta$. If β is a limit ordinal, let $\mathcal{V}_\beta = \cup_{\alpha<\beta}\mathcal{V}_\alpha$. If β is not a limit ordinal, let $\mathcal{V}_\beta = \mathcal{V}_{\beta-1} \cup \{U_F: F \in \mathcal{F}(\partial U),$ $U \in \mathcal{V}_{\beta-1}\} \cup \{U_F: F \in \mathcal{F}(\overline{\cup\mathcal{V}_{\beta-1}}\setminus\cup\mathcal{V}_{\beta-1})\}$.

Clearly, $\mathcal{V}_{\beta-1} \subset \mathcal{V}_\beta$ and $\overline{\cup\mathcal{V}_{\beta-1}} \subset \cup\mathcal{V}_\beta$. We claim that $\mathcal{V}_\beta$ is countable. Suppose not. Then $\mathcal{F}(\overline{\cup\mathcal{V}_{\beta-1}}\setminus\cup\mathcal{V}_{\beta-1})$ is uncountable, and so there is $n \in \omega$ such that uncountably many elements of $\mathcal{F}_n$ meet $\overline{\cup\mathcal{V}_{\beta-1}}\setminus\cup\mathcal{V}_{\beta-1}$. Thus there is a set $\mathcal{F}_n' \subset \mathcal{F}_n$, with $|\mathcal{F}_n'| = \omega_1$, such that each $F \in F_n'$ meets $\overline{\cup\mathcal{V}_{\beta-1}}\setminus\cup\mathcal{V}_{\beta-1}$. For each $F \in \mathcal{F}_n'$, choose a point $x_F \in F \cap (\overline{\cup\mathcal{V}_{\beta-1}}\setminus\cup\mathcal{V}_{\beta-1})$. From Lemma 1, we see there is a discrete collection $\{W_F: F \in \mathcal{F}_n'\}$ of connected open sets such that $x_F \in W_F$ for each $F \in F_n'$. Since W_F is connected, and $W_F \cap (\cup\mathcal{V}_{\beta-1}) \neq \phi$, there is $U \in \mathcal{V}_{\beta-1}$ such that $W_F \cap (\partial U) \neq \phi$. But $\mathcal{V}_{\beta-1}$ is countable so there must be $U' \in \mathcal{V}_{\beta-1}$ such that uncountably many W_F's meet $\partial U'$. This contradiction completes the inductive step.

It is easy to check that $0 = \cup_{\alpha<\omega_1}(\cup\mathcal{V}_\alpha)$ is a connected open set. To finish the proof of the lemma, we will show that 0 is closed. Suppose not. Then there exists $x_0 \in \overline{0}\setminus 0$. First we claim that if $x_0 \in V$, V rim-compact, then $(\partial V) \cap 0$ is closed. This is equivalent to the existence of an $\alpha_0 < \omega_1$ such that $(\partial V) \cap 0 \subset \mathcal{V}_{\alpha_0}$. Let $S = \{\alpha < \omega_1: \beta < \alpha$ implies $(\partial V) \cap (\overline{\cup\mathcal{V}_\alpha}\setminus\cup\mathcal{V}_\beta) \neq \phi\}$. S is a closed subset of ω_1, for if $\alpha_n \to \alpha$ and $x_n \in (\partial V) \cap (\overline{\cup\mathcal{V}_{\alpha_n}}\setminus\cup\mathcal{V}_{\alpha_{n-1}})$, and x is a cluster point of $\{x_1, x_2, \cdots\}$ then $x \in (\partial V) \cap (\overline{\cup\mathcal{V}_\alpha}\setminus\cup\mathcal{V}_\alpha)$. This also shows that if α is a limit ordinal in S, then there exists $F_\alpha \in \mathcal{F}(\overline{\cup\mathcal{V}_\alpha}\setminus\cup\mathcal{V}_\alpha)$ with $F_\alpha \cap (\partial V) \neq \phi$. Notice that if α and α' are distinct limit ordinals in S, then $F_\alpha \neq F_{\alpha'}$, since $F_\alpha \subset \cup\mathcal{V}_{\alpha+1}$. Since only countably many F's can meet ∂V, S must be countable. If $\alpha > \sup S$, then there is $\beta(\alpha) < \alpha$ such that $(\partial V) \cap (\overline{\cup\mathcal{V}_\alpha}\setminus\cup\mathcal{V}_{\beta(\alpha)}) = \phi$. By the pressing down lemma, there is an uncountable set $A \subset \omega_1$ and $\alpha_0 < \omega_1$ such that $\beta(\alpha) = \alpha_0$ whenever $\alpha \in A$. Then $(\partial V) \cap (0 \setminus \cup \mathcal{V}_{\alpha_0}) = \phi$, and so $(\partial V) \cap 0 \subset \cup \mathcal{V}_{\alpha_0}$, as claimed.

Now let $V_0 \supset \overline{V}_1 \supset V_1 \supset \overline{V}_2 \supset \cdots$ be a decreasing sequence of connected rim-compact neighborhoods of x_0 such that $V_n \cap (\cup\{F \in F_i: i \leq n, x_0 \notin F\}) = \phi$. There exists $\beta_0 < \omega_1$ such that $\cup_{i\in\omega}((\partial V_i) \cap 0) \subset \cup \mathcal{V}_{\beta_0}$, and so there exists a rim-compact open set W containing x_0 such that $\overline{W} \cap (\partial V_i) \cap 0 = \phi$, for all $i \in \omega$. Then for each $n \in \omega$, $V_n \cap (\partial W) \cap 0 \neq \phi$, for otherwise $V_n \cap W \cap 0$ is clopen in the connected set 0. Thus there is a point $y \in \cap_{n\in\omega}(\overline{V}_n \cap (\partial W) \cap 0)$. Suppose $y \in U \in \mathcal{V}_\alpha$. Since V_n is connected, $V_n \cap \partial U \neq \phi$, so there exist a point $z \in (\cap_{n\in\omega}\overline{V}_n) \cap \partial U$. There is $F_z \in \mathcal{F}(\partial U)$ with $z \in F_z$. But since $V_n \cap (\cup\{F \in \mathcal{F}_i: i \leq n, x_0 \notin F\} = \phi$, it must be true that $x_0 \in F_z$. But $F_z \subset \cup\mathcal{V}_{\alpha+1}$, a contradiction which completes the proof of Lemma 4.

Proof of Theorem 2. We may assume X is connected. Let $\mathcal{U}$ be an open cover of X by rim-compact open sets, with $|\mathcal{U}| \leq \omega_1$. Let $\mathcal{F} = \cup_{n\in\omega}\mathcal{F}_n$ be a σ-discrete closed refinement of $\mathcal{U}$, with $|\mathcal{F}| \leq \omega_1$. Fix $n \in \omega$. For each $x \in F \in \mathcal{F}_n$, choose a rim-compact open set containing x whose closure does not meet any other element of $\mathcal{F}_n$. Let $\mathcal{V}_n$ be the collection of these open sets, together with $X \setminus \cup\mathcal{F}_n$. Let $\mathcal{V}_n'$ be a subcover of $\mathcal{V}_n$ with $|\mathcal{V}_n'| \leq \omega_1$. Let $\mathcal{G}_n = \cup_{m\in\omega}\mathcal{G}_{n,m}$ be a σ-discrete closed refinement of $\mathcal{V}_n'$, with $|\mathcal{G}_n| \leq \omega_1$. Let $\mathcal{G} = \cup_{n\in\omega}\mathcal{G}_n$.

For $F \in \mathcal{F}_n$, let $F_m = \{x \in F$: there exists $G \in \mathcal{G}_{n,m}$

with $x \in G\}$. We will show that the collection $\{F_m: F \in \mathcal{F}_n\}$ can be separated by disjoint open sets. Since each F_m is contained in some element of $\mathcal{U}$, we can then use normality to get a discrete collection of open sets refining U and covering $\{F_m: F \in \mathcal{F}_n\}$. Since $\{F_m: F \in \mathcal{F}_n; m,n \in \omega\}$ covers X, this would show that $\mathcal{U}$ has a σ-discrete open refinement, and Theorem 2 would be proven.

Let $\mathcal{F}_n = \{F_\alpha: \alpha \in A\}$ and let $\mathcal{G}_{n,m} = \{G_\beta: \beta \in B\}$, with $|A \cup B| \leq \omega_1$. Thinking of A and B as subsets of the irrationals, we see that there exists a sequence $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \cdots$ of partitions of $A \times B$ such that

(i) $\mathcal{B}_i = \{B_{i,0}, B_{i,1}, \cdots, B_{i,k_i}\}$;

(ii) $B_{n+1,k} \cap B_{n,k'} \neq \phi$ implies $B_{n+1,k} \subset B_{n,k'}$;

(iii) $(a,b), (a',b') \in A \times B$ and $(a,b) \neq (a',b')$ implies there are $n,k \in \omega$ with $(a,b) \in B_{n,k}$ and $(a',b') \notin B_{n,k}$; and

(iv) $a \neq a' \in A$ implies there is $n \in \omega$ such that if $(a,b) \in B_{n,k}$, then $(\{a'\} \times B) \cap B_{n,k} = \phi$.

By the normality, there are open sets $B^*_{n,k}$ containing $\cup\{F_\alpha \cap G_\beta: (\alpha,\beta) \in B_{n,k}\}$ such that $B^*_{n,k} \cap B^*_{n,k'} = \phi$ whenever $k \neq k'$.

For each $\beta \in B$, pick $V_\beta \in \mathcal{V}'_n$ with $G_\beta \subset V_\beta$. For each $(\alpha,\beta) \in A \times B$ and $x \in F_\alpha \cap G_\beta$, pick a rim-compact open set 0_x with $x \in 0_x \subset V_\beta$, such that $0_x \cap (F_{\alpha'} \cap G_{\beta'}) = \phi$ unless $\alpha' = \alpha$ and $\beta' = \beta$. Let $\mathcal{H} = \cup_{n \in \omega} \mathcal{H}_n$ be a σ-discrete refinement of $\{0_x: (\alpha,\beta) \in A \times B \text{ and } x \in F_\alpha \cap G_\beta\} \cup \{X \backslash \cup\{F_\alpha \cap G_\beta: (\alpha,\beta) \in A \times B\}\}$.

For each $(\alpha,\beta) \in A \times B$ and $x \in F_\alpha \cap G_\beta$, choose a sequence

$\{N_n(x)\}_{n\in\omega}$ of connected neighborhoods of x such that

(i) $V_\beta \supset \overline{N_0(x)} \supset N_0(x) \supset \overline{N_1(x)} \supset \cdots$;

(ii) $N_m(x) \cap (\cup\{H \in \mathcal{H}_i: i \leq m, x \notin H\}) = \phi$;

(iii) $N_m(x) \subset \cap\{B^*_{i,k}: (\alpha,\beta) \in B_{i,k}, i \leq m\}$;

(iv) $Cl(\cup\{N_{m+1}(x): x \in F_\alpha \cap G_\beta, (\alpha,\beta) \in B_{m+1,k}\}) \subset \cup\{N_m(x): x \in F_\alpha \cap G_\beta, (\alpha,\beta) \in B_{m+1,k}\}$.

We claim that for each $x \in F_\alpha \cap G_\beta$, there is $n_x \in \omega$ such that $N_{n_x}(x) \cap N_{n_x}(y) = \phi$ whenever $y \in F_{\alpha'} \cap G_{\beta'}$ with $\alpha' \neq \alpha$. If this is true, we can let $U_F = \cup\{N_{n_x}(x): x \in F_m\}$. Then $F_m \subset U_F$, and $U_F \cap U_{F'} = \phi$ if $F \neq F'$, and Theorem 2 then follows.

Suppose the claim is false. Then there exists $(\alpha,\beta) \in A \times B$, a point $x \in F_\alpha \cap G_\beta$, and $y_n \in F_{\alpha_n} \cap G_{\beta_n}$ with $\alpha_n \neq \alpha$, such that $N_n(y_n) \cap N_n(x) \neq \phi$, $n \in \omega$. Since F_α is the only element of $\mathcal{V}'_n$ that $\overline{V}_\beta$ meets, we have $y_n \notin \overline{V}_\beta$. Since $N_n(y_n)$ is connected there exists $z_n \in (\partial V_\beta) \cap N_n(y_n)$. Since ∂V_β is compact, there exists a cluster point z of $\{z_0, z_1, z_2, \cdots\}$. Now z is in some element H of $\mathcal{H}$, say $H \in \mathcal{H}_r$. Let $\mathcal{C} = \{(\gamma,\delta) \in A \times B: H \cap F_\gamma \cap G_\delta \neq \phi\}$. Then $|\mathcal{C}| \leq 1$. $H \cap F_\alpha \cap G_\beta = \phi$, for otherwise $H \subset 0_y \subset V_\beta$ for some $y \in F_\alpha \cap G_\beta$, contradicting $z \in H \cap \partial V_\beta$. Thus $(\alpha,\beta) \notin \mathcal{C}$. If $m \geq r$, by (ii) above we have $\{(\alpha',\beta'): z \in N_m(y)$ for some $y \in F_{\alpha'} \cap G_{\beta'}\} \subset \mathcal{C}$. There exists $r' \geq r$ and $k \in \omega$ such that $(\alpha,\beta) \in B_{r',k}$ and $\mathcal{C} \cap B_{r',k} = \phi$. Then $z \notin \cup\{N_{r'}(y): y \in F_{\alpha'} \cap G_{\beta'}, (\alpha',\beta') \in B_{r',k}\}$. There exists $k' \in \omega$ with $(\alpha,\beta) \in B_{r'+1,k'} \subset B_{r',k}$, so by (iv), $z \notin Cl(\cup\{N_{r'+1}(y): y \in F_{\alpha'} \cap G_{\beta'}, (\alpha',\beta') \in B_{r'+1,k'}\} = E$.

Pick $p \geq r' + 1$ such that $z_p \notin E$. Then if $y_p \in F_{\alpha_p} \cap G_{\beta_p}$ and $(\alpha_p, \beta_p) \in B_{r'+1,k'}$, then $z_p \notin N_p(y_p)$, since $N_p(y_p) \subset N_{r'+1}(y_p) \subset E$, a contradiction. Thus $(\alpha_p, \beta_p) \notin B_{r'+1,k'}$. Then $N_p(y_p) \subset B^*_{r'+1,k''}$, with $k'' \neq k'$. Then since $N_p(x) \subset B^*_{r'+1,k'}$, we have $N_p(y_p) \cap N_p(x) = \phi$, a contradiction which finishes the proof.

Final remarks. Some of the general ideas of our proofs are similar to ideas in the proof of Reed and Zenor's theorem that a normal, locally connected, locally compact Moore space is metrizable, especially as presented by M. E. Rudin in [R].

F. Tall has asked (see problems section in [R]) whether a normal, locally compact, metacompact space is paracompact.[1] As a corollary to Theorem 1, we see the answer is yes if the space is locally connected (an admittedly strong hypothesis).

We mention again that we do not know if the hypothesis of subparacompactness in Theorem 2 can be replaced by metacompactness or θ-refinability. However, Heikki Junnila has informed the author of a result of his which says that in a normal θ-refinable space, every open cover or cardinality $\leq c$ has a σ-discrete closed refinement. Thus the answer is affirmative if one can show that each component is c-Lindelöf. This will be true, as Junnila points out, at least in some special cases, e.g., if each point has a weak neighborhood base of cardinality less than c.

[1] Steve Watson has recently shown that the answer is yes under V = L.

References

[CZ] J. Chaber and P. Zenor, *On perfect subparacompactness and a metrization theorem for Moore spaces*, Top. Proc. 2, No. 2 (1977), 401-407.

[RZ] G. M. Reed and P. Zenor, *Metrization of Moore spaces and generalized manifolds*, Fund. Math. XCl (1976), 203-210.

[R] M. E. Rudin, *Lectures on set theoretic topology*, Regional Conference Series in Mathematics CBMS 23, Amer. Math. Soc. (1975).

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