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A COMPLEMENT THEOREM FOR CONTINUA IN A MANIFOLD

I. Ivanšić and R. B. Sher[1]

1. Introduction

Since the appearance of the well known complement theorem of Chapman [C] for compacta in the Hilbert cube, a number of analogous results have been obtained for compacta in euclidean space E^n. In [I-S-V], connectivity conditions are used to obtain a complement theorem for ILC embedded continua of cofundamental dimension 3 in E^n, $n \geq 5$; this theorem subsumes most of the previously known results in this area (many of which are listed in the bibliography of [I-S-V]).

Here we use some of the techniques developed in [I-S-V] to establish a complement theorem (Theorem 3) in manifolds other than E^n. Our main tool is Theorem 1, in which we obtain nice defining sequences for certain continua in piecewise linear manifolds; this should be of further use in studying problems involving embedded continua. In Section 4 we use these results to establish a piecewise linear embedding-up-to-homotopy result (Theorem 4) and obtain as a consequence a result on the existence of deleted product neighborhoods.

We assume that the reader is familiar with the basic notions of shape theory, as found for example in [B] or [D-S], and some of the basic techniques of piecewise linear topology as found in [H] or [Z].

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[1]This research was carried out while the first named author was visiting the Univ. of N.C. at Greensboro.

If X is a compactum in the piecewise linear manifold M, then X is said to satisfy the *inessential loops condition*, ILC, if for each neighborhood U of X in M there exists a neighborhood V of X in U such that each loop in V-X which is nullhomotopic in V is also nullhomotopic in U-X. The *fundamental dimension* of the compactum X is min {dim Y: Sh(X) = Sh(Y), Y a compactum}. In [V_2], ILC was studied as it relates to the problem of finding a small polyhedral neighborhood of X having spine whose dimension does not exceed the fundamental dimension of X.

All continua considered in this paper will be pointed 1-movable. It follows from Theorem 7.1.3 of [D-S] that shape morphisms between such continua may be regarded as *pointed* morphisms. We use this fact throughout, assuming that *all* shape morphisms are pointed; however, we shall suppress base points from our notation.

Finally, let us recall that a map f: X → Y between ANR's is r-*connected* if $f_\#: \pi_i(X) \to \pi_i(Y)$ is an isomorphism when $0 \leq i \leq r - 1$ and an epimorphism when i = r. With this in mind, we say that the shape morphism $\underline{f}$: X → Y between pointed 1-movable continua is *shape* r-*connected* if $\underline{f}_\#$: pro-$\pi_i(X) \to$ pro-$\pi_i(Y)$ is an isomorphism of pro-groups for $0 \leq i \leq r - 1$ and an epimorphism for i = r. We also recall that a pro-group $\underline{G} = \{G_\alpha, g_{\alpha\beta}, A\}$ is *stable* if $\underline{G}$ is isomorphic in the category pro-groups to a group, and that $\underline{G}$ satisfies the *Mittag-Leffler condition* if for each $\alpha \in A$ there exists $\beta \geq \alpha$ such that for all $\gamma \geq \beta$, $g_{\alpha\gamma}(G_\gamma) = g_{\alpha\beta}(G_\beta)$.

2. Defining Sequences for ILC Embedded Continua

If X is a compactum lying in the interior of the piecewise linear n-manifold M, a *defining sequence* for X is a sequence $\{U_i\}_{i=1}^{\infty}$ of compact piecewise linear n-manifolds in M such that $X = \cap_{i=1}^{\infty} U_i$ and, if $j = 1,2,\cdots$, $U_{j+1} \subset \text{int } U_j$. In Theorem 2 of [I-S-V] it was shown that under certain conditions an r-shape connected continuum lying in the interior of a piecewise linear manifold has a defining sequence whose members are r-connected. The following provides a generalization. We call a defining sequence $\{U_i\}_{i=1}^{\infty}$ *r-connected* if for $j = 1,2,\cdots$, the inclusion of U_{j+1} into U_j is an r-connected mapping.

Theorem 1. *Suppose* X *is a continuum of fundamental dimension at most* k *lying in the interior of the piecewise linear* n*-manifold* M *and satisfying* ILC*, where* $n \geq 5$ *and* $k \leq n - 3$*. Suppose* pro-$\pi_i(X)$ *is stable for* $0 \leq i \leq r - 1$ *and satisfies the Mittag-Leffler condition for* $i = r < n - 3$*. Then there exists an* r*-connected defining sequence* $\{U_i\}_{i=1}^{\infty}$ *for* X *such that if* $j = 1,2,\cdots$*, then* U_j *has a spine of dimension at most* $k' = \max(k,r+1)$.

Proof. Fix s, $0 \leq s < r$, and inductively assume that there exists an s-connected defining sequence $\{V_i\}_{i=0}^{\infty}$ for X such that if $j = 0,1,2,\cdots$, then V_j has a spine $K_j \subset \text{int } V_j$ of dimension at most k'. The induction begins, when s = 0, by Theorem 4.1 of $[V_2]$. Since pro-$\pi_s(X)$ is stable, an easy argument using Theorems 6 and 7 of [M] shows that the inclusion of V_{j+1} into V_j induces an isomorphism of $\pi_s(V_{j+1})$ onto $\pi_s(V_j)$. Let $r_j\colon V_j \to K_j$ denote the retraction induced by a collapse of V_j onto K_j, and let $s_j = r_j|K_{j+1}$. Since pro-$\pi_{s+1}(X)$ satisfies the Mittag-Leffler condition we may assume, by taking a subsequence if necessary, that if $j = 1,2,\cdots$, then $(s_j)_\#$ carries $(s_{j+1})_\#(\pi_{s+1}(K_{j+2}))$ onto $(s_j)_\#(\pi_{s+1}(K_{j+1}))$. By the proof of Theorem 4 of [F] (cf. Theorem 3.1 of [K] and Theorem 5 of [H-I]) there exist compact polyhedra $L_0, L_1, L_2, \cdots$ and mappings $g_j\colon L_{j+1} \to L_j$ such that if $j = 0,1,2,\cdots$, then

(1) $K_j \subset L_j$,

(2) $g_j(L_{j+1}) \subset K_j$,

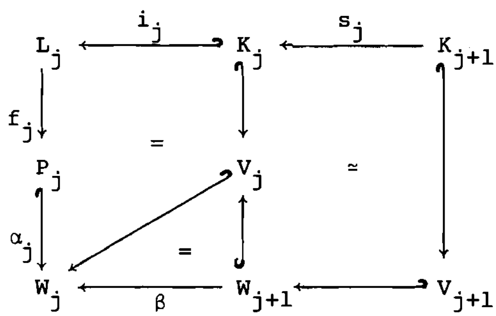
(3) $g_j i_{j+1} = i_j s_j$, where $i_m\colon K_m \to L_m$ denotes the inclusion, $m = 0,1,2,\cdots$,

(4) $(i_j s_j)_\#\colon \pi_{s+1}(K_{j+1}) \to \pi_{s+1}(L_j)$ is an epimorphism, and

(5) L_j is obtained from K_j by attaching finitely many $(s+2)$-cells.

Now fix $j \geq 1$, and define $h_{j-1}\colon L_j \to V_{j-1}$ by $h_{j-1}(x) = g_{j-1}(x)$ for all $x \in L_j$. Then $h_{j-1}(x) = s_{j-1}(x)$ for all $x \in K_j$. Noting that $s_{j-1} \simeq id_{K_j}$ in V_{j-1}, it follows from the Borsuk Homotopy Extension Theorem, (5.13) on pg. 22 of [B], that there exists a map $k_{j-1}\colon L_j \to \text{int } V_{j-1}$ such that $k_{j-1} \simeq h_{j-1}$ and $k_{j-1}(x) = x$ for all $x \in K_j$. We may further assume that k_{j-1} is piecewise linear and in general position. Note that k_{j-1} is s-connected and has singular set of dimension at most $s - 1 < n - 5$; Theorem 4.3 of [St] thus applies, and yields an at most k'-dimensional polyhedron $P_j \subset \text{int } V_{j-1}$ such that $k_{j-1}(L_j) \subset P_j$ and the map $f_j\colon L_j \to P_j$ defined by $f_j(x) = k_{j-1}(x)$ for all $x \in L_j$ is a simple homotopy equivalence. Since $K_j \subset P_j$, there exists a regular neighborhood W_j of P_j such that int $V_{j-1} \supset W_j \supset$ int $W_j \supset V_j$.

It is claimed that the defining sequence $\{W_i\}_{i=1}^{\infty}$ is (s+1)-connected, thereby allowing us to continue our induction. If j = 1,2,···, let β denote the inclusion of W_{j+1} into W_j. By our construction β induces an isomorphism of $\pi_i(W_{j+1})$ onto $\pi_i(W_j)$ for $1 \leq i \leq s$, and so it remains to be shown that β induces an epimorphism of $\pi_{s+1}(W_{j+1})$ onto $\pi_{s+1}(W_j)$. To verify the latter, consider the following diagram.



By (4), the fact that f_j is a homotopy equivalence, and the fact that W_j is a regular neighborhood of P_j, it follows that $(\alpha_j f_j i_j s_j)_\#: \pi_{s+1}(K_{j+1}) \to \pi_{s+1}(W_j)$ is an epimorphism. This, along with the homotopy commutativity of the outermost rectangle of the diagram, shows that $\beta_\#: \pi_{s+1}(W_{j+1}) \to \pi_{s+1}(W_j)$ is an epimorphism.

In Theorem 1 we hypothesize that pro-$\pi_i(X)$ is stable for $0 \leq i \leq r - 1$ and satisfies the Mittag-Leffler condition for i = r. This is equivalent to assuming that pro-$\pi_i(X)$ is stable for $0 \leq i \leq r - 1$ and that X is pointed r-movable. These conditions hold when X has shape finite r-skeleton, and the converse holds provided $r \geq 2$ (cf. Theorem 5 of [H-I]);

the converse does not hold when $r = 1$, as seen by letting X be the "Hawaiian earring" (pg. 100 in [D-S]).

We now make two brief and easy observations which shall be required in the next section.

Observation 1. If $\{U_i\}_{i=1}^{\infty}$ is an r-connected defining sequence for the continuum X and $i = 1,2,\cdots$, then the inclusion of X into U_i is shape r-connected. Hence, under the hypothesis of Theorem 1, X has arbitrarily small neighborhoods U in M such that the inclusion of X into U is shape r-connected.

Observation 2. Suppose U and V are ANR's, X is a continuum, $X \subset V \subset U$, and the inclusion of X into each of V and U is shape r-connected. Then the inclusion of V into U is r-connected.

Finally, we note in the following that Theorem 1 may be improved in the case $k = 1 = r$ by obtaining $k' = 1$.

Theorem 2. *Suppose* X *is a pointed* 1-*movable continuum of fundamental dimension* $k \leq 1$ *lying in the interior of the piecewise linear* n-*manifold* M *and satisfying* ILC, *where* $n \geq 5$. *Then there exists a* 1-*connected defining sequence* $\{U_i\}_{i=1}^{\infty}$ *for* X *such that if* $j = 1,2,\cdots$, *then* U_j *has a spine of dimension* k.

Proof. If $k = 0$, then X is cellular in M. If $k = 1$, let U be a compact piecewise linear manifold neighborhood of X having 1-dimensional spine. By uniqueness of regular neighborhoods, U is an n-cell with finitely many 1-handles.

Then U embeds in E^n, so we may simply assume $X \subset E^n$. By Theorem 7.3.3 of [D-S], there exists a bouquet of circles $Y \subset E^2 \subset E^n$ such that ShX = ShY. It is easy to verify that Y has a 1-connected defining sequence in E^n each member of which has 1-dimensional spine. The proof of the complement theorem (e.g. Theorem 1 of $[V_1]$) shows, as noted in Section 6 of [I-S-V], that X also has such a defining sequence.

3. A Complement Theorem for Continua in a Piecewise Linear Manifold

The main result of this section is the complement theorem, Theorem 3 below. It generalizes one part of Theorem A of [I-S-V], and its proof is essentially the same as the proof of that theorem, only using the following result in place of Lemma 1 of [I-S-V]. This result and those that follow use the notion of *relative shape*, treated in [C].

Lemma 1. *Let* X_1 *and* X_2 *be continua in the interior of the piecewise linear* n*-manifold* M *such that for* $j = 1$ *or* 2, X_j *has fundamental dimension at most* k, X_j *satisfies* ILC, *and* pro-$\pi_i(X_j)$ *is stable for* $0 \leq i \leq r - 1$ *and satisfies the Mittag-Leffler condition for* $i = r < n - 3$, *where* $n \geq \max(2k+2-r, k+3, 5)$. *Let* $\underline{f} = \{f_i, X_1, X_2, G\}$ *and* $\underline{f}' = \{f_i', X_2, X_1, H\}$ *be relative fundamental sequences in* M *such that* $\underline{f}'\underline{f} \simeq \underline{id}_{X_1}$ *and* $\underline{f}\underline{f}' \simeq \underline{id}_{X_2}$. *Let* U_0 *be a compact piecewise linear manifold neighborhood of* X_1 *for which the inclusion of* X_1 *into* U_0 *is shape* r*-connected, and let* $h\colon M \to M$ *be a piecewise linear homeomorphism homotopic to the identity such that* $X_2 \subset \operatorname{int} h(U_0)$ *and such that* $h^{-1}|W_0 \simeq f_i'|W_0$ *in* U_0 *for some neighborhood* W_0 *of* X_2 *and for almost all* i. *Then for every* *neighborhood* V_0 *of* X_2*, there exist a compact piecewise linear manifold neighborhood* V *of* X_2 *lying in* $V_0 \cap h(U_0)$ *for which the inclusion of* X_2 *into* V *is shape* r*-connected, a piecewise linear homeomorphism* $q: M \to M$ *homotopic to the identity, and a neighborhood* U_1 *of* X_1 *such that*

(1) $q|M - U_0 = h|M - U_0$,

(2) $q(X_1) \subset V$, *and*

(3) $q|U_1 \simeq f_i|U_1$ *in* V *for almost all* i.

Proof. As in the proof of Lemma 3 of [I-S-V], we note that the proof of Lemma 1 of [I-S-V] will apply provided we can find an arbitrarily small compact piecewise linear manifold neighborhood V of X_2 for which the inclusion of X_2 into V is shape r-connected and the pair $(h(U_0),V)$ is $(2k+2-n)$-connected.

By Observation 1 of the preceding section, there exists a small neighborhood V of X_2 in int $h(U_0)$ for which the inclusion of X_2 into V is shape r-connected. Since $r \geq 2k + 2 - n$ it follows from Observation 2 of the preceding section that to complete the argument we need only show that the inclusion of X_2 into $h(U_0)$ is shape r-connected. To this end, let $\underline{h}^{-1}: h(U_0) \to U_0$, $\underline{j}: X_1 \to U_0$, and $\underline{k}: X_2 \to h(U_0)$ be relative fundamental sequences generated by $h^{-1}: h(U_0) \to U_0$ (it is here we require $h \simeq id_M$), and the inclusions $j: X_1 \to U_0$ and $k: X_2 \to h(U_0)$. Since $h^{-1}|W_0 \simeq f_i'|W_0$ in U_0 for almost all i, $\underline{jf}' \simeq \underline{h}^{-1}\underline{k}$, and so $\underline{hjf}' \simeq \underline{k}$. This shows that $\underline{k}$ is an r-connected shape morphism, and the proof is complete.

Lemma 1 is used to maintain an inductive argument to prove Theorem 3, as Lemma 1 of [I-S-V] was used to prove Theorem 3 of [I-S-V]. Our hypotheses have been tailored so that the induction may be started by letting $h = id_M$ and $U_0 = M$.

Theorem 3. *Let* X_1 *and* X_2 *be continua in the interior of the piecewise linear* n*-manifold* M *such that for* $j = 1$ *or* 2, X_j *has fundamental dimension at most* k, X_j *satisfies* ILC, *and* pro-$\pi_i(X_j)$ *is stable for* $0 \leq i \leq r - 1$ *and satisfies the Mittag-Leffler condition for* $i = r < n - 3$, *where* $n \geq \max(2k+2-r, k+3, 5)$. *Suppose the inclusion of* X_1 *into* M *is shape* r*-connected and that* X_1 *and* X_2 *have the same shape relative to* M. *Then* $M - X_1 \cong M - X_2$.

We note that the hypothesis that the inclusion of X_1 into M be shape r-connected is necessary in Theorem 3. In Counterexample 6, Chapter 8 of [Z] is shown the existence, for $m \geq 2$, of an m-sphere S_1^m inessentially (piecewise linearly) embedded in $\text{int}(B^{2m} \times S^1)$ so that S_1^m does not bound an (m+1)-cell in $B^{2m} \times S^1 = M$. If S_2^m is a piecewise linear m-sphere lying in the interior of a (2m+1)-cell in M, then it is easily seen that $M - S_1^m \not\cong M - S_2^m$.

As in Section 6 of [I-S-V] we may obtain from the proof of Theorem 3 a result on Borsuk's notion of *position* (see Chapter XI of [B]) which we state here as follows.

Addendum to Theorem 3. *Under the hypotheses of Theorem* 3, $\text{pos}(M, X_1) = \text{pos}(M, X_2)$.

4. A Piecewise Linear Embedding-up-to-Simple-Homotopy Theorem

Let X be a compactum lying in the interior of the piecewise linear manifold M, and let K be a compact ANR. If $\{U_i\}_{i=1}^{\infty}$ is a defining sequence for X, every shape morphism $\underline{f}$: K → X may be represented by a system map $\underline{f} = \{f_i\}_{i=1}^{\infty}$, f_i: K → U_i (cf. Chapter IX of [B]). Suppose Y is a compactum in M and g: Y → K is a map. Since K ∈ ANR, g extends to g*: G → K for some neighborhood G of Y in M. Then the sequence $\{f_i g^*\}_{i=1}^{\infty}$ satisfies condition (2) of the definition of relative fundamental sequence given on pg. 188 of [C]. If G may be so chosen that this sequence also satisfies condition (1) of that definition, then we say that $\{f_i g^*\}_{i=1}^{\infty}$, denoted by $\underline{fg}$, is a relative fundamental sequence *induced* by the map g and the shape morphism $\underline{f}$.

Lemma 2. *Suppose* X *is a continuum lying in the interior of the piecewise linear* n-*manifold* M *such that* X *satisfies* ILC *and has the shape of a* k-*dimensional polyhedron* K, *where* n ≥ 5 *and* k ≤ n − 3. *Then there exist a defining sequence* $\{U_i\}_{i=1}^{\infty}$ *for* X, k-*dimensional polyhedra* $K_i \subset U_i$, *and simple homotopy equivalences* $\tilde{f}_i$: K → K_i *such that*

(1) if f_i: K → U_i *is the composition* $\tilde{f}_i$: K → K_i ↪ U_i, *then* f_i *is* (k-1)-*connected and* $\underline{f} = \{f_i\}_{i=1}^{\infty}$: K → $\{U_i\}$ *is a system map which is a shape equivalence from* K *to* X,

(2) if i ≥ j > ℓ, *then* K_i *is a strong deformation retract of* U_j *in* U_ℓ, *and*

(3) if i > ℓ *and* ϕ_i: K_i → K *is a homotopy inverse of* $\tilde{f}_i$, *then* ϕ_i *and* f *induce a relative shape equivalence of* K_i *to* X *in* U_ℓ.

Proof. Theorem 1 yields a (k-1)-connected defining sequence for X in M. Since Sh(X) = Sh(K), there exists a shape equivalence $\underline{f}' = \{f_i'\}_{i=1}^{\infty}$ of K to X, where f_i': K $\to U_i$. It is easy to see that each of the maps f_i' is (k-1)-connected. Let f_i: K $\to U_i$ be a piecewise linear general position map such that $f_i(K) \subset \text{int } U_i$ and $f_i \simeq f_i'$. Then f_i is a (k-1)-connected map whose singular set has dimension at most $2k - n \leq k - 3 \leq n - 5$; Theorem 4.3 of [St] then applies to yield a k-dimensional polyhedron $K_i \subset \text{int } U_i$ such that $f_i(K) \subset K_i$ and the composition $\tilde{f}_i$: $K \xrightarrow{f_i} f_i(K) \hookrightarrow K_i$ is a simple homotopy equivalence. Let β_{ji}: $U_i \to U_j$ denote the inclusion map, where $i \geq j$. Since $\underline{f} = \{f_i\}_{i=1}^{\infty}$ is a shape equivalence, we may assume that there are maps g_i: $U_i \to K$ such that if $j < i$, then $f_j g_i \simeq B_{ji}$, and if $j \leq i$, $g_j \beta_{ji} f_i \simeq id_K$ and $\beta_{ji} f_i \simeq f_j$.

We note that (1) holds. The proof of (2) is similar to the proof of Lemma 2.1 of [C-D-D]. Specifically, if $\ell < j \leq i$, then $g_j \beta_{ji} f_i \simeq id_K$, and so $\tilde{f}_i g_j \beta_{ji} f_i \simeq \tilde{f}_i$. Letting ϕ_i denote a homotopy inverse of $\tilde{f}_i$, this yields $\tilde{f}_i g_j \beta_{ji} f_i \phi_i \simeq id_{K_i}$. But $f_i \phi_i \simeq id_{K_i}$ in K_i, and so $\tilde{f}_i g_j | K_i \simeq id_{K_i}$. This implies that the map $\tilde{f}_i g_j$: $U_j \to K_i$ is a weak retraction of U_j to K_i. Since $K_i \in$ ANR, it follows from the Borsuk Homotopy Extension Theorem that K_i is a retract of U_j. Since $\beta_{\ell i} f_i g_j \simeq f_\ell g_j \simeq \beta_{\ell j}$ and $\beta_{\ell i} f_i g_j (U_j) \subset K_i$, U_j deforms into K_i in U_ℓ; since $U_\ell \in$ ANR, a modification of Theorem 11 on pg. 31 of [Sp] shows that K_i is a strong deformation retract of U_j in U_ℓ.

It now remains to verify (3). Let $i > \ell$ and ϕ_i: $K_i \to K$ be a homotopy inverse of $\tilde{f}_i$. Let $G \subset \text{int } U_i$ be a regular neighborhood of K_i and $\rho\colon G \to K_i$ be the retraction induced by a collapse of G onto K_i. Now, $\beta_{\ell i} f_i \phi_i \rho\colon G \to U_\ell$ is homotopic to the inclusion of G into U_ℓ. Thus, if $m \geq i$, $\beta_{\ell m} f_m \phi_i \rho\colon G \to U_\ell$ is homotopic to the inclusion of G into U_ℓ. This implies that

$$\underline{h} = \{\beta_{\ell m} f_m \phi_i \rho, K_i, X, G\}_{m=i}^{\infty}$$

is a relative fundamental sequence in U_ℓ induced by the map ϕ_i and the shape morphism $\underline{f}$. (We index $\underline{h}$ by $m \geq i$ rather than $m \geq 1$, but this is an unimportant technical matter.)

If $m > i$, then $\beta_{im} \simeq f_i g_m\colon U_m \to U_i$. By the Borsuk Homotopy Extension Theorem, it follows that there exists an extension $\psi_m\colon U_{i+1} \to U_i$ of $f_i g_m$ so that $\psi_m \simeq \beta_{i,i+1}$. By (2), there exists $\tilde{r}\colon U_i \to K_i$ such that $\tilde{r}$ is the final stage of a strong deformation retraction of U_i to K_i in U_ℓ. Let $r\psi_m\colon U_{i+1} \to U_\ell$ denote the map $\tilde{r}\psi_m$ composed with the inclusion of K_i into U_ℓ. Then $r\psi_m \simeq \beta_{\ell,i+1}$ and $r\psi_m \simeq r\psi_{i+1}$ in K_i. It follows that

$$\underline{k} = \{r\psi_m, X, K_i, U_{i+1}\}_{m \geq i+1}$$

is a relative fundamental sequence in U_ℓ.

It now remains to be seen that $\underline{h}$ and $\underline{k}$ are mutually inverse. To see that $\underline{kh} \simeq \mathrm{id}_{K_i}$, let V be a neighborhood of K_i, let G' be a regular neighborhood of K_i in $V \cap G$, let $\alpha\colon K_i \to U_i$ denote the inclusion, and note that if $m \geq i + 1$, then $(r\psi_m)(\beta_{\ell m} f_m \phi_i \rho)|G' = r f_i g_m f_m \phi_i \rho|G' \simeq r f_i \phi_i \rho|G' = r\alpha \tilde{f}_i \phi_i \rho|G' \simeq r\alpha\rho|G' = \rho|G' \simeq \mathrm{id}|G'$ in G'. To see that $\underline{hk} \simeq \underline{\mathrm{id}}_X$, let V be a neighborhood of X and choose $s \geq i + 1$ such that $U_s \subset V$. Then, if $m > s + 1$, $(\beta_{\ell m} f_m \phi_i \rho)(r\psi_m)|U_{s+1}$ $\simeq \beta_{sm} f_m \phi_i \rho r \beta_{i,s+1} \simeq \beta_{sm} f_m \phi_i \rho r \psi_{s+1}|U_{s+1} = \beta_{sm} f_m \phi_i \rho r f_i g_{s+1} =$

$\beta_{sm}f_m\phi_i r\alpha\tilde{f}_i g_{s+1} = \beta_{sm}f_m\phi_i\tilde{f}_i g_{s+1} \simeq \beta_{sm}f_m g_{s+1} \simeq f_s g_{s+1} \simeq$ $\beta_{s,s+1} = id|U_{s+1}$, all homotopies occurring in $U_s \subset V$. This completes the proof.

We are now prepared to state our embedding theorem. It generalizes the Corollary established in [V_3].

Theorem 4. *Let* X *be a continuum in the interior of the piecewise linear* n*-manifold* M*,* K *be a* k*-dimensional polyhedron, and* $\underline{f}$: K → X *be a shape equivalence, where* $n \geq 5$ *and* $k \leq n - 3$. *Then for each neighborhood* U *of* X *in* M *there exists a* k*-dimensional polyhedron* K' ⊂ U *and a simple homotopy equivalence* h: K → K' *so that the homotopy inverse of* h *and* $\underline{f}$ *induce a relative shape equivalence of* K' *and* X *in* U.

Proof. By the main result of [V_3] there exists X' ⊂ int M such that X' satisfies ILC and such that X and X' have the same relative shape in U via relative fundamental sequences $\underline{g}$: X → X' and $\underline{k}$: X' → X. Part (3) of Lemma 2 (as applied to X') shows that there exist a neighborhood U_ℓ of X' in U, a k-dimensional polyhedron $K_{\ell+1}$, and a simple homotopy equivalence $\tilde{f}_{\ell+1}$: K → $K_{\ell+1}$ so that a homotopy inverse $\phi_{\ell+1}$ of $\tilde{f}_{\ell+1}$ induces, along with $\underline{gf}$, a relative shape equivalence of $K_{\ell+1}$ and X' in U_ℓ. But a relative shape equivalence in U_ℓ is clearly a relative shape equivalence in U, and so $\phi_{\ell+1}$ and $\underline{kgf}$ (which is homotopic to $\underline{f}$) induce a relative shape equivalence of $K_{\ell+1}$ and X in U.

If X is a compactum in the interior of the piecewise linear n-manifold M, a *deleted product neighborhood* of X is a compact piecewise linear manifold neighborhood N of X in

M such that $N - X \cong \partial N \times [0,1)$. The following generalizes Theorem 2.4 of [C-D-D] and Theorem 3 of [L]. Since deleted product neighborhoods are I-*regular* neighborhoods [Si] we also obtain the fact, established in [S-G-H], that I-regular neighborhoods exist for ILC embedded continua in the interior of a piecewise linear n-manifold, $n \geq 5$, which have the shape of a codimension 3 polyhedron.

Corollary 1. *Let* X *be a continuum in the interior of the piecewise linear* n-*manifold* M *such that* X *satisfies* ILC *and has the shape of a* k-*dimensional polyhedron* K, *where* $n \geq 5$ *and* $k \leq n - 3$. *Then* X *has a deleted product neighborhood in* M.

Proof. By Observation 1 of Section 2, there exists a neighborhood U of X in M such that the inclusion of X into U is shape (k-1)-connected. By Theorem 4, there exists a k-dimensional polyhedron $K' \subset U$ such that X and K' have the same relative shape in U. By Theorem 3, $U - X \cong U - K'$. A careful examination of the proof of Theorem 3 shows that in the first step of the induction we may push X into a regular neighborhood N of K' by a homeomorphism $q: U \to U$, the remainder of the proof showing that $N - K' \cong N - q(X)$. Then $q^{-1}(N)$ is a deleted product neighborhood of X.

References

[B] K. Borsuk, *Theory of shape*, Mathematical Monographs, vol. 59, Polish Scientific Publishers, Warsaw, 1975.

[C] T. A. Chapman, *On some applications of infinite-dimensional manifolds to the theory of shape*, Fund. Math 76 (1972), 181-193. MR47#9530.

[C-D-D] D. Coram, R. J. Daverman, and P. F. Duvall, Jr., *A loop condition for embedded compacta*, Proc. Amer. Math. Soc. 53 (1975), 205-212. MR51#14071.

[D-S] J. Dydak and J. Segal, *Shape theory, an introduction*, Lecture Notes in Mathematics No. 688, Springer-Verlag, New York, 1978.

[F] S. Ferry, *A stable converse to the Vietoris-Smale theorem with applications to shape theory* (preprint).

[H] J. F. P. Hudson, *Piecewise linear topology*, W. A. Benjamin Inc., New York, 1969.

[H-I] L. S. Husch and I. Ivanšić, *Shape domination and embedding up to shape*, Composito Math. (to appear).

[I-S-V] I. Ivanšić, R. B. Sher, and G. A. Venema, *Complement theorems beyond the trivial range*, Illinois J. Math. (submitted).

[K] J. Krasinkiewicz, *Continuous images of continua and 1-movability*, Fund. Math. 98 (1978), 141-164.

[L] V.-T. Liem, *Certain continua in* S^n *of the same shape have homeomorphic complements*, Trans. Amer. Math. Soc. 218 (1976), 207-217. MR53#1595.

[M] S. Mardešić, *On the Whitehead theorem in shape theory II*, Fund. Math. 91 (1976), 93-103. MR54#8562.

[S-G-H] L. Siebenmann, L. Guillou, and H. Hähl, *Les voisinages ouverts réguliers: critères homotopiques d'existance*, Ann. Sci. École Norm. Sup. (4) 7 (1974), 431-462. MR50#14766.

[Si] L. C. Siebenmann, *Regular (or canonical) open neighborhoods*, General Topology and Appl. 3 (1973), 51-61. MR51#6831.

[Sp] E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

[St] J. R. Stallings, *The embedding of homotopy into manifolds*, Mimeographed Notes, Princeton University, 1965.

[V_1] G. A. Venema, *Embeddings of compacta with shape dimension in the trivial range*, Proc. Amer. Math. Soc. 55 (1976), 443-448. MR53#1596.

[V_2] ________, *Neighborhoods of compacta in Euclidean space*, Fund. Math. (to appear).

[V_3] ________, *An approximation theorem in the shape category* (preprint).

[Z] E. C. Zeeman, *Seminar on combinatorial topology*, Mimeographed Notes, Institut Hautes Études Sci., Paris, 1963.

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