
TOPOLOGY PROCEEDINGS



Volume 4, 1979

Pages 463–478

<http://topology.auburn.edu/tp/>

INSERTION OF A CONTINUOUS FUNCTION

by

ERNEST P. LANE

Topology Proceedings

Web: <http://topology.auburn.edu/tp/>

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

ISSN: 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

INSERTION OF A CONTINUOUS FUNCTION

Ernest P. Lane

1. Introduction

This paper presents new results concerning the insertion of a continuous function between two comparable real-valued functions that possess any combination in three classes of semicontinuous properties, and it summarizes known results in the area. Since the notation and terminology of [9] is used throughout, it is convenient to recall that notation here. Write $g \leq f$ [respectively, $g < f$] if $g(x) \leq f(x)$ [respectively, $g(x) < f(x)$] for all x in X . Let P_1 and P_2 denote any properties that are defined relative to real-valued functions f on a space X . A space X has the *weak C insertion* [respectively, *C insertion*] *property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$ [respectively, $g < f$], g has property P_1 and f has property P_2 , then there exists a continuous function h on X such that $g \leq h \leq f$ [respectively, $g < h < f$]. A space X has the *strong C insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2 , then there exists a continuous function h on X such that $g \leq h \leq f$ and such that for any x in X for which $g(x) < f(x)$ then $g(x) < h(x) < f(x)$. The classes of functions considered consist of lower and upper semicontinuous (lsc and usc), normal lower and upper semicontinuous (nlsc and nusc), z lower and upper semicontinuous (zlsc and zusc), and continuous functions. If f_*

[respectively f^*] denotes the lower [respectively, upper] limit function of f , then f is nlsc in case $f = (f^*)_*$ and f is nusc in case $f = (f_*)^*$. See Dilworth [5] for details. A function f is zlsc (respectively, zusc) in case $\{x|f(x) \leq r\}$ [respectively, $\{x|f(x) \geq r\}$] is a zero-set for each real number r . These functions have been considered by Stone [17] and by Blatter and Seever [3].

For our consideration of strong C insertion and C insertion in the third and fourth sections, the following concept is useful. A property P defined relative to a real-valued function on a topological space is a C property provided any constant function has property P and provided the sum of a function with property P and a continuous function also has property P .

If f is a real-valued function on a space X , then for any real number t , A is a *lower cut set* for f at level t in case

$$\{x|f(x) < t\} \subset A \subset \{x|f(x) \leq t\}.$$

(In later applications of this notion, the notation $A(f,t)$ is used for a uniquely specified lower cut set.)

The Lebesgue sets of f are sets of the form

$$L_t(f) = \{x|f(x) \leq t\} \text{ and } L^t(f) = \{x|f(x) \geq t\}.$$

If $\{A_n\}$ is a sequence of sets such that $\bigcap_{n=1}^{\infty} A_n = \emptyset$, then write $\{A_n\} \rightarrow \emptyset$; if in addition $A_n \supset A_{n+1}$ for each n , write $\{A_n\} \times \emptyset$. Recall that X is an F -space in case disjoint cozero-sets are completely separated, and X is a P -space in case every zero-set in X is also a cozero-set.

2. Weak C Insertion

The following theorem gives a necessary and sufficient condition for the weak C insertion of a continuous function; this result was announced in [11]. The proof that is presented here depends on Theorem 1 of [8], which was derived from the results of Katětov [7]. This theorem has been obtained independently by R. L. Blair [2], who used techniques of Tong [18].

Theorem 2.1. Let g and f be real-valued functions on a space X such that $g \leq f$. The following are equivalent:

(i) There exists a continuous function h on X such that $g \leq h \leq f$.

(ii) For each rational number t there exist lower cut sets $A(g,t)$ and $A(f,t)$ such that if $t_1 < t_2$ then $A(f,t_1)$ and $X - A(g,t_2)$ are completely separated.

(iii) For rational numbers t_1 and t_2 such that $t_1 < t_2$ the Lebesgue sets $L_{t_1}(f)$ and $L_{t_2}^2(g)$ are completely separated.

Proof. (ii) \Rightarrow (i). Let g and f be real-valued functions on a space X such that $g \leq f$. Define a binary relation ρ on the power set of X as follows. For subsets A and B of X , let

$$A \rho B \text{ if and only if } A \subset Z \subset C \subset B$$

for some zero-set Z and some cozero-set C .

For any space X it is possible to show that ρ is a strong binary relation on the power set of X . (That is, ρ satisfies the following three properties:

(1) If $A_i \rho B_j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, then there is a subset C of X such that $A_i \rho C$ and $C \rho B_j$ for each i

and j .

(2) If $A \subset B$, then $D \rho A$ implies $D \rho B$ and $B \rho E$ implies $A \rho E$.

(3) If $A \rho B$, then $\text{cl}(A) \subset \text{int}(B)$.

By Theorem 1 of [8], there exists a continuous function h on X such that $g \leq h \leq f$.

(i) \Rightarrow (iii). If h is a continuous function such that $g \leq h \leq f$, then for $t_1 < t_2$,

$$L_{t_1}(f) \subset L_{t_1}(h) \subset X - L^{t_2}(h) \subset X - L^{t_2}(g).$$

Since $L_{t_1}(h)$ is a zero-set and $X - L^{t_2}(h)$ is a cozero-set, then $L_{t_1}(f)$ and $L^{t_2}(g)$ are completely separated.

(iii) \Rightarrow (ii). For any rational number t , let $A(f,t) = L_t(f)$ and $A(g,t) = X - L^t(g)$.

Let $B(X)$ denote the Banach lattice of all bounded real-valued functions on X . If C is a sublattice of the power set of X to which \emptyset and X belong, the smallest convex cone in $B(X)$ which contains the constant functions and the characteristic functions 1_D , $D \in C$, is denoted by $\text{cn}(C)$ and its closure is denoted by $\overline{\text{cn}}(C)$. Blatter and Seever proved the following result (Theorem 3.8 of [4]): Let A and B be sublattices of the power set of X that contain \emptyset and X . If g is in $\overline{\text{cn}}(A)$, if f is in $\overline{\text{cn}}(B)$, and if $g \leq f$, then there is a function h in $\overline{\text{cn}}(A) \cap \overline{\text{cn}}(B)$, such that $g \leq h \leq f$ if and only if whenever $C \in A$, $D \in B$ and $C \subset D$, there exist $C' \in A$ and $D' \in B$ such that $C \subset D' \subset C' \subset D$. In some cases, Theorem 2.1 above is equivalent to this result. (In particular if $g \leq f$, if g satisfies a property P_1 , and if f

satisfies a property P_2 , then the result of Blatter and Seever may be used if the class of functions that satisfy P_1 [respectively, P_2] can be characterized as $\overline{cn}(A)$ [respectively, $\overline{cn}(B)$] for some sublattices A and B of the power set of X .) However, in some of the applications of Theorem 2.1 that are given below, it appears that the result of Blatter and Seever does not apply since the property being considered is not preserved under sum or supremum. In particular, neither the sum nor the least upper bound of two nlsc functions need be nlsc.

Corollary 2.2. A space X satisfies the weak C insertion property for:

(a) *(usc, lsc) if and only if X is normal (Katětov [7], Tong [18]).*

(b) *(lsc, usc) if and only if X is extremally disconnected (Stone [17]).*

(c) *(nusc, nlsc) if and only if X is mildly normal (Lane [8]).*

(d) *(zusc, zlsc) if X is any space (Blatter and Seever [3]).*

(e) *(zlsc, zusc) if and only if X is an F -space (Seever [16]).*

(f) *(lsc, zusc) [resp., (zlsc, usc)] if and only if X is basically disconnected (Stone [17]).*

(g) *(zusc, lsc) [resp., (usc, zlsc)] if and only if X is δ -normally separated.*

(h) *(zusc, nlsc) [resp., (nusc, zlsc)] if and only if X is weakly δ -normally separated.*

(i) $(nlsc, nusc)$ if and only if the closure of each regular open subset of X is open.

(j) $(nlsc, usc)$ [resp., $(lsc, nusc)$] if and only if disjoint open sets, at least one of which is regular open, are completely separated.

(k) $(nlsc, zusc)$ [resp., $(zpsc, nusc)$] if and only if disjoint open sets, one of which is regular open and the other a cozero-set, are completely separated.

Proof. The proofs of (a) through (f) are omitted since these results are known.

(g) Assume that X is δ -normally separated and let g and f be functions on X such that $g \leq f$, g is $zusc$, and f is lsc . For each rational number t , $A(f, t) = \{x | f(x) \leq t\}$ is closed and $A(g, t) = \{x | g(x) < t\}$ is a cozero-set. If $t_1 < t_2$ then $A(f, t_1) \subset A(g, t_2)$, and since X is δ -normally separated the closed set $A(f, t_1)$ and the zero-set $X - A(g, t_2)$ are completely separated. By Theorem 2.1, there is a continuous function h on X such that $g \leq h \leq f$.

Conversely, assume that X satisfies the weak C insertion property for $(zusc, lsc)$. If F is closed and disjoint from the zero-set Z , then l_Z is $zusc$, l_{X-F} is lsc , and $l_Z \leq l_{X-F}$. If h is a continuous function such that $l_Z \leq h \leq l_{X-F}$, then Z and F are contained in disjoint zero-sets determined by h . Hence F and X are completely separated and consequently X is δ -normally separated.

The proof for $(usc, zpssc)$ is similar, or may be deduced from the above result.

The proof of the sufficiency is sketched in the remaining cases:

(h) If $g \leq f$, g is zusc, and f is nlsc, then $A(f,t) = \text{cl}\{x|f(x) < t\}$ is regular closed and $A(g,t) = \{x|g(x) \leq t\}$ is a cozero-set. For $t_1 < t_2$ the regular closed set $A(f,t_1)$ is completely separated from the zero-set $X - A(g,t_2)$.

(i) If $g \leq f$, g is nlsc, and f is nusc, then $A(f,t) = X - \text{cl}\{x|f(x) > t\}$ is a regular open lower cut set for f , and $A(g,t) = \text{cl}\{x|g(x) < t\}$ is a regular closed lower cut set for g . If $t_1 < t_2$, and if the closure of a regular open set is open, then

$$A(f,t_1) \subset \text{cl } A(f,t_1) \subset \text{int } A(g,t_2) \subset A(g,t_2),$$

and the open and closed sets $\text{cl } A(f,t_1)$ and $X - \text{int } A(g,t_2)$ are completely separated. Hence $A(f,t_1)$ and $X - A(g,t_2)$ are completely separated.

(j) If $g \leq f$, g is nlsc, and f is usc, let $A(g,t) = \text{cl}\{x|g(x) < t\}$ and $A(f,t) = \{x|f(x) < t\}$. If $t_1 < t_2$, the open set $A(f,t_1)$ and the regular open set $X - A(g,t_2)$ are completely separated.

(k) If $g \leq f$, g is nlsc and f is zusc, then $A(f,t) = \{x|f(x) < t\}$ is a cozero-set and $A(g,t) = \text{cl}\{x|g(x) < t\}$ is regular closed. If $t_1 < t_2$, then the cozero-set $A(f,t_1)$ and the regular open set $X - A(g,t_2)$ are completely separated.

The proof of the necessity in cases h, i, j, and k is omitted.

Case (g) above answers a question posed by Blatter and Seever [3].

3. Strong C Insertion

Michael's work on continuous selections [15] proved that a space X satisfies the strong C insertion property for

(usc,lsc) if and only if X is perfectly normal. Blatter and Seever [3, 4] prove general necessary and sufficient conditions for strong C insertion that yield theorems for the cases (zusc,zlsc), (lsc,usc), (lsc,zusc), (zlsc,zusc), (lsc,continuous), and (zlsc,continuous). Using the following two results, additional special cases are obtained.

Theorem 3.1 [9]. Let P_1 and P_2 be C properties and consider the following condition:

(α) *If g and f are functions on X such that $g \leq f$, g satisfies property P_1 and f satisfies property P_2 , then there exists a sequence $A(f - g, 2^{-n})$ of lower cut sets for $f - g$ and there exists a sequence $\{F_n\}$ of subsets of X such that*

$$(i) \{x \mid (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n, \text{ and}$$

(ii) *for each n the sets $A(f - g, 2^{-n})$ and F_n are completely separated.*

If X satisfies the weak C insertion property for (P_1, P_2) and if X satisfies condition (α), then X satisfies the strong C insertion property for (P_1, P_2) . Conversely, if X satisfies the strong C insertion property for (P_1, P_2) and if $f - g$ satisfies property P_1 , then X satisfies condition (α).

Theorem 3.2 [10]. Let P_1 and P_2 be C properties and assume that the space X satisfies the weak C insertion property for (P_1, P_2) . The space X satisfies the strong C insertion property for (P_1, P_2) if and only if X satisfies the strong C insertion property for $(P_1, \text{continuous})$ and for $(\text{continuous}, P_2)$.

The known cases that were mentioned above and new results are given in the following theorem.

Theorem 3.3. A space X satisfies the strong C insertion property for:

(a) (usc, lsc) [resp., $(usc, continuous)$, $(continuous, lsc)$] if and only if X is perfectly normal (Michael [15]).

(b) $(zusc, zlsc)$ if X is any space (Blatter and Seever [3]).

(c) $(zusc, lsc)$ [resp., $(usc, zlsc)$] if and only if X is perfectly normal (Blatter and Seever [3]).

(d) (lsc, usc) [resp., $(lsc, zusc)$, $(zlsc, usc)$, $(lsc, continuous)$, $(continuous, usc)$] if and only if each open subset of X is closed (Blatter and Seever [3]).

(e) $(zlsc, zusc)$ [resp., $(zlsc, continuous)$, $(continuous, zusc)$] if and only if X is a P -space (Blatter and Seever [3]).

(f) $(nusc, nlsc)$ [resp., $(nusc, continuous)$, $(continuous, nlsc)$, $(zusc, nlsc)$, $(nusc, zlsc)$] if and only if X is an Oz space.

(g) $(nlsc, nusc)$ [resp., $(nlsc, continuous)$, $(continuous, nusc)$] if and only if each regular open subset of X is closed.

(h) $(nlsc, zusc)$ [resp., $(zlsc, nusc)$] if and only if each open subset of X is closed.

(i) $(nlsc, zusc)$ [resp., $(zlsc, nusc)$] if and only if X is a P -space such that each regular open subset is closed.

Proof. The proof of the necessity is in each case similar to known cases and is omitted.

(f) The first three cases were proved in [10] (where the property Oz, which has been studied by Blair [1], was inadvertently called pm-normal; X is an Oz space in case each regular closed subset is a zero-set). The following observations prove the case for (zusc,nlsc) (and the proof for (nusc,zlsc) follows from this): If X is an Oz space then X is weakly δ -normally separated, and hence X satisfies the weak C insertion property for (zusc,nlsc) by Corollary 2.2 (h). Now any space satisfies the strong C insertion property for (zusc,continuous), by case (b) above, and since X is an Oz space, X satisfies the strong C insertion property for (continuous,nlsc). By Theorem 3.2, X satisfies the strong C insertion property for (zusc,nlsc).

(g) Assume that each regular open subset of X is closed and let $g \leq f$, where g is nlsc and f is continuous. Since $f - g$ is nusc, the lower cut set for $f - g$ that is defined by

$$A(f - g, 2^{-n}) = X - \text{cl}\{x \mid (f-g)(x) > 2^{-n}\}$$

is regular open; by hypothesis $A(f - g, 2^{-n})$ is therefore closed, and consequently $A(f - g, 2^{-n})$ is a zero-set. Similarly, if

$$F_n = \text{cl}\{x \mid (f-g)(x) > 2^{-n}\},$$

then each F_n is a zero-set. Now

$$\{x \mid (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$$

and each F_n and $A(f - g, 2^{-n})$ are completely separated. By Theorem 3.1, X satisfies the strong C insertion property for (nlsc,continuous). The case for (continuous,nusc) is equivalent to this. By Theorem 3.2 it follows that X satisfies the strong C insertion property for (nlsc,nusc).

Case (h) follows from (d), and (i) follows from the results for (continuous,zusc) and (nlsc,continuous) and Theorem 3.2. This concludes the proof of Theorem 3.3.

4. C Insertion

The following result, which was proved in [9], is restated here since it will be used repeatedly below.

Theorem 4.1. Let X be a space that satisfies the weak C insertion property for C properties P_1 and P_2 . The space X has the C insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a decreasing sequence D_n of subsets of X with empty intersection and such that for each n, $X - D_n$ and $A(f - g, 3^{-n+1})$ are completely separated.

Blatter and Seever give in Theorem 3.11 of [4] a necessary and sufficient condition for C insertion. However, as in the case noted above for weak C insertion, it appears that Theorem 4.1 applies in situations where the result of Blatter and Seever does not because of their hypothesis concerning lattice properties. It would be beneficial to have a version of Theorem 4.1 that used lower cut sets for f and g rather than lower cut sets for f - g. But as will be illustrated below, this theorem applies in situations where f - g fails to have the same property that f and -g possess. With one exception, the following theorem gives a necessary and sufficient condition for C insertion for any pairwise combination of the properties continuous, lsc, usc, zlsc, zusc, nlsc, and nusc.

Theorem 4.2. A space X satisfies the C insertion property for:

(a) (usc, lsc) if and only if X is normal and countably paracompact (Dowker [6], Katětov [7]).

(b) $(zusc, zlsc)$ if X is any space (Blatter and Seever [3]).

(c) (lsc, usc) if and only if X is an extremally disconnected P -space that satisfies Baire's condition (Blatter and Seever [3]).

(d) $(zlsc, zusc)$ if and only if X is a P -space (Blatter and Seever [3]).

(e) $(usc, continuous)$ [resp., $(continuous, lsc)$] if and only if X is a cb -space (Mack [12]).

(f) $(nusc, continuous)$ [resp., $(continuous, nlsc)$] if and only if X is a weak cb -space (Mack and Johnson [14]).

(g) $(zusc, lsc)$ [resp., $(usc, zlsc)$] if and only if X is δ -normally separated and if $\{F_n\}$ is any sequence of closed sets such that $\{F_n\} \times \emptyset$ then there exists a sequence $\{C_n\}$ of cozero-sets such that $C_n \supset F_n$ for each n and such that $\{C_n\} \rightarrow \emptyset$.

(h) $(lsc, zusc)$ [resp., $(zlsc, usc)$] if and only if X is basically disconnected and for any sequence $\{G_n\}$ of open sets such that $\{G_n\} \times \emptyset$ then there exists a sequence $\{C_n\}$ of cozero-sets such that $C_n \supset G_n$ and such that $\{cl(C_n)\} \rightarrow \emptyset$.

(i) $(zusc, nlsc)$ [resp., $(nusc, zlsc)$] if and only if X is a weak cb -space.

(j) $(nlsc, nusc)$ if and only if each regular open set is closed.

(k) $(nlsc, usc)$ [resp., $(lsc, nusc)$] if and only if

disjoint open sets, at least one of which is regular open, are completely separated and if for any sequence $\{G_n\}$ of open sets such that $\{G_n\} \times \emptyset$ there exists a sequence $\{F_n\}$ of regular closed sets such that $F_n \supset G_n$ for each n and such that $\{F_n\} \rightarrow \emptyset$.

(l) (lsc, continuous) [resp., (continuous, usc)] if and only if X is a P -space such that if $\{G_n\}$ is any sequence of open sets such that $\{G_n\} \times \emptyset$ then there exists a sequence $\{C_n\}$ of cozero-sets such that $C_n \supset G_n$ for each n and such that $\{C_n\} \rightarrow \emptyset$.

(m) (zlsc, continuous) [resp., (continuous, nusc)] if and only if X is a P -space.

(n) (nlsc, continuous) [resp., (continuous, nusc)] if and only if any regular open set in X is completely separated from any zero-set disjoint from it and if for any sequence $\{G_n\}$ of regular open sets such that $\{G_n\} \times \emptyset$ then there exists a sequence $\{C_n\}$ of cozero-sets such that $C_n \supset G_n$ for each n and such that $\{C_n\} \rightarrow \emptyset$.

Proof. The proofs of the known results (a) through (f) are omitted.

(g) If $g < f$, g is zusc, and f is lsc, let

$$A(f - g, 3^{-n+1}) = \{x \mid (f - g)(x) \leq 3^{-n+1}\}.$$

Since $f - g > 0$ and since $f - g$ is lsc, each $A(f - g, 3^{-n+1})$ is a closed set and $\{A(f - g, 3^{-n+1})\} \times \emptyset$. By hypothesis

there exists a sequence $\{C_n\}$ of cozero-sets such that $C_n \supset A(f - g, 3^{-n+1})$ for each n and such that $\{C_n\} \rightarrow \emptyset$.

Since X is δ -normally separated the closed set $A(f - g, 3^{-n+1})$ and the zero-set $X - C_n$ are completely separated, and by

Corollary 2.2(g) above, X satisfies the weak C insertion property for $(zusc, lsc)$. By Theorem 4.1, there is a continuous function h on X such that $g < h < f$.

Conversely, assume that X satisfies the C insertion property for $(zusc, lsc)$. If $f = -1$ on F , $f = 2$ on $X - F$, $g = -2$ on $X - Z$, and $g = 1$ on Z , then g is $zusc$, f is lsc and $g < f$. If h is a continuous function such that $g < h < f$, then the zero-sets $\{x | h(x) \leq -1\}$ and $\{x | h(x) \geq 1\}$ contain F and Z respectively. Hence F and Z are completely separated and thus X is δ -normally separated. If $\{F_n\}$ is a sequence of closed sets such that $\{F_n\} \times \emptyset$, let $f(x) = 1$ for x in $X - F_1$ and $f(x) = 1/n$ for x in $F_n - F_{n+1}$. Since $f > 0$ and since f is lsc , there is by hypothesis a continuous function h on X such that $g > h > 0$. If $C_n = \{x | h(x) < 1/n\}$, then $\{C_n\}$ is a sequence of cozero-sets such that $C_n \supset F_n$ and such that $\{C_n\} \rightarrow \emptyset$.

In each of the cases (h) through (n) the proof that the respective C insertion property implies the other condition may be accomplished by an argument similar to that given in case (g) above. Consequently these arguments are omitted. The proof of the sufficiency of the given property for C insertion in cases (h), (k), and (l) is essentially the same argument used in case (g) above and thus is not repeated. In case (n) where $g < f$, g is $nlsc$, and f is continuous, the same argument as in case (g) works, except the sequence of lower cut sets for $f - g$ is defined by

$$A(f - g, 3^{-n+1}) = X - cl\{x | (f - g)(x) > 3^{-n+1}\};$$

since $f - g$ is $nusc$, $A(f - g, 3^{-n+1})$ is a regular open lower

cut set for $f - g$. Case (l) is a consequence of (h), and (m) follows from (d). Case (j) is a corollary of Theorem 3.3(g) above since strong C insertion for (nlsc,nusc) trivially implies C insertion for (nlsc,nusc). The following argument proves case (i): Assume that X is a weak cb-space, and let $g < f$ where g is zusc and f is nlsc. Since a weak cb-space is weakly δ -normally separated (by Lemma 10 of Mack [15]), there is a continuous function k on X such that $g \leq k \leq f$ by Corollary 2.2(h) above. By Theorem 3.3(b), any space satisfies the strong C insertion property for (zusc,continuous); hence there is a continuous function k_1 on X such that $g \leq k_1 \leq k$ and if $g(x) < k(x)$ then $g(x) < k_1(x) < k(x)$. Let $k_2 = (k_1 + k)/2$. Then k_2 is continuous and $k_2 < f$. Since X is a weak cb-space there is, by case (f) above, a continuous function h on X such that $k_2 < h < f$. Hence $g < h < f$, and it follows that X satisfies the C insertion property for (nusc,zlsc). This concludes the proof of Theorem 4.2.

We note that cases (g) and (h) above answer questions left open by Blatter and Seever in [3].

Added in proof: The condition in (g) of Theorem 4.2 is equivalent to X is a cb-space.

References

1. R. L. Blair, *Spaces in which special sets are z-imbedded*, Can. J. Math. 28, No. 4 (1976), 673-690.
2. _____, *Extensions of Lebesgue sets and of real-valued functions* (to appear).
3. J. Blatter and G. L. Seever, *Interposition of semi-continuous functions by continuous functions*, *Analyse*

- Fonctionnelle et Applications (Comptes Rendus du Colloque d'Analyse, Rio de Janeiro 1972), Hermann, Paris, 1975, 27-51.
4. _____, *Interposition and lattice cones of functions*, Trans. Amer. Math. Soc. 222 (1976), 65-96.
 5. R. P. Dilworth, *The normal completion of the lattice of continuous functions*, Trans. Amer. Math. Soc. 68 (1950), 427-438.
 6. C. H. Dowker, *On countably paracompact spaces*, Can. J. Math. 3 (1951), 219-224.
 7. M. Katětov, *On real-valued functions in topological spaces*, Fund. Math. 38 (1951), 85-91.
 8. E. P. Lane, *A sufficient condition for the insertion of a continuous function*, Proc. Amer. Math. Soc. 49 (1975), 90-94.
 9. _____, *Insertion of a continuous function*, Pacific J. Math. 66 (1976), 181-190.
 10. _____, *PM-normality and the insertion of a continuous function*, Pacific J. Math. 82 (1979), 155-162.
 11. _____, *Weak C insertion of a continuous function*, Notices Amer. Math. Soc. 26 (1979), A-231.
 12. J. E. Mack, *On a class of countably paracompact spaces*, Proc. Amer. Math. Soc. 16 (1976), 467-472.
 13. _____, *Countable paracompactness and weak normality properties*, Trans. Amer. Math. Soc. 148 (1970), 265-272.
 14. _____ and D. G. Johnson, *The Dedekind completion of $C(X)$* , Pacific J. Math. 20 (1967), 231-243.
 15. E. Michael, *Continuous selections I*, Ann. of Math. 63 (1956), 361-382.
 16. G. L. Seever, *Measures on F-spaces*, Trans. Amer. Math. Soc. 133 (1968), 267-280.
 17. M. H. Stone, *Boundedness properties in function lattices*, Can. J. Math. 1 (1949), 176-186.
 18. H. Tong, *Some characterizations of normal and perfectly normal spaces*, Duke Math. J. 19 (1952), 289-292.

Appalachian State University
Boone, North Carolina 28607