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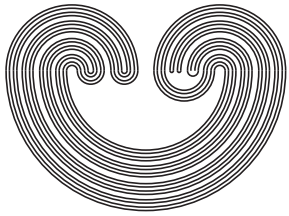

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SHRINKABLE DECOMPOSITIONS, CRITERIA AND GENERALIZATIONS

by

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1. Introduction

Almost thirty years ago, R. H. Bing described upper semicontinuous decompositions G of E^3 and defined homeomorphisms of E^3 onto itself which shrank the collections H of the nondegenerate elements of G to small size. He shrank the members of H to points with a sequence $\{h_i\}$ of "shrinking" homeomorphisms such that $\{h_i\}$ converged uniformly to a continuous mapping f of E^3 onto E^3 where G was identical to the collection $\{f^{-1}(x) \mid x \in E^3\}$. Some of this work appeared in print in 1952 (see [3]). Various other papers followed which used this concept of shrinkability. See [4; 5; 6; 7].

I was the first person to formalize the concept of *shrinkability* in a paper [14] which appeared in 1961. I also formalized other concepts such as that of a collection being *countably shrinkable* and a continuum being *locally shrinkable*. These concepts readily generalize to metric spaces (and more general spaces). The first theorems proved for upper semicontinuous (usc) decompositions of *metric spaces* which possessed one of these properties of shrinkability appeared in [14; 15]. In [15], I needed the usual concept of upper semicontinuity (Whyburn [22]) rather than my own more general concept. The usual concept is equivalent to requiring that the quotient mapping P (or projection mapping) be closed and point compact, i.e., point inverses

under P are compact.

In my thesis (1954) and in [15], I defined upper semi-continuous decompositions using purely topological properties. Such decompositions do not necessarily yield closed mappings $p: X \to X/G$. In fact, M. J. Reed [18] has used my definition of upper semicontinuity to say formally that a T_1-topological space Y is Mc if and only if for points $x,y \in Y$, $x \neq y$, and each of x and y is a limit point of a subset A of Y, then there exists a subset B of A such that x is a limit point of B and y is not a limit point of B.

A decomposition G of a T_1-topological space (X,T) is said to be *Mc upper semicontinuous* iff (a) G is a collection of pairwise disjoint closed subsets which covers X, (b) the decomposition space X/G is given a topology by declaring a subcollection C of G to be open iff C* (the union of the elements of C) is open in X [note that the projection $p: X \to X/G$ (defined by $p(x) = g \in G$ iff $x \in g$) is continuous], and (c) X/G is Mc.

In 1971, Myra Reed improved the results in [17]. This work comprises Chapter III of her thesis. Proofs are provided with clarity and details. For the sake of having a detailed proof of an important theorem for shrinkable decompositions of metric spaces, I hope that her proof will appear in the proceedings of this conference (Ohio University, March, 1979).

It was observed by Reed that the concept of *shrinkability* was independent of the metric used. This was also pointed out by Edwards and Glaser in 1972 [11].

The concept of a shrinkable decomposition G of a space M has had a major impact on the study of the topology of manifolds, in particular, over the last ten years. The various important papers which have appeared in recent years are too numerous to mention here.

2. Definitions of Shrinkable Decompositions

Although no attempt is made to list all definitions of shrinkable decompositions, some examples are given below.

A subset K of a metric space (M,d) is *locally shrinkable* iff for each open set $U \supset K$ and each $\varepsilon > 0$, there exists a homeomorphism $h: M \Rightarrow M$ such that $h = id.$ off U and diam $h(K) < \varepsilon$. (McAuley)

Suppose that G is an usc decomposition of a metric space (M,d) and that H is the collection of all nondegenerate elements of G. Also, let $P: M \Rightarrow M/G$ denote the quotient or projection mapping of M onto the (metric) decomposition space M/G. An open covering U of open sets in M is *saturated* iff for each $0 \in U$, $0 = P^{-1}P(0)$. Reed says that 0 is P-open, while Whyburn calls 0 an open inverse set.

(Reed) The collection H is *tightly shrinkable in* M iff given a saturated open covering U of H* (the union of the elements of H), $\varepsilon > 0$, and a homeomorphism h of M onto M, there is a saturated open covering V of H* which refines U and a homeomorphism f of M onto M such that

(1) $f = h$ off V^*,

(2) for each $g \in H$, diam $f(g) < \varepsilon$, and

(3) for each $v \in V$, there exists $u \in U$ such that $h(v) \cup f(v) \subset h(u)$.

The collection H is *weakly tightly shrinkable* iff the above holds for $h = id_M$.

(McAuley's definition as stated by Edwards and Glaser in [11].) The decomposition G is *shrinkable* if given any map $\varepsilon: M \to (0,\infty)$ and any saturated open cover U of M, there is an isotopy $f_t: M \Rightarrow M$, $t \in [0,1]$, such that f_0 = identity and for each $g \in G$,

(1) there is $u \in U$ such that $u \supset g \cup f_t(g)$ for all $t \in [0,1]$, and

(2) diam $f_1(g) < \inf \varepsilon(g)$.

An equivalent definition given by Edwards and Glaser [11] shows that the notion of shrinkability is independent of the metric d chosen for M.

The collection G is *shrinkable* iff for any two open coverings U and V of M where U is saturated, there is an isotopy $f_t: M \Rightarrow M$, $t \in [0,1]$, f_0 = identity, and such that for each $g \in G$,

(1) there is $u \in U$ such that $u \supset g \cup f_t(g)$ for all t, and

(2) there is $v \in V$ such that $f_1(g) \subset v$.

3. Shrinkable Decompositions—Some Theorems

One of the most general and useful theorems about shrinkable decompositions is the following. The version stated here comes from the paper [11] by Edwards and Glaser. They removed the local compactness hypothesis required in my theorem in [15].

Theorem (Bing and McAuley). *Suppose that* G *is an usc decomposition of the complete metric space* (M,d) *and that* G *is shrinkable. If* U *is any given saturated open covering of* M, *then there is a homotopy* h_t: M ⇒ M, t ∈ [0,1], *such that*

(1) h_0 = *identity,*

(2) h_t *is a homeomorphism for* $0 \leq t < 1$ *(called a pseudo-isotopy),*

(3) for each g ∈ G, *there is* u ∈ U *such that* u ⊃ g ∪ $h_t(g)$ *for all* t ∈ [0,1],

(4) h_1 *takes* M *into itself, and*

(5) the collection $\{h_1^{-1}(x) \mid x \in M\}$ *is identical to* G.

Corollary. Under the hypothesis given above, M/G *is homeomorphic to* M.

It should be noted that Edwards and Glaser remark in [11] that the theorem above may hold for a wider class of spaces, in particular, paracompact spaces which admit a complete gauge structure [9].

Although shrinkability is a sufficient condition that an usc decomposition G of E^3 have the property that E^3/G be homeomorphic to E^3, it is not necessary. The so-called figure eight decomposition of E^3 by Bing in [6] is a counter-example.

One is led naturally to the following:

Question. Suppose that G is an usc decomposition of a metric space (M,d) and M/G is homeomorphic to M. Is G shrinkable?

The first partial answer is due to Armentrout. In 1969, he proved [2] that if G is a cellular 0-dimensional usc decomposition of E^3 and E^3/G is homeomorphic to E^3, then G is shrinkable.

Here, G is a 0-dimensional usc decomposition iff P(H) had dimension 0 in E^3/G where P is the projection mapping.

Next, Price proved in 1969 [17] that if G is a cellular usc decomposition of S^3 such that S^3/G is a 3-manifold, then there is a pseudo-isotopy of S^3 onto itself which shrinks each element of G to a point.

Soon afterwards, Voxman proved in [21], which appeared in 1970, that if G is a cellular usc decomposition of a 3-manifold M, then M/G is homeomorphic to M iff G is shrinkable.

Later, Siebenmann proved in [20] that if M is a manifold without boundary (dim M $\neq$ 4) and G is an usc cell-like decomposition of M and M/G is a manifold, then G is shrinkable.

As far as I know, the general question remains unanswered. That is, it is unknown for what spaces M and what usc decompositions G of M it is true that M/G is homeomorphic to M iff G is shrinkable.

4. Generalizations—The End of a Pseudo-Isotopy

In 1967, Ross L. Finney published a paper [12] entitled "Psuedo-isotopies and cellular sets." A pseudo-isotopy h shrinks the elements of a decomposition G of M iff

(1) $h_0 = id_M$ and

(2) for each $x \in M$, $h_1^{-1}(x) \in G$.

The mapping h_1 is called the *end* of the pseudo-isotopy where $h: M \times I \Rightarrow M$ is a homotopy, $h_t = h|M \times t$ is onto, and h_t for $t < 1$ is a homeomorphism.

Suppose that $f: M \Rightarrow M$ is a mapping and that $G_f = \{f^{-1}(x) | x \in M\}$. Thus, f induces a decomposition G_f of M. If f is closed, then G_f is an usc decomposition of M.

The following proposition is proved by Finney [12].

Proposition (Finney). *Suppose that* X *is a compact Hausdorff space. A mapping* f *of* X *onto itself is the end of a pseudo-isotopy on* X *iff there exists some pseudo-isotopy on* X *that shrinks the elements of* G_f *to points.*

Question (Finney). Suppose that f is a cellular mapping of a compact manifold M onto M. Is f the end of a pseudo-isotopy?

The following is a partial answer to this question.

Theorem (Finney [12]). *Let* f *be a cellular map of a triangulated compact* 3*-manifold onto itself. If* f *is simplicial, then* f *is the end of a pseudo-isotopy on* M.

The results of Voxman and Siebenmann stated in the previous section provide more complete answers to Finney's question. However, the question has not been fully answered as far as I know.

5. Concerning the Shrinkability of Countable Collections H

There are numerous theorems which give conditions under

which an usc decomposition G of $E^3(S^3)$ is shrinkable when the collection H of nondegenerate elements is *countable*. One theorem, in my opinion, offers some hope of simplifying a very chaotic situation. It is Woodruff's 2-sphere theorem, which is stated below:

Theorem (Woodruff [23]). *Suppose that* G *is an usc decomposition of* S^3 *and that for each* $p \in P(H)$ *(where* $P: S^3 \Rightarrow S^3/G$*) and each open set* U *containing* p, *there is an open set* V *such that* $p \in V \subset U$ *and* Bd V *is a 2-sphere missing* P(H). *Then* S^3/G *is homeomorphic to* S^3.

Question. Is there some *useful and easily applied criterion* which can distinguish *the shrinkable countable collections* H from the *nonshrinkable* ones? Here, again, H denotes the collection of nondegenerate elements of an usc continuous decomposition of a manifold (metric space, etc.).

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