# TREE-LIKE CONTINUA AS LIMITS OF CYCLIC GRAPHS

by

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A continuum X is a compact, connected metric space with more than one point.  A continuum is tree-like if it is homeomorphic to the limit of an inverse sequence of trees.

A map is a continuous function.  A map f: X → X has a fixed point if there exists a point x in X such that f(x) = x. A continuum X has the fixed-point-property if each map f: X → X has a fixed point.

R. H. Bing [2] has raised the following question:

*Question.*  Does each tree-like continuum have the fixed-point-property?

Bing's question has recently been answered by David Bellamy [1].

*Theorem 1.  There exists a tree-like continuum X without the fixed-point-property.*

Here is the gist of Bellamy's proof:  Take a certain solenoid.  Remove an arc and recompactify with the suspension of an infinite set.  The example X is the orbit space of a certain involution on this modified solenoid.

This example has sparked a renewed interest in a second, venerable question.

*Question.*  Does there exist a planar tree-like continuum without the fixed-point-property?

Bellamy's continuum X is not planar.  If we apply the
Fugate-Mohler technique [3] to X, we get a tree-like con-
tinuum $Y = \lim\{X_n\}$, where each factor space $X_n$ is a copy of
X.  The embedding properties of Y are not obvious.

Hence we are interested in the following question:

*Question*.  How can we find a geometric description of
a tree-like continuum without the fixed-point-property, so
we can study embeddings?

Before considering possible answers to this question,
let us formulate the following definition:

*Definition* (Mioduszewski).  Let $\{\varepsilon_n\}$ be a sequence of
positive numbers with $\varepsilon_n \to 0$.  Suppose X is the inverse
limit of the sequence $(X,\sigma)$.  Let $\langle\gamma_n\colon X_n \to X_n\rangle$ be a se-
quence of maps such that each diagram

(1)

$$
\begin{array}{ccc}
X_n & \longleftarrow & X_m \\
\downarrow\gamma_n & & \downarrow\gamma_m \\
X_k \longleftarrow X_n & \longleftarrow & X_m
\end{array}
$$

is $\varepsilon_n$-commutative.  Define $\gamma\colon X \to X$ by $\gamma(x_1,x_2,x_3,\cdots) =$
$(y_1,y_2,y_3,\cdots)$, where

$$y_k = \lim_{m\to\infty} \sigma_k^m \gamma_m(x_m).$$

It was proved by Mioduszewski [4] that the map $\gamma$ is
well defined and continuous.  The map $\gamma$ is said to be almost
a simple induced map.  (Observe if $\varepsilon_n = 0$ for each n, then
$\gamma$ is a (regular) simple induced map.)

One obstruction to obtaining an inverse limit (or,
equivalently, a tree-chain) description of a fixed-point-free

map on a tree-like continuum is the next theorem.

   *Theorem 2.  Let the continuum X be the limit of the inverse sequence $(X,\sigma)$, where each factor space $X_n$ has the fixed-point-property.  Then each almost simple induced map has a fixed point.*

   *Proof.*  Suppose the almost simple induced map $\gamma: X \to X$ has no fixed point and let $\{\gamma_n\}$ and $\{\varepsilon_n\}$, $\varepsilon_n \to 0$, be as in the definition of $\gamma$.  There exists an $\varepsilon > 0$ such that $d(x,\gamma(x)) > \varepsilon$ for each x in X.  Recall that

$$d(\langle x_i \rangle, \langle y_i \rangle) = \sum_{i=1}^{\infty} \frac{d_i(x_i,y_i)}{2^i},$$

where we assume that $\mathrm{diam}(X_n) \leq 1$ $(n = 1,2,\cdots)$.  Hence there exists an n such that:

   (1)  $\displaystyle\sum_{i=n}^{\infty} \frac{d_i(x_i,y_i)}{2^i} < \varepsilon/2$ and

   (2)  $\varepsilon_n < \varepsilon/2$.

Let $x_n$ be a fixed point of $\gamma_n: X_n \to X_n$.  Let $x = (x_1,x_2,\cdots, x_n,\cdots)$ be any point of X whose $n^{th}$ coordinate is $x_n$ and let $\gamma(x) = y = (y_1,y_2,\cdots)$.  It follows from the $\varepsilon_n$-commutativity in (1) that for $1 \leq k \leq n$ and $m > n$

$$d_k(x_k, \sigma_k^m \gamma_m(x_m)) = d_k(\sigma_k^n \gamma_n \sigma_n^m(x_m), \sigma_k^m \gamma_m(x_m)) < \varepsilon_n.$$

   Hence, by definition of $y_k$, $d_k(x_k,y_k) \leq \varepsilon_n$.

   Hence

$$d(x,\gamma(x)) = \sum_{k=1}^{\infty} \frac{d_k(x_k,y_k)}{2^k} =$$

$$\sum_{k=1}^{n} \frac{d_k(x_k,y_k)}{2^k} + \sum_{k=n+1}^{\infty} \frac{d_k(x_k,y_k)}{2^k} \leq$$

$$\varepsilon_n \cdot \sum_{k=1}^{n} \frac{1}{2^k} + \frac{\varepsilon}{2} < \varepsilon.$$

This is a contradiction.

In spite of the fact that each almost simple induced map has a fixed point, one would like to consider maps that, in some sense, preserve the index, for otherwise it is more difficult to verify that a map is fixed-point-free.

One possibility is to drop the restriction that each $X_n$ has the fixed-point-property.  Let us see how we are led to this choice in another way.

Let I be the interval $[0,6]$.  Let $g: I \to I$ be the roof-top function whose graph is pictured in Figure 1, and let $f: I \to I$ be the three-rooftops function whose graph is pictured in Figure 1.
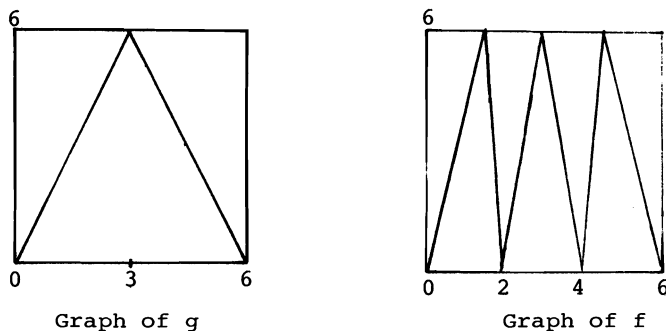


Graph of g                              Graph of f

Figure 1

Note the following facts about the maps f and g:

*Fact* 1.  The maps f and g commute.

*Fact* 2.  The set of fixed points of g is $\{0,4\}$.

*Fact* 3.  The set $f(\{0,4\}) = \{0\}$.

It is a corollary to Theorem 2 and Fact 1 that the map $g_\infty: X \to X$ defined on the arc-like continuum $X = \lim(I,f)$ by the diagram

$$(2) \quad \begin{array}{ccccccccc} I & \xleftarrow{\ f\ } & I & \xleftarrow{\ f\ } & I & \longleftarrow & \cdots & & : X \\ {\scriptstyle g}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle g_\infty} \\ I & \xleftarrow{\ f\ } & I & \xleftarrow{\ f\ } & I & \longleftarrow & \cdots & & : X \end{array}$$

has a fixed point. What is it? It follows from Facts 2 and 3 that $\langle 0,0,\cdots\rangle$ is the only fixed point of g.

A person unaware of Theorem 2 might be tempted to modify Diagram 2 by removing a small arc containing 0 and replacing it with a V. Such a person would then have a diagram of trees

(3)



where the maps σ and γ satisfy the conditions

$$\pi_2 \circ \sigma(x,y) = f(y)$$

and

$$\pi_2 \circ \gamma(x,y) = g(y)$$

There will now be a left zero and a right zero, denoted OL and OR respectively, and the map γ will interchange OL and OR. Any map induced by the γ's will interchange the points $\langle OL,OL,\cdots\rangle$ and $\langle OR,OR,\cdots\rangle$, and hence will have no fixed point.

Those of us in the know on Theorem 2 will seek the part

of the hypothesis of Theorem 2 that has been violated.

Indeed, it is the commutativity of the diagram.

Consider $\sigma(4)$. We must decide whether to define

$\sigma(4)$ = OL or $\sigma(4)$ = OR. Without loss of generality, define

$\sigma(4)$ = OL. Then $\gamma \circ \sigma(4)$ = OR, while $\sigma \circ \gamma(4) = \sigma(4)$ = OL.

There seems to be no alternative other than to have two

points called "4," namely 4L and 4R.

Introducing two "4's" will cause two problems. First,

the factor space $X_n$, pictured below, is no longer acyclic.

4L $\Diamond$ 4R

OL     OR
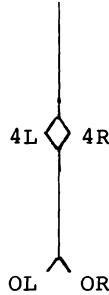
Hence, there appears the new problem of proving that the

inverse limit space is tree-like.

Secondly, a check of the commutativity of the diagram

for the six points in $\sigma^{-1}(4)$ reveals that because of the

two "4's," it will be necessary to install two "2's." The

two "2's" force us to split still other points into left and

right components, and commutativity of the square is never

attained.

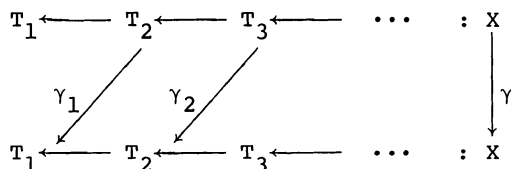In spite of these problems, the following theorem is

proved in [5].

*Theorem 3. There exists an inverse sequence of plane*

*curves $(X,\sigma)$ whose inverse limit X is tree-like, and a*

*sequence $\{\gamma_n : X_n \to X_n\}$ of mappings that defines a*

*fixed-point-free map* $\gamma: X \to X$. *The map* $\gamma$ *is "almost" a simple induced map with respect to* $\{\gamma_n\}$. *X has the properties that*

(1) *X is atriodic*

(2) *Each proper subcontinuum is an arc*

(3) *The complement of any nonempty open set can be embedded in the plane.*

Alternatively, if one is willing to forego the property of preserving indices, then the following theorem can be proved (see [6]):

*Theorem 4. There exists an inverse sequence* $(T,\sigma)$ *of trees and a sequence* $\{\gamma_n: T_{n+1} \to T_n\}$ *of maps making parallelograms commute*

$$
\begin{array}{ccccccccc}
T_1 & \leftarrow & T_2 & \leftarrow & T_3 & \leftarrow & \cdots & & : X \\
& \gamma_1 \nearrow & & \gamma_2 \nearrow & & & & & \downarrow \gamma \\
T_1 & \leftarrow & T_2 & \leftarrow & T_3 & \leftarrow & \cdots & & : X
\end{array}
$$

*such that the sequence* $\{\gamma_n\}$ *induces a fixed-point-free map* $\gamma: X \to X$ *on the tree-like continuum* $X = \lim (T,\sigma)$.

The apparent simplicity of this last theorem is somewhat deceiving, for the price for commutativity is "high-powered" bonding maps, which make it necessary to develop an apparatus similar to the proof of the previous theorem in order to prove that the map $\gamma$ has no fixed point.

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