TOPOLOGY PROCEEDINGS
Volume 4, 1979
Pages 515–532
http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

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1. Introduction

There are various definitions of shrinkable decompositions. The talk given by McAuley entitled "Shrinkable Decompositions, Criteria, and Generalizations" gave a survey of some of the definitions and results. See [3].

The primary purpose of this paper is to give the first detailed proof of a theorem of McAuley involving the *local shrinkability* of individual elements of an upper semicontinuous decomposition G to obtain the *shrinkability* of the *entire decomposition* G.

This paper is essentially Chapter III of my thesis "Decomposition Spaces and Separation Properties," SUNY-Binghamton, 1971.

2. Preliminaries

The following definition is due to McAuley [1].

A subset K of a metric space (M,d) is *locally shrinkable* iff for each open set $U \supset K$ and $\varepsilon > 0$, there exists a homeomorphism $h: M \Rightarrow M$ such that $h = \mathrm{id}$ off U and diam $hK < \varepsilon$.

As originally stated in [2], the theorem: If G is a McAuley-upper semicontinuous (Mc--rather than Whyburn) decomposition of a complete metric space (M,d) such that H_G (the collection of all nondegenerate elements of G) is countable, H_G is a G_δ collection (H_G^*, the union of the elements of H_G, is a G_δ set), *and each element* $g \in H_G$ is a locally shrinkable continuum which lies in an open set with compact closure, then M is homeomorphic to the decomposition space I = M/G, is false. See Example C, section 2.3 of [4] where I is not First Axiom. The theorem fails when there exists a point which is a degenerate limit of elements having diameters bounded away from zero. This cannot happen if p is closed, but, as the example shows, it is not a violation of Mc. The hypotheses of the theorem and the condition that there be no such "bad" points guarantee the map p is closed. *The theorem is true if McAuley--usc is replaced by Whyburn--usc (*p *closed)* and we will obtain this form from a more general proposition which restates another of McAuley's theorems.

If G is a decomposition of X, we call a subset U of X p-*open* if it is an open inverse set (for p), i.e., U is open and $p^{-1}p(U) = U$. Some authors say that U is a *saturated open set* in X.

Definition. If G is a decomposition of a metric space M, H is tightly shrinkable in M (tsh) iff given any p-open cover U of H*, $\varepsilon > 0$, and h: $M \approx M$, there exists a p-open (refinement of U) V covering H* and a homeomorphism f: $M \approx M$ such that 1) f = h off V*, 2) for each $g \in H$, diam $f(g) < \varepsilon$ and 3) for each $v \in V$ there exists $u \in U$ such that $h(v) \cup f(v) \subset h(u)$.

H is *weakly tsh* if the above holds for the special case of $h = id_M$.

3. A Convergence Theorem

We will make use of the following theorem of McAuley, slightly revised.

Convergence Theorem (McAuley). *If* M *is a metric space,* $\sum \varepsilon_n < \infty$ $(\varepsilon_n > 0)$, *for each* n, $f_n\colon M \approx M$, $f_0 = \mathrm{id}$, *for each* $n \geq 1$, V_n *is a collection of open sets with compact closure and* $V_n^* \supset V_{n+1}^*$, *for each* $n \geq 0$, $f_{n+1} = f_n$ *off* V_{n+1}^*, $D \in V_{n+1} \Rightarrow \operatorname{diam} f_n D < \varepsilon_n$, *and* $x \in V_{n+1}^* \Rightarrow$ *there exists* $D \in V_{n+1}$ *such that* $f_n D \supset f_n x \cup f_{n+1} x$, *then* $\{f_n\}$ *are uniformly Cauchy and if* $\{f_n(x)\}_{n=1}^{\infty}$ *converges for each* $x \in \Delta = \cap V_n^*$ *then* $f_n \to f$ [unif], $f\colon M \to M$ *is continuous and onto, and* f *is* 1 - 1 *off* Δ. *Furthermore, if* M *is locally compact on* $\overline{V_1^*}$, *then* f *is closed.*

Proof. First, we show that $\{f_n\}$ are uniformly Cauchy. Let $\varepsilon > 0$. For some N, $\sum_{n=N}^{\infty} \varepsilon_n < \varepsilon$. Let $x \in M$. For each n, if $x \notin V_{n+1}^*$ then $f_{n+1}x = f_n x$. If $x \in V_{n+1}^*$ then there exists $D \in V_{n+1}$ such that $f_n D \supset f_n x \cup f_{n+1} x$, but $\operatorname{diam} f_n D < \varepsilon_n$. So, in either case, $d(f_n x, f_{n+1} x) < \varepsilon_n$. So for $m > N$,

$$d\{f_N x, f_m x\} < \sum_{i=N}^{m} \varepsilon_i < \sum_{i=N}^{\infty} \varepsilon_i < \varepsilon.$$

$\{f_n x\}$ converges for $x \notin \Delta = \cap V_n^*$, for if $x \notin V_{J+1}^*$ then $f_n x = f_J x$ for $n > J$, i.e., $\{f_n x\}$ is ultimately constant. So if $\{f_n x\}$ converges for $x \in \Delta$ then we have pointwise convergence everywhere. And since $\{f_n\}$ are uniformly Cauchy, $f_n \to f = \lim f_n$ [unif], and f is continuous.

To show f is onto, let $p \in M$. Let $z_n = f_n^{-1} p$. It suffices to show $\{z_n\}$ has a convergent subsequence, since if $z_{n_i} \to x$ then continuity gives $f z_{n_i} \to fx$ while $d\{f_{n_i} z_{n_i}, f z_{n_i}\} < \varepsilon$ for large i by uniform convergence. So $f z_{n_i} \to p$ and hence $p = fx$. Now, if $p \notin V_1^*$ then for each n, $f_n p = p$.

Thus $\cup f_n^{-1}p = \{p\}$. If $p \in V_1^*$, $p \in D \in V_1$ with $\overline{D}$ compact. Choose $\delta > 0$ such that $N_\delta(p) \subset D$. By the uniform convergence there exists N such that $n > N \Rightarrow f_N z \in N_\delta f_n z$ for all $z \in M$. So $f_N z_n \in N_\delta f_n z_n = N_\delta(p) \subset D$. Hence $\{f_N z_n\}_{n=N}^{\infty} \subset D$ and $\{z_n\}_{n=N}^{\infty} \subset f_N^{-1}D$. Since f_N is a homeomorphism, $\overline{f_N^{-1}D}$ is compact and so $\{z_n\}$ has a convergent subsequence.

Now we suppose that M is locally compact at each point of $\overline{V_1^*}$. To show f is closed, let D be a closed subset of M and $y_n \to y$ with $y_n \in fD$. We must show $y \in fD$. There exists $x_n \in D$ with $y_n = fx_n$. If $\{x_n\}$ has a convergent subsequence, we are done, since if $x_{n_i} \to x$ then $x \in D$ and $fx_{n_i} = y_{n_i} \to fx$ by continuity. Hence $fx = y$. Furthermore, if M is locally compact at y, we can choose $\varepsilon > 0$ so that $\overline{N_\varepsilon y}$ is compact. By uniform convergence there exists I so that for every $x \in M$, $f_I x \in N_{\varepsilon/2} fx$. In particular, for each n, $f_I x_n \in N_{\varepsilon/2} fx_n$. But there exists N such that for $n > N$, $fx_n \in N_{\varepsilon/2} y$. So $f_I x_n \in N_{\varepsilon/2} fx_n \subset N_\varepsilon y$, which has compact closure. So $\{f_I x_n\}$ has a convergent subsequence and thus $\{x_n\}$ does also, as f_I is a homeomorphism.

We may suppose then that $y \notin \overline{V_1^*}$. Now $f_j(V_1^*) = V_1^*$ for each j since f_j is a homeomorphism which is the identity off V_1^*. For some $\varepsilon > 0$, $N_\varepsilon y$ misses $\overline{V_1^*}$ and for large n, $y_n \in N_{\varepsilon/2} y$. For large i, $f_i x_n \in N_{\varepsilon/2} y_n \subset N_\varepsilon y$ so $f_i x_n \notin V_1^*$ and thus $x_n \notin V_1^*$. So $fx_n = x_n$ and since $fx_n \to y$, we have $x_n \to y$.

4. A Theorem for Tightly Shrinkable Decompositions

The following theorem is proved.

Theorem T. *If* M *is a metric space,* G *a decomposition of* M *such that* p *is closed and point-compact,* H *is tightly shrinkable in* M, *and* M *is locally compact at* H*, *then* $I \approx M$.

Proof. For each $g \in H$, let $w_1(g)$ be a p-open set containing g such that $\overline{w_1(g)}$ is compact $\subset N_{1/2}(g)$. Let $W_1 = \{w_1(g): g \in H\}$. Let U_1 be a star refinement of W_1 by p-open sets. (I is metrizable, hence paracompact, by Stone's Theorem [16].) By tsh, there exists $f_1: M \approx M$ and V_1 a p-open refinement of U_1 covering H^* such that:

$$f_1 = \text{id off } V_1^*$$
$$g \in H \Rightarrow \text{diam } f_1 g < \frac{1}{2}$$
$$v \in V_1 \Rightarrow \text{there exists } u \in U_1 \text{ such that } v \cup f_1 v \subset u.$$

For each $g \in H$, choose $v_1(g) \in V_1$ containing g and let $w_2(g)$ be p-open containing g so that $\overline{w_2(g)}$ compact $\subset N_{1/2^2}(g) \cap v_1(g) \cap f_1^{-1}(N_{1/2^2} f_1 g)$. Let $W_2 = \{w_2(g): g \in H\}$. Let U_2 be a star refinement of W_2 by p-open sets. By tsh there exists $f_2: M \approx M$ and V_2 a p-open refinement of U_2 covering H^*, satisfying

$$f_2 = f_1 \text{ off } V_2^*$$
$$g \in H \Rightarrow \text{diam } f_2 g < \frac{1}{2^2}$$
$$v \in V_2 \Rightarrow \text{there exists } u \in U_2 \text{ such that } f_1 v \cup f_2 v \subset f_1 u.$$

Inductively, given $f_{n-1}: M \approx M$, V_{n-1} a p-open refinement of U_{n-1} covering H^* with

$$f_{n-1} = f_{n-2} \text{ off } V_{n-1}^*$$
$$g \in H \Rightarrow \text{diam } f_{n-1} g < \frac{1}{2^{n-1}}$$

$$v \in V_{n-1} \Rightarrow \text{ there exists } u \in U_{n-1} \text{ with } f_{n-2}v \cup f_{n-1}v \subset f_{n-2}u,$$

for each $g \in H$, choose $v_{n-1}(g) \in V_{n-1}$ containing g and let $w_n(g)$ be p-open containing g so that $\overline{w_n(g)}$ is compact $\subset N_{1/2^n}(g) \cap v_{n-1}(g) \cap f_{n-1}^{-1}(N_{1/2^n}f_{n-1}g)$. Let $W_n = \{w_n(g) : g \in H\}$ and U_n a star-refinement of W_n by p-open sets. By tsh there exists $f_n\colon M \approx M$ and V_n a p-open refinement of U_n covering H^*, satisfying:

$$f_n = f_{n-1} \text{ off } V_n^*$$

$$g \in H \Rightarrow \text{ diam } f_n g < \frac{1}{2^n}$$

$$v \in V_n \Rightarrow \text{ there exists } u \in U_n \text{ such that } f_{n-1}v \cup f_n v \subset f_{n-1}u.$$

It is clear that this construction gives for each n, $g \in G \Rightarrow f_{n-1}V_n^*(g) \cup f_nV_n^*(g) \subset f_{n-1}U_n^*(g) \subset f_{n-1}w_n(g') \subset f_{n-1}v_{n-1}(g') \cap N_{1/2^n}f_{n-1}(g')$, this last set having diameter $< \frac{1}{2^{n-2}}$, for some $g' \in H$.

Also, we have for each $g \in G$, for each $k \geq 1$ and $n \geq k$, $f_k(V_n^*(g)) \cup f_n(V_n^*(g)) \subset f_k(V_k^*(g))$. To see this, let $k \geq 1$ and induct on n: For $n = k$ the statement is trivial. Suppose it holds for some $n \geq k$. Now, $f_n(V_{n+1}^*(g)) \cup f_{n+1}(V_{n+1}^*(g)) \subset f_n(U_{n+1}^*(g))$ by construction and this is a subset of $f_n(w_{n+1}(g'))$, for some $g' \in H$, which in turn lies in $f_n(v_n(g'))$. We assume $g \in p(V_{n+1}^*)$ since otherwise the statement is trivial. So $g \subset V_{n+1}^*(g) \subset v_n(g')$. Hence $v_n(g') \in V_n(g)$ and $v_n(g') \subset V_n^*(g)$. So $f_n(v_n(g')) \subset f_n(V_n^*(g)) \subset f_k(V_k^*(g))$ by the inductive hypothesis. Also, since $V_{n+1}^*(g) \subset V_n^*(g)$, $f_k(V_{n+1}^*(g)) \subset f_k(V_k^*(g))$ also by the inductive hypothesis and this establishes the corresponding statement for the case of n+1.

We may restate the last result: for each $g \in G$, for each $k \geq 1$, $\cup_{n=k}^{\infty} f_n(V_n^*(g)) \subset f_k(V_k^*(g))$. In particular, for each $g \in \cap p(V_n^*)$ (where $g \subset V_n^*(g)$ for each n), $\cup_{n=k}^{\infty} f_n(g) \subset \cup_{n=k}^{\infty} f_n V_n^*(g) \subset f_k V_k^*(g)$.

The result is sequences $f_n: M \approx M$ and $\{U_n\}$ such that each U_n is a collection of p-open sets with compact closure, $f_{n+1} = f_n$ off U_{n+1}^* (actually off $V_{n+1}^* \subset U_{n+1}^*$). Furthermore, $x \in U_{n+1}^* \Rightarrow$ there exists $u \in U_{n+1}$ with $f_n u \supset f_n x \cup f_{n+1} x$, since if $x \notin V_{n+1}^*$, $f_{n+1} x = f_n x$, which is in the image under f_n of whichever element of U_{n+1} contains x. And if $x \in V_{n+1}^*$, $x \in$ some $v \in V_{n+1}$ but $f_n v \cup f_{n+1} v \subset f_n u$ for some $u \in U_{n+1}$.

For each $u \in U_{n+1}$, diam $f_n u < \frac{1}{2^{n-1}}$. And since $\sum \frac{1}{2^{n-1}} < \infty$, we have verified all of the conditions we need of the Convergence Theorem except convergence itself at points of $\cap U_n^*$. But suppose $x \in \cap U_n^* = \cap V_n^*$. $p(x) = g \subset \cap V_n^*$ so $g \subset V_n^*(g)$ for each n, while $\cup_{n=1}^{\infty} f_n(V_n^*(g)) \subset f_1(V_1^*(g)) \subset U_1^*(g)$, which has compact closure. So $\{f_n x\}_{n=1}^{\infty}$ lies in a compact set. Thus it has a convergent subsequence. But the sequence $\{f_n x\}$ is Cauchy and hence converges.

So by the Convergence Theorem, $f_n \to f: M \to M$ [unif], f is continuous, onto and f is 1-1 off $\Delta = \cap U_n^*$.

We now establish that for each $g \in H$, $f(g)$ is a point. For each k and $n \geq k$, $f_n(g) \subset f_k(V_k^*(g))$. So for each k, $f(g) \subset \overline{f_k(V_k^*(g))}$. Thus $f(g) \subset \cap_{k=1}^{\infty} \overline{f_k(V_k^*(g))}$, while the sets in this intersection have diameters tending to zero as k increases, so $f(g) = \cap_{k=1}^{\infty} \overline{f_k V_k^*(g)}$ = a point.

We claim also, $g \neq g' \in G \Rightarrow$ for some N, $\overline{V_N^*(g)} \cap \overline{V_N^*(g')} = \phi$. To prove this, note that since g and g' are compact, there exists $\varepsilon_1 > 0$ such that $\overline{N_{2\varepsilon_1}}(g) \cap \overline{N_{2\varepsilon_1}}(g') = \phi$. Let U and V be p-open with $g \subset U \subset N_{\varepsilon_1} g$ and $g' \subset V \subset N_{\varepsilon_1} g'$. So $N_{\varepsilon_1} U \subset N_{2\varepsilon_1} g$ and $N_{\varepsilon_1} V \subset N_{2\varepsilon_1} g'$ and $\overline{N_{\varepsilon_1} U} \cap \overline{N_{\varepsilon_1} V} = \phi$. Choose $\varepsilon > 0$ so that $\varepsilon < \varepsilon_1$ and $N_\varepsilon g \subset U$, $N_\varepsilon g' \subset V$. Choose N so $\frac{1}{2^N} < \varepsilon$. Then $\overline{W_N^*(g)} \cap \overline{W_N^*}(g') = \phi$. For if $w \in W_N(g)$, $g \subset w = w_N(g_0)$, some $g_0 \in H$, $\subset N_{1/2^N}(g_0) \subset N_\varepsilon(g_0)$. So g_0 meets $N_\varepsilon g$ and thus $g_0 \subset U$. So $w \subset N_\varepsilon g_0 \subset N_\varepsilon U \subset N_{\varepsilon_1} U$. Thus $W_N^*(g) \subset N_{\varepsilon_1} U$. Similarly, if $w' \in W_N(g')$, $w' \subset N_{\varepsilon_1} V$. So $W_N^*(g') \subset N_{\varepsilon_1} V$. So $\overline{W_N^*}(g)$ and $\overline{W_N^*}(g')$ are disjoint and as V_N refines W_N, $\overline{V_N^*}(g) \cap \overline{V_N^*}(g') = \phi$.

We can now show that fx = fy iff px = py. If px = py = g then since f(g) is a single point, fx = fy. Now suppose fx = fy and px = g $\neq$ py = g'. Since f is 1-1 off $\cap V_n^*$ we may assume at least one of g and g' is in $\cap pV_n^*$. In case both g and g' are in $\cap pV_n^*$, choose N so that $\overline{V_N^*(g)} \cap \overline{V_N^*(g')} = \phi$. Then $f_N\, \overline{V_N^*}(g) \cap f_N\, \overline{V_N^*}(g') = \phi$, while the first of these sets contains f(g) and the second contains f(g'), contradicting f(g) = f(g'). Now assume that $g \notin \cap pV_n^*$ while $g' \in \cap pV_n^*$. For some M, $g \notin pV_M^*$ and $f(g) = f_M(g) = f_k(g)$ for $k \geq M$. There exists $N > M$ such that $g \notin \overline{V_N^*}(g')$ so $f_N(g) \notin f_N\, \overline{V_N^*}(g')$ but $f_N(g) = f(g)$ while $f(g') \in f_N\, \overline{V_N^*}(g')$.

So fp^{-1} is a homeomorphism of I onto M iff f is quasi-compact.

We will show f is closed but first we will prove: if

$y \notin U_1^*$ (so $fy = f_j y = y$ for each j) and if $fz_n \to y$ with each $z_n \in \cap V_n^*$ then $z_n \to y$. Let $p(z_n) = g_n$. So $f(z_n) = f(g_n)$. Since each $g_n \in \cap pV_n^*$, $f(g_n) = \cap_{k=1}^{\infty} f_k(\overline{V_k^*}(g_n))$. So for each k,n $f(g_n) \in f_k \overline{V_k^*}(g_n)$. But $f(g_n) \to y$. So $y \ell p \cup_{n=1}^{\infty} f_k \overline{V_k^*}(g_n)$ and since f_k is a homeomorphism, $f_k^{-1} y \ell p \cup_{n=1}^{\infty} V_k^*(g_n)$. i.e., for each k, $y \ell p \cup_{n=1}^{\infty} \overline{V_k^*}(g_n)$. Now $y \ell p \cup g_n$. For suppose not. Then there exists $\varepsilon > 0$ such that $N_\varepsilon y$ misses $\cup g_n$. There exists $\varepsilon_1 > 0$ such that if $g \in G$ meets $N_{\varepsilon_1} y$ then $g \subset N_{\varepsilon/2} y$. Choose K so that $\frac{1}{2^K} < \frac{\varepsilon_1}{2}$. Since $y \ell p \cup_{n=1}^{\infty} \overline{V_K^*}(g_n)$ there is a point $x \in \cup_{n=1}^{\infty} \overline{V_K^*}(g_n) \cap N_{\varepsilon_1/2} y$, say $x \in \overline{V_K^*}(g_N) \cap N_{\varepsilon_1/2} y$. But by construction, $\overline{V_K^*}(g_N) \subset N_{1/2^K}(g_N') \subset N_{\varepsilon_1/2}(g_N')$, some $g_N' \in H$. So there exists $z \in g_N'$ such that $d(x,z) < \frac{\varepsilon_1}{2}$, while $d(x,y) < \frac{\varepsilon_1}{2}$, so $d(x,z) < \varepsilon_1$. Thus g_N' meets $N_{\varepsilon_1} y$ and $g_N' \subset N_{\varepsilon/2} y$. Meanwhile $g_N \subset N_{1/2^K}(g_N') \subset N_{\varepsilon_1}(g_N') \subset N_{\varepsilon/2}(g_N') \subset N_\varepsilon y$, which contradicts the choice of $N_\varepsilon y$.

So $\{y\}$ ℓp $\{g_n\}$ in I by continuity of p. Hence $z_n \to y$ since p is closed and $g_n = p(z_n)$ and the argument applies as well to any subsequence z_{n_i}.

To show f is closed, let D be closed $\subset$ M and suppose $y_n \to y$ with $y_n \in fD$. Let $x_n \in D$ such that $y_n = f(x_n)$. As in the proof of the last part of the convergence theorem, it suffices to have M locally compact at y or that $\{x_n\}$ has a convergent subsequence. So we may assume $y \notin U_1^*$ since U_1 is a collection of open sets which have compact closure. Then for each j, $f_j y = y = fy$. If for some J, $\{x_n\}$ is frequently not in U_J^*, then for a subsequence $\{x_{n_i}\} \subset M \setminus U_J^*$,

$f(x_{n_i}) = f_J x_{n_i}$ for each i. So $f_J(x_{n_i}) \to y$ hence $x_{n_i} \to f_J^{-1} y = y$. So we may suppose $\{x_n\}$ is ultimately in each U_J^*. There is a subsequence $\{x_{n_i}\}$ with $x_{n_i} \in U_i^*$. Since it is only subsequences we are interested in, let us assume $x_n \in U_{n+1}^*$. Now, since U_{n+1} refines W_{n+1}, there exists $g_n \in H$ such that $x_n \in w_{n+1}(g_n) \subset N_{1/2^{n+1}}(g_n) \cap v_n(g_n)$. So $g_n \in H$, $d(x_n, g_n) < \frac{1}{2^{n+1}}$ and $x_n \in V_n^*(g_n)$. Thus for each j, $f_j x_n \in f_j V_n^*(g_n)$.

Let $\varepsilon > 0$. Choose N so that $n > N \Rightarrow fx_n \in N_{\varepsilon/4} y$, since $fx_n \to y$. By uniform convergence there exists J such that $j > J \Rightarrow f_j x \in N_{\varepsilon/4} fx$ for $x \in M$. So $n > M$, $j > J \Rightarrow f_j x_n \in N_{\varepsilon/2} y$. But for each $g \in H$ and each k, diam $f_k V_k^*(g) < \frac{1}{2^{k-2}}$. So there exists K such that $k > K \Rightarrow$ diam $f_k V_k^*(g) < \frac{\varepsilon}{4}$ and since $V_\ell^*(g) \subset V_k^*(g)$ for $\ell \geq k$, for each $\ell \geq k$, diam $f_k V_\ell^*(g) < \frac{\varepsilon}{4}$. Choose $I > J,K$; then for $n > I,N$, $f_I x_n \in N_{\varepsilon/2} y$ and diam $f_I V_n^* g < \frac{\varepsilon}{4}$ for $g \in H$. But $f_I x_n \in f_I V_n^*(g_n)$. So $f_I V_n^*(g_n) \subset N_{3\varepsilon/4} y$, and since $I > J$, $f(g_n) \in f(V_n^*(g_n)) \subset N_{\varepsilon/4} f_I V_n^*(g_n) \subset N_\varepsilon y$. We have shown: given $\varepsilon > 0$ there exists M such that $n > M \Rightarrow f(g_n) \in N_\varepsilon y$. So $f(g_n) \to y$. But $d(x_n, g_n) < \frac{1}{2^{n+1}}$. Choose $z_n \in g_n$ such that $d(x_n, z_n) < \frac{1}{2^{n+1}}$. Now $f(z_n) \to y$ and $z_n \in H^*$. So $z_n \to y$, as we have already proved. But $d(x_n, z_n) \to 0$ so $x_n \to y$ also. This completes the proof of Theorem T.

5. A Proof of McAuley's Theorem for p Closed

We will use Theorem T to establish McAuley's Theorem in case p is closed. Some further observations will be useful.

First, if G is a decomposition of a metric space M, then H_G is tsh iff for each homeomorphism $h: M \approx M$, $H_{h(G)}$ is weakly tsh. This is an immediate consequence of the definitions and the fact that under a homeomorphism $h: M \approx M$, $h(H_G) = H_{h(G)}$ and if $p': M \to M/h(G)$ is the quotient map and u a p-open set then $h(u)$ is p'-open. This enables us to carry maps and coverings back and forth via the given homeomorphism. The details are straightforward and omitted here.

Consequently, if we find a set of purely topological conditions on a decomposition G (preserved under homeomorphisms on M) which yield H_G is weakly tsh, then H_G is tsh also.

We also note that local shrinkability of continua is topological, i.e., if M and M' are metric, h a homeomorphism of M onto M' and C a locally shrinkable continuum in M, then $h(C)$ is a locally shrinkable continuum in M'.

Proof. Trivially, hC is a continuum. Since C is locally shrinkable in M, for each positive integer k there exists $f_k: M \approx M$ such that $f_k = \text{id}$ off $N_{1/k}C$ and $\text{diam } f_kC < \frac{1}{k}$. $C_k = f_kC \subset N_{1/k}C$. Each open set containing C contains C_k ultimately as C is compact. There exists $x \in C$ such that each neighborhood of x meets C_k for infinitely many k, again by compactness of C. Since M is metric a subsequence $C_{k_i} \to x$, i.e., each neighborhood of x meets C_{k_i} ultimately. And since $\text{diam } C_{k_i} \to 0$ each neighborhood of x contains C_{k_i} ultimately. Now, since h is a homeomorphism $hC_{k_i} \to hx \in hC$. Also $\text{diam } hC_{k_i} \to 0$ since if V is any neighborhood of $h(x)$, $h^{-1}V$

is a neighborhood of x and contains C_{k_i} ultimately. Then V ultimately contains hC_{k_i}. Since we may choose neighborhoods V of h(x) with arbitrarily small diameter, diam hC_{k_i} must tend to zero. Now let U open $\supset hC$, $\varepsilon > 0$. Then $h^{-1}U$ is open $\supset C$. Choose I so that diam $hC_{k_I} < \varepsilon$ and $N_{1/k_I}C \subset h^{-1}U$. Then $f_{k_I}: M \approx M$, $f_{k_I} = \text{id}$ off $h^{-1}U$, $f_{k_I}C = C_{k_I}$. Let $h' = hf_{k_I}h^{-1}: M' \approx M'$ so $h' = \text{id}$ off U and $h'(hC) = hf_{k_I}C = hC_{k_I}$ has diameter $< \varepsilon$, which means hC is locally shrinkable.

We need the following theorem of McAuley:

Theorem H (McAuley). *If* M *is a metric space,* $\{f_i\}: M \approx M$, $\{U_i\}$ *a sequence of open subsets of* M *such that* $U_i \supset \overline{U}_{i+1}$, $\cap U_i = \phi$, $f_i = f_{i-1}$ *off* U_i, $f_0 = \text{id}$, *and for each* $p \in M$, $\cup_{i=1}^{\infty} f_i^{-1}p$ *has compact closure then* $\{f_i\} \to f: M \approx M$.

Remark. Excluding the last hypothesis of Theorem H yields $f = \lim f_i$ continuous, 1-1 and open. This last condition provides that f is onto.

Theorem H' (McAuley, revised). *If* G *is a decomposition of a metric space* M *satisfying*

1) p *is closed and point-compact,*

2) *each element of* H *is locally shrinkable,*

3) H *is countable and* G_δ,

4) M *is locally compact at* H*,

then H *is weakly tsh in* M.

Proof. In this proof the notation $\langle 0,D \rangle$ is used to replace the sequence of symbols: $0p\text{-open} \subset \overline{0} \subset Dp\text{-open} \subset \overline{D}$

compact. By hypothesis, $H = \{C_j\}_{j=1}^{\infty}$, $H^* = \cap_{i=1}^{\infty} G_i$, G_i open $\supset G_{i+1}$. Let $\mathcal{A}$ be a p-open cover of H^*, $\varepsilon > 0$. For each j, choose $A_j \in \mathcal{A}$ with $C_j \subset A_j$. Let $h_0 = \text{id}$.

Let $H_1 = \{C \in H\colon \text{diam } C \geq \varepsilon\}$. By usc, H_1^* is closed. If $H_1 \neq \phi$, let k_1 be least such that $C_{k_1} \in H_1$. So $C_j \notin H_1$ for $j < k_1$. $H_1^* \subset W_1$ open such that W_1 misses C_j for $j < k_1$. $H_1^* \subset U_1$ open such that $\overline{U}_1 \subset W_1 \cap G_1$. Let $C_{k_1} \subset \langle 0_1, D_1 \rangle \subset U_1 \cap A_{k_1}$ and let $h_1\colon M \approx M$ such that $h_1 = \text{id}$ off 0_1 and diam $h_1 C_{k_1} < \varepsilon$.

Let $H_2 = \{C \in H\colon \text{diam } h_1 C \geq \varepsilon\}$. H_2^* is closed $\subset U_1$. If $H_2 \neq \phi$, let k_2 be least such that $C_{k_2} \in H_2$. Then $k_2 > k_1$. $H_2^* \subset W_2$ open such that W_2 misses C_j for $j < k_2$. $H_2^* \subset U_2$ open such that $\overline{U}_2 \subset U_1 \cap W_2 \cap G_2$. Let $C_{k_2} \subset \langle 0_2, D_2 \rangle \subset U_2 \cap A_{k_2}$ and such that if $C_{k_2} \cap \overline{0}_1 = \phi$, we select D_2 so that $\overline{D}_2 \cap \overline{0}_1 = \phi$, while if $C_{k_2} \cap \overline{0}_1 \neq \phi$, then choose D_2 so that $\overline{D}_2 \subset D_1$. Let $h_2\colon M \approx M$ such that $h_2 = h_1$ off 0_2 and h_2 shrinks C_{k_2} to diameter $< \varepsilon$, (hence C_j for $j \leq k_2$).

Inductively, given $h_\ell\colon M \approx M$ for $0 \leq \ell \leq i$ such that for $1 \leq \ell \leq i$ $h_\ell = h_{\ell-1}$ off 0_ℓ, W_ℓ is open missing C_j for $j < k_\ell$, $C_{k_\ell} \subset \langle 0_\ell, D_\ell \rangle \subset U_\ell \cap A_{k_\ell} \subset U_\ell$ open $\subset \overline{U}_\ell \subset U_{\ell-1} \cap G_\ell \cap W_\ell$ and $\overline{D}_\ell \cap \overline{0}_j = \phi$ or $\overline{D}_\ell \subset D_j$ (and $\overline{0}_\ell \cap \overline{0}_j \neq \phi$) for all $j < \ell$, and, h_ℓ shrinks C_j for $j \leq k_\ell$.

Let $H_{i+1} = \{C \in H\colon \text{diam } h_i C \geq \varepsilon\}$. Then H_{i+1}^* is closed $\subset U_i$. If $H_{i+1} \neq \phi$ let k_{i+1} be least such that $C_{k_{i+1}} \in H_i$. Then $k_{i+1} > k_i$ and $C_j \notin H_i$ for $j < k_{i+1}$. $H_{i+1}^* \subset W_{i+1}$ open such that W_{i+1} misses C_j for $j < k_{i+1}$. $H_{i+1}^* \subset U_{i+1}$ open

$\subset \overline{U}_{i+1} \subset U_i \cap W_{i+1} \cap G_{i+1}$. Let $C_{k_{i+1}} \subset \langle 0_{i+1}, D_{i+1} \rangle \subset U_{i+1} \cap A_{k_{i+1}}$ and such that for each ℓ, $1 \leq \ell \leq i$, if $C_{k_{i+1}} \cap \overline{0}_\ell \neq \phi$, choose $\overline{D}_{i+1} \subset D_\ell$ and if $C_{k_{i+1}} \cap \overline{0}_\ell = \phi$, choose D_{i+1} so that $\overline{D}_{i+1} \cap \overline{0}_\ell = \phi$ also. (So we have $\overline{D}_j \cap \overline{0}_\ell = \phi$ or $\overline{D}_j \subset D_\ell$ and $\overline{0}_j \cap \overline{0}_\ell \neq \phi$ for each $j \leq i + 1$ and $\ell < j$.) Let $h_{i+1}: M \approx M$ such that $h_{i+1} = h_i$ off 0_{i+1} and h_{i+1} shrinks $C_{k_{i+1}}$ to diameter $< \varepsilon$ (hence C_j for $j \leq k_{i+1}$).

If $H_i = \phi$ for some i, let $h = h_{i-1}$. This gives a homeomorphism $h: M \approx M$, without appeal to Theorem H, which shrinks each element of G to diameter $< \varepsilon$. And we can construct a p-open refinement V of $\mathcal{A}$ as required for weakly tsh in the same way as for the case that $\{H_i\}$ is infinite which follows.

If $H_i \neq \phi$ for each i, then we have a sequence of homeomorphisms h_i of M onto M and open sets U_i such that $U_i \supset \overline{U}_{i+1}$, $h_i = h_{i-1}$ off U_i (actually off 0_i), $\cap U_i = \phi$ (since $\cap U_i \subset \cap G_i = H^* = \cup C_j$, but each j, U_{j+1} misses C_j, so $H^* \cap (\cap U_i) = \phi$). So we have verified conditions of Theorem H which give $h_i \to h: M \to M$, with h 1-1, continuous and open.

We must show h is onto. Prior to this, we list some properties of the construction:

Lemma 1. *For each* i, $h_i A = h_{i-1} A$ *for any set* A *containing* 0_i. *In particular,* $h_i \overline{0}_i = h_{i-1} \overline{0}_i \subset h_{i-1} D_i = h_i D_i$.

Lemma 2.1. *For each* $i < j$ *if* $x \notin 0_\ell$ *for* $i < \ell \leq j$ *then* $h_j x = h_i x$.

Lemma 2. *For each* i *there exists* $L(i) \leq i$ *such that*

$\cup_{\ell=0}^{i} h_\ell(D_i) \subset D_{L(i)}$.

Proof. The statement holds for $i = 1$ since $h_1 D_1 = h_0 D_1 = D_1$. Let $L(1) = 1$. Assume for each $j < i$ that there exists $L(j) \leq j$ such that $\cup_{\ell=0}^{j} h_\ell D_j \subset D_{L(j)}$. If D_i misses $\overline{0}_j$ for each $j < i$ then $h_\ell D_i = D_i$ for $\ell < i$ by Lemma 2.1. But $h_i D_i = h_{i-1} D_i$ by Lemma 1 so $h_i D_i = D_i$ also. And $\cup_{\ell=0}^{i} h_\ell D_i = D_i$. Let $L(i) = i$. If D_i meets some $\overline{0}_j$ for $j < i$, let J be the largest such j. Then by construction $D_i \subset D_J$ and by our inductive assumption, there exists $L(J) \leq J$ such that $\cup_{\ell=0}^{J} h_\ell D_J \subset D_{L(J)}$. But $\cup_{\ell=0}^{J} h_\ell D_i \subset \cup_{\ell=0}^{J} h_\ell D_J$ and since D_i misses $\overline{0}_j$ for $J < j < i$, $h_\ell D_i = h_J D_i$ pointwise for $J < \ell \leq i-1$ by Lemma 2.1. So we also have $\cup_{\ell=0}^{i-1} h_\ell D_i \subset D_{L(J)}$. And by Lemma 1, $h_{i-1} D_i = h_i D_i$. Hence $\cup_{\ell=0}^{i} h_\ell D_i \subset D_{L(J)}$. So we let $L(i) = L(J) \leq J < i$.

Now it is easy to show h is onto. Let p be any point of M. If $p \notin \cup 0_i$ then $h_i p = p$ for each i and $hp = p$. So suppose $p \in \cup 0_i$ and let I be least such that $p \in 0_I$. We will show that $\{h_i^{-1}p\}_{i \geq I} \subset \cup_{i=1}^{I} D_i$. Otherwise, there exists a least J such that $h_J^{-1}p \notin \cup_{i=1}^{I} D_i$. Let $z = h_J^{-1}p$. If $z \notin 0_J$ then $p = h_J z = h_{J-1} z$ so $z = h_{J-1}^{-1}p$ contrary to the choice of J. So $z \in 0_J$. But $z \cup p = h_0 z \cup h_J z \subset D_{L(J)}$ for some $L(J)$ by Lemma 2. So $D_{L(J)}$ meets $\overline{0}_I$ in p. If $L(J) > I$ then by construction $D_{L(J)} \subset D_I$. If $L(J) \leq I$, we still have $z \in \cup_{i=1}^{I} D_i$ which is a contradiction. So $\{h_i^{-1}p\}_{i \geq I} \subset \cup_{i=1}^{I} D_i$, which is a finite union of sets having compact closures. So we have confirmed the last hypothesis of Theorem H and we have $h_i \to h\colon M \approx M$.

Lemma 3.1. *For each* i *and* j *with* $i < j$ *if* $\overline{0}_i$ *and* $\overline{0}_j$

are disjoint then no $\overline{0}_\ell$ *can meet them both for* $\ell \geq j$.

Proof. If $\overline{0}_\ell$ meets both $\overline{0}_i$ and $\overline{0}_j$ with $\ell \geq j > i$ then $\overline{0}_\ell \subset D_\ell$ is chosen so that $D_\ell \subset D_i \cap D_j$. But D_j was chosen to miss $\overline{0}_i$.

Lemma 3.2. *If* A *is any set which contains each* $\overline{0}_i$ *for* $I \leq i \leq J$ *which* A *intersects, then* $h_I A = h_J A$.

Proof. Suppose not. Let L be least such that $h_L A \neq h_I A$ with $I < L \leq J$. Then $h_{L-1}A = h_I A$. But if $h_L A \neq h_{L-1}A$ then A meets $\overline{0}_L$ so $\overline{0}_L \subset A$. Hence $h_L A = h_{L-1}A$ by Lemma 1.

Lemma 3. *For each* I *and* $J \geq I$, $h_J\overline{0}_I \subset h_I D_I$.

Proof. For J = I the statement is trivial. Given J > I, let $Q = \{\overline{0}_i: I \leq i \leq J\}$. Let $A = \{0 \in Q$: there exists a (finite) sequence of elements of Q, consecutively intersecting and of increasing index from $\overline{0}_I$ to $0\}$. Clearly, $\overline{0}_I \in A$, and $A^* \subset D_I$, for otherwise if there exists an element $\overline{0}_i \in A$ with $\overline{0}_i \not\subset D_I$ then $D_i \not\subset D_I$. Let K be least such that $\overline{0}_K \in A$ and $D_K \not\subset D_I$. There is a sequence from $\overline{0}_I$ to $\overline{0}_K$, as described above. An element $\overline{0}_j$ of this sequence meets $\overline{0}_K$ with $j < K$. So $D_j \subset D_I$ but also by construction $D_K \subset D_j$. Hence $D_K \subset D_I$. Furthermore, A^* contains each element of Q which A^* intersects. For if $\overline{0}_i \in Q$ and $\overline{0}_i$ meets A^*, let J be least such that $\overline{0}_J \in A$ and $\overline{0}_i$ meets $\overline{0}_J$. Now if J < i, augmenting the sequence from $\overline{0}_I$ to $\overline{0}_J$ by $\overline{0}_i$ gives a sequence from $\overline{0}_I$ to $\overline{0}_i$, placing $\overline{0}_i \in A$. So suppose J > i. Let $\overline{0}_K$ be the element of the sequence from $\overline{0}_I$ to $\overline{0}_J$ which meets $\overline{0}_J$. Then k < J. So $\overline{0}_i$ does not meet $\overline{0}_k$. But $\overline{0}_J$ cannot meet both of the disjoint sets $\overline{0}_i$ and $\overline{0}_k$ by Lemma 3.1. Now by Lemma 3.2 $h_I(A^*) = h_J(A^*)$. And since $\overline{0}_I \subset A^*$, $h_J(\overline{0}_I) \subset h_J(A^*) =$

$h_I(A^*) \subset h_I D_I$, and Lemma 3 is proved.

Now, $\{\overline{U}_i\}$ is a locally finite collection since $U_i \supset \overline{U}_{i+1}$ and $\cap U_i = \phi$. $\{\overline{0}_i\}$ is locally finite, as $\overline{0}_i \subset U_i$. Since each $\overline{0}_j$ is compact, it meets at most a finite number of elements of $\{0_i\}$. So for each j there exists $N(j) \geq j$ such that $\overline{0}_j \subset M \setminus \cup_{i \geq N(j)} \overline{0}_i$. Then $h\overline{0}_j = h_{N(j)}\overline{0}_j \subset h_j D_j$ by Lemma 3, while $D_j \cup h_j D_j \subset D_{L(j)}$ for some $L(j) \leq j$ by Lemma 2. Thus $\overline{0}_j \cup h\overline{0}_j \subset D_{L(j)} \subset A_{L(j)}$. For each $C \in H \setminus \cup p0_i$, $hC = C$ and diam $C < \varepsilon$. Suppose $C = C_J$ so that $C \subset A_J$. Then $A_J \cap h^{-1}A_J$ contains a p-open set $N(C)$ containing C and we have $N(C) \cup hN(C) \subset A_J$.

Let $V = \{0_j\}_{j=1}^{\infty} \cup \{N(C): C \in H \setminus \cup p0_i\}$. Then V is a p-open refinement of $\mathcal{A}$, h = id off V^*, h shrinks each element of H to diameter $< \varepsilon$, and $v \in V \Rightarrow$ there exists $A \in \mathcal{A}$ with $A \supset v \cup hv$. Thus, H is weakly tsh.

Since the hypotheses of Theorem H' are topological, we have immediately that H is tsh. Hence, by Theorem T,

Corollary H' (McAuley). *Under the hypotheses of Theorem* H', $I \approx M$.

References

1. L. F. McAuley, *Some upper semicontinuous decompositions of* E^3 *into* E^3, Annals of Math. 73 (1961), 437-457.
2. ________, *Upper semicontinuous decompositions of* E^3 *into* E^3 *and generalizations to metric spaces*, Topology of 3-manifolds and related topics, Prentice-Hall, Englewood Cliffs, N.J., 1962, 21-26.
3. ________, *Shrinkable decompositions, criteria, and generalizations*, these proceedings.
4. M. J. Reed, *Decomposition spaces and separation*

properties, Ph.D. Dissertation, SUNY-Binghamton, June, 1971.

5. ________, *Hausdorff-like separation properties and generalizations of the first countability axiom*, Tamkang J. Math. 5 (1974), 197-201.

6. G. T. Whyburn, *Analytic topology*, AMS Colloq. Publ. 28 (1942), Providence, R.I.

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