
TOPOLOGY PROCEEDINGS



Volume 4, 1979

Pages 533–540

<http://topology.auburn.edu/tp/>

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Topology Proceedings

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ISSN: 0146-4124

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MONOTONE MAPS AND ϵ -MAPS

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In this paper, a *compactum* is a compact metric space; a *continuum* is a connected compactum, and a *map* is a continuous function. A map $f: X \rightarrow Y$ is an ϵ -map, for $\epsilon > 0$, if and only if $\text{diam}(f^{-1}(y)) < \epsilon$ for each $y \in Y$. If X is a continuum, and \mathcal{P} is a class of continua, then X is \mathcal{P} -like if and only if there is an ϵ -map from X onto some element of \mathcal{P} for every $\epsilon > 0$. If $\mathcal{P} = \{Y\}$ is degenerate, then X is also said to be Y -like. A *graph* is a space homeomorphic to the space of a one-dimensional finite complex. It will usually be assumed that the graph is triangulated, and the 1-simplexes will also be called *edges* and the 0-simplexes, *vertices*.

In [2], Carlisle proved a theorem that related the problem of determining which graphs are "like" which other graphs to the intuitively simpler problem of determining which graphs are monotone images of which other graphs.

Theorem A (Carlisle). If G and G' are graphs, then G is G' -like if and only if G is a monotone image of G' .

The proof in [2] involves the use of piecewise linear maps and is somewhat long, but other interesting results follow from the theorems used. The purpose of this note is to give a complete and fairly short argument for a generalization of Theorem A. The theorems used here are of independent interest. It follows from Theorem 1, for example, that

no continuum that admits a monotone map onto an arc is circle-like. Theorem 3 implies each of the following results: every monotone image of an arc-like (or circle-like) continuum is arc-like (or circle-like) (see [1], Theorem 3, p. 47), and if M is a locally connected continuum which admits a monotone map onto an arc then the arc is M -like (in particular, the arc is disk-like, torus-like, etc.).

Theorem 1. Suppose k is a positive integer. If there exists a monotone map m from the continuum M onto the graph G , then there exists a positive number ϵ such that if there is an ϵ -map from M onto a graph G with no more than k vertices, then there is a monotone map from G' onto G .

Proof. Suppose not. Then there exists, for each positive integer i , a $(1/i)$ -map f_i from M onto a graph G_i with no more than k vertices such that G_i admits no monotone map onto G . For each i , let v_1^i, \dots, v_k^i denote a sequence of distinct points of G_i which includes all the vertices of G_i . By choosing a subsequence and relabeling, we may arrange that the sequence $f_1^{-1}(v_j^1), f_2^{-1}(v_j^2), \dots$ converges to a point p_j of M for all $1 \leq j \leq k$.

For each edge E of G , let $Q(E)$ denote a subinterval of E which misses both end points of E and also $m(p_j)$ for $1 \leq j \leq k$. Then, because of the uniform continuity of m , there is a positive integer i' such that if $i > i'$, then $Q(E)$ misses $m(f_i^{-1}(v_j^i))$ for each j . The closure of $E - Q(E)$ is the union of two nonintersecting intervals, E_s and E_t , containing the end points s and t of E , respectively.

Pick $i > i'$ sufficiently large that (letting $z(H) = f_i(m^{-1}(H))$ for each $H \subseteq G$) if E and F are distinct edges of G' , then $z(Q(E)) \cap z(Q(F)) = \emptyset$, and, if E is an edge of G with endpoints s and t , then $z(E_s)$ misses $z(E_t)$. Since m is monotone, $z(Q(E))$ is a continuum, and it intersects no vertex of G_i ; hence it lies in a single edge of G_i . Using the fact that $z(E_s)$ and $z(E_t)$ are nonintersecting continua which intersect $z(Q(E))$ but also contain points outside of $z(Q(E))$ (namely, the points of $z(s)$ and $z(t)$, respectively), it is not hard to see that $P(E) = z(E) - z(E_s) \cup z(E_t)$, a subset of $z(Q(E))$, is connected, and that $\overline{P(E)}$ is therefore an arc lying in a single edge of G_i .

We are now prepared to define a monotone map from G_i onto G , which gives a contradiction. If $x \in z(E_s)$ for some vertex s of some edge E of G , define $g'(x) = s$. Each of the remaining points of G_i lies in $\overline{P(E)}$ for some edge E of G , and these sets are all nonintersecting arcs. Hence g' extends to a map g from G_i onto G such that $g|_{\overline{P(E)}}$ is a homeomorphism onto E for each edge E of G . The map g is clearly monotone.

To see that the bound on the number of vertices of the graphs G' in this lemma is necessary, consider the following example. Let $M = G = [0,1]$, and m denote the identity on M . Then, if $\epsilon > 0$, there is a positive integer n so that there is an ϵ -map from M onto the set A_n :

$$\{(x,y) \mid 0 \leq x \leq n \text{ and } y = 0 \text{ or } y = 1\} \cup \bigcup_{i=0}^n \{(x,y) \mid x = i \text{ and } 0 \leq y \leq 1\}.$$

However, there is no monotone map from A_n onto G .

Lemma 2. Suppose m is a monotone map from the continuum M onto the continuum X , and G is a finite open cover of X of order ≤ 1 . Then there is a number $\varepsilon > 0$ so that if f is an ε -map from M onto a locally connected continuum L , then there exist a finite open refinement G' of G of order ≤ 1 , a finite open cover H of L of order ≤ 1 with connected sets, and a 1-1 function u from G' onto H so that if g and g' belong to G' , then $g \cap g' \neq \emptyset$ implies $u(g) \cap u(g') \neq \emptyset$ and $u(g) \cap u(g') \neq \emptyset$ implies $\text{diam}(g \cup g') \leq 2 \cdot \text{mesh}(G)$.

Proof. Let $\{v(g) \mid g \in G\}$ be an open cover of X so that $\overline{v(g)} \subseteq g$ for each $g \in G$ and $v(g) \cap v(g') = \emptyset$ if and only if $g \cap g' = \emptyset$. Pick $\varepsilon > 0$ so small that, if f is an ε -map from M onto a space L , then, for each $g \in G$,

$$(a) \quad f(m^{-1}(\overline{v(g)})) \text{ misses } f(m^{-1}(X-g)).$$

Suppose f is an ε -map from M onto the locally connected continuum L , and denote $f(m^{-1}(H))$ by $z(H)$ for each $H \subseteq X$.

Pick $\delta > 0$ so that

$$(b) \quad \delta \leq d(zv(g), z(X-g)) \text{ for each } g \in G,$$

and let H' denote a finite cover of L with connected open sets of diameter $< \delta$. Let H'' denote the collection of all sets h such that, for some element $w(h)$ of G , h is the union of the elements of a subcollection of H' maximal with respect to the property that each element of it intersects $zvw(h)$ and the union of all the elements of it is connected. For each h in H'' , let $o(h)$ denote the open set $X - mf^{-1}(L-h)$, and let $G' = \{o(h) \mid h \in H''\}$. It is easily verified that

$$(c) \quad o(h) \subseteq mf^{-1}(h) \subseteq w(h) \text{ for each } h \text{ in } H''$$

(the second inclusion follows from the fact that, using (b), h misses $fm^{-1}(X-w(h))$). Also,

(d) $o(h) = \{x \in X \mid z(x) \subseteq h\}$ for each h in H'' .

Clearly, H' covers L . Since m is monotone, it follows that, for each x in X , $z(x)$ is a continuum lying in $zv(g)$, for some g in G . Some h in H'' is constructed from g (although g may not be chosen as $w(h)$), and $z(x) \subseteq h$. It follows from (d) that $x \in o(h)$, and so G' covers X . Since, for all h and h' in H'' , $w(h) = w(h')$ implies $h \cap h' = \emptyset$, it follows from (c) that H'' is of order ≤ 1 , and that G' is of order ≤ 1 .

Now, for each $g \in G'$, pick $u(g) = h \in H''$ so that $g = o(h)$ and let $H = \{u(g) \mid g \in G'\}$. Since G' covers X , $g = o(u(g))$ for each g in G' , and, by (d), $z(g) \subseteq u(g)$, it follows that H covers L . So H is an open cover of L of order ≤ 1 with connected open sets, and u is 1-1 from G' onto H . The conclusion of the Lemma now follows from the fact that, if g and g' are in G' , then (1) $z(g \cap g') \subseteq u(g) \cap u(g')$, from (d), and (2) $mf^{-1}(u(g) \cap u(g')) \subseteq w(u(g)) \cap w(u(g'))$ and $g \cup g' \subseteq w(u(g)) \cup w(u(g'))$, by (c).

Theorem 3. Suppose that \mathcal{P} is a class of locally connected continua, M is a \mathcal{P} -like continuum, and m is a monotone map from M onto the 1-dimensional continuum X . Then X is \mathcal{P} -like.

Proof. We will show that, for every $\delta > 0$, there is a δ -map from X onto some element of \mathcal{P} . Suppose $\delta > 0$, and let G denote a finite open cover of X of mesh $< \delta/2$ and order 1. Let ϵ be as given in Lemma 2. There is an ϵ -map from M onto some element L of \mathcal{P} . Let G' , H , and u be as given in Lemma 2. Let N denote the nerve of G' , N' denote the first barycentric subdivision of N , and, for each simplex s of N ,

let b_s denote the barycenter of s . For each $g \in G'$, let $u'(g)$ denote a non-empty open subset of L so that $\overline{u'(g)} \subseteq u(g)$, $u'(g)$ intersects $u'(g')$ whenever $u(g)$ intersects $u(g')$, and $H' = \{u'(g) \mid g \in G'\}$ covers L . Let h denote a canonical map from X onto $|N|$ (see [5], Lemma 6, p. 79). If the elements g and g' of G' are vertices of a 1-simplex s of N , let b'_s denote some point of $u'(g) \cap u'(g')$. If s is a 0-simplex, $\{g\}$, of N , let $b'_s \in u'(g)$.

Now, using the fact that the Cantor set maps onto every compactum, and that $\overline{u'(g)}$ is a non-empty compact subset of the connected and locally connected set $u(g)$ for each $g \in G'$, it follows from Theorem 5, p. 253 of [4] that there is a map j from $|N|$ into L so that, for each $g \in G'$, $\overline{u'(g)} \subseteq j(\overline{st_N(g)}) \subseteq u(g)$ and, for each simplex s of N , $j(b'_s) = b'_s$. The map j is onto because H' covers L . So jh is onto. To see that jh is a δ -map, suppose that $x \in L$. Since H is of order 1, there are elements g and g' of G' (not necessarily distinct) so that $x \in u(g) \cap u(g')$, but if g'' is any other element of G' , then $x \notin u(g'') \supseteq j(st_N(g''))$. Consequently, $j^{-1}(x) \subseteq st_N(g) \cup st_N(g') \subseteq st_N(g) \cup st_N(g')$. So, since h is canonical, $h^{-1}j^{-1}(x) \subseteq h^{-1}(st_N(g) \cup st_N(g')) = h^{-1}(st_N(g) \cup h^{-1}(st_N(g'))) \subseteq g \cup g'$. Since $\text{diam}(g \cup g') < 2(\delta/2) = \delta$ by Lemma 2, jh is the desired δ -map from X onto L .

The requirement that X be 1-dimensional in Theorem 3 is necessary, since there is a monotone map from the universal planar curve C (obtained by removing the interiors of a dense null-sequence of nonintersecting square disks from the unit disk in the plane) onto the unit disk. Now if \mathcal{P} is

the class of all compact 1-dimensional polyhedra, then C is \mathcal{P} -like, but the disk, being 2-dimensional, cannot be.

Theorem 4. *If there is an ε -map g from the continuum M onto the graph G , and the graph G' admits a monotone map m onto G , then there is an ε -map from M onto G' .*

Proof. It is well known that if g is an ε -map from M onto the continuum G , then there exists a positive number δ such that if h is a δ -map from G onto any continuum, then hg is an ε -map. By Theorem 3, there exists such a map h from G onto G' . So hg satisfies the requirements of the theorem.

A map $f: X \rightarrow Y$ is *refinable* [3] if and only if, for each $\varepsilon > 0$, there is an ε -map h from X onto Y so that $d(f, h) < \varepsilon$, i.e., f is a uniform limit of ε -maps for each $\varepsilon > 0$. All near homeomorphisms are refinable, but there are many refinable maps onto graphs from continua with very bad local properties; indeed, it is shown in [3] that every hereditarily decomposable arc-like continuum admits a refinable map onto an arc. Still, a continuum which admits a refinable map onto a graph shares a number of properties with that graph, one of which is given by the following theorem. This theorem generalizes Carlisle's theorem if M is taken to be G , and the refinable map is taken to be the identity on M .

Theorem 5. *If the continuum M admits a refinable map onto a graph G , and G' is a graph, then M is G' -like if and only if G is a monotone image of G' .*

Proof. If G is a monotone image of G' , then M is G' -like by Theorem 4. If M is G' -like, then there is a monotone map from G' onto G by Theorem 1.

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